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Coalgebra structures in algebras

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Abstract. Historically, the interest in coalgebras grew out from the study of the notion of *Hopf algebras* introduced in topology. Now, in the definition of Hopf algebras, the coalgebraic part is formulated explicitly and is by itself the basis of a rich theory. Separable, Azumaya, and Frobenius algebras are usually introduced as algebraic structures without referring to the notion of a coalgebra. In this survey we reveal the internal coalgebra structure in these algebras which may also be used to characterise them and to describe their properties. All these classes of algebras – including the Hopf algebras – have an associative multiplication and a coassociative comultiplication; they are distinguished by requiring different compatibility conditions and properties for units and counits.

1 Modules and algebras over commutative rings

Throughout this paper R will denote a commutative ring with unit. Firstly we recall the basic notions of algebras and modules over R in a way which allows for an easy translation to coalgebras and comodules. For this the language of category theory is extremely helpful and for convenience we provide the basic notions needed (from [9], [21]).

1.1. Categories. A *category* $\mathbb{A}$ consists of a class of *objects* $\text{Obj}(\mathbb{A})$ and, for any $A, A' \in \text{Obj}(\mathbb{A})$, a (possibly empty) set $\text{Mor}_{\mathbb{A}}(A, A')$ of *morphisms* which allow for an associative composition. Furthermore, for any $\text{Mor}_{\mathbb{A}}(A, A)$ the existence of an *identity morphism* is required which we denote by I_A (or just I).

A *covariant functor* $F : \mathbb{A} \to \mathbb{B}$ between categories $\mathbb{A}$ and $\mathbb{B}$ consists of assignments $\text{Obj}(\mathbb{A}) \to \text{Obj}(\mathbb{B})$, $A \mapsto F(A)$, and

$$\text{for all } A, A' \in \text{Obj}(\mathbb{A}),\ \text{Mor}_{\mathbb{A}}(A, A') \to \text{Mor}_{\mathbb{B}}(F(A), F(A')),\ f \mapsto F(f),$$

respecting the identity morphism and the composition of morphisms. F is a *contravariant functor* if it reverses the composition of morphisms.

Given two functors $F, G : \mathbb{A} \to \mathbb{B}$, a natural transformation $\gamma : F \to G$ is defined by a family of morphisms $\gamma_A : F(A) \to G(A)$, $A \in \text{Obj}(\mathbb{A})$, such that any morphism $f : A \to A'$ in $\mathbb{A}$ induces commutativity of the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_{A'}} \\ G(A) & \xrightarrow{G(f)} & G(A'). \end{array}$$

For functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ between any categories $\mathbb{A}$ and $\mathbb{B}$, a *pairing* is defined by maps, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\text{Mor}_{\mathbb{B}}(L(A), B) \xrightarrow{\alpha_{A,B}} \text{Mor}_{\mathbb{A}}(A, R(B)) \xrightarrow{\beta_{A,B}} \text{Mor}_{\mathbb{B}}(L(A), B).$$

Such a pairing is determined by the images of the identity morphisms of $L(A)$ and $R(B)$, respectively,

$$\eta_A := \alpha_{A,L(A)}(I_{L(A)}) : A \to RL(A),$$
$$\varepsilon_B := \beta_{R(B),B}(I_{R(B)}) : LR(B) \to B,$$

corresponding to natural transformations

$$\eta : I_{\mathbb{A}} \to RL, \quad \varepsilon : LR \to I_{\mathbb{B}},$$

which are called *quasi-unit* and *quasi-counit* of (L, R, α, β), respectively. They allow to reconstruct α and β.

(L, R) is said to be an *adjoint pair* provided $\alpha \circ \beta$ and $\beta \circ \alpha$ yield the identity maps and this corresponds to the equalities

$$L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L = I_L, \quad R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R = I_R,$$

known as *triangular identities*.

Notice that so far we only have put up a framework without using deeper results from category theory. In the course of this talk we will encounter several more concrete examples of these abstract notions.

1.2. Category of R-modules. For the ring R, denote by $\mathbb{M}_R$ the category of R-modules, taking for objects the R-modules and for morphisms the R-linear maps. This is a category with products and coproducts, kernels and cokernels, and the R-module R as a projective generator.

For any R-modules M, N, there is the tensor product $M \otimes_R N$ yielding the functors

$$M \otimes_R - : \mathbb{M}_R \to \mathbb{M}_R, \quad N \mapsto M \otimes_R N,$$
$$\mathrm{Hom}_R(M, -) : \mathbb{M}_R \to \mathbb{M}_R, \quad N \mapsto \mathrm{Hom}_R(M, N),$$

which form an adjoint pair by the bijection (property of tensor product)

$$\mathrm{Hom}_R(M \otimes_R N, K) \xrightarrow{\simeq} \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, K)), \quad \varphi \mapsto [n \mapsto \varphi(- \otimes n)], \tag{1.1}$$

and unit and counit of this adjunction come out as

$$\eta_N : N \to \mathrm{Hom}_R(M, M \otimes N), \quad n \mapsto [m \mapsto m \otimes n],$$
$$\varepsilon_N : M \otimes \mathrm{Hom}_R(M, N) \to N, \quad m \otimes f \mapsto f(m).$$

The functors $M \otimes_R -$ and $\mathrm{Hom}_R(M, -)$ are naturally isomorphic if and only if M is a finitely generated and projective R-module: the isomorphism implies that $\mathrm{Hom}_R(M, -)$ preserves epimorphisms and direct sums (since $M \otimes_R -$ does so) and hence M has the properties required.

Furthermore, M and N can be interchanged by the twist map

$$\tau_{M,N} : M \otimes_R N \to N \otimes_R M, \quad m \otimes n \mapsto n \otimes m,$$

which obviously satisfies $\tau_{N,M} \circ \tau_{M,N} = I_{M \otimes N}$.

For modules over a field R all these properties are well-known from elementary linear algebra and the corresponding proofs hold for any commutative base rings.

1.3. Algebras over R. An R-algebra $(A, m, 1_A)$ is defined as R-module A with an associative R-bilinear multiplication $\overline{m} : A \times A \to A$ (usually written as $(a, b) \mapsto ab$) and unit element 1_A satisfying $1_A a = a 1_A$ for all $a \in A$.

By the properties of the tensor product, the bilinear map $\overline{m}$ can be replaced by an R-linear map $m : A \otimes_R A \to A$, and 1_A defines an R-linear map $\eta : R \to A, r \mapsto r1_A$. With this terms, associativity and unitality conditions required for an algebra are expressed by commutativity of the diagrams (writing $\otimes$ for $\otimes_R$)

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes I_A"] \arrow[d, "I_A \otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A, \end{tikzcd} \qquad \begin{tikzcd} A \otimes R \arrow[r, "I_A \otimes \eta"] \arrow[dr, "="'] & A \otimes A \arrow[d, "m"] & R \otimes A \arrow[l, "\eta \otimes I_A"'] \arrow[dl, "="] \\ & A & \end{tikzcd} .$$

Defining R-algebras (A, m, η) in this way we are only using objects and morphisms in the category $\mathbb{M}_R$.

As well known, the tensor product $A \otimes B$ of two R-algebras A and B is again an R-algebra by componentwise multiplication. For this definition, the twist map $\tau_{B,A} : B \otimes A \to A \otimes B$ is needed. Analysing the setting shows that there may be other R-linear maps $\lambda : B \otimes A \to A \otimes B$ leading to an associative algebra structure on $A \otimes B$. These are examples of *distributive laws* known from general category theory (e.g. [2]).

1.4. Tensor product of algebras. Consider two R-algebras (A, m, η) and (B, m', η'). Multiplication and unit on $A \otimes B$ can be defined by

$$\begin{gathered} m_{AB} : A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \tau_{B,A} \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m'} A \otimes B, \\ a \otimes b \otimes c \otimes d \longmapsto a \otimes c \otimes b \otimes d \longmapsto ab \otimes cd, \end{gathered}$$

$$\eta_{AB} : R \xrightarrow{\eta \otimes \eta'} A \otimes B, \qquad 1_R \mapsto 1_A \otimes 1_B,$$

making $(A \otimes B, m_{AB}, \eta_{AB})$ an associative unital algebra.

Replacing $\tau_{B,A}$ by some R-linear map $\lambda : B \otimes A \to A \otimes B$, we observe:

1.5. Distributive laws. Let (A, m, η) and (B, m', η') be R-algebras with some R-linear map $\lambda : B \otimes A \to A \otimes B$. Defining a product on $A \otimes B$ by

$$m_\lambda : A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \lambda \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m'} A \otimes B,$$

the triple $(A \otimes B, m_\lambda, \eta \otimes \eta')$ is an (associative and unital) R-algebra if and only if it induces commutativity of the diagrams

$$\begin{tikzcd} B \otimes A \otimes A \arrow[r, "\lambda \otimes A"] \arrow[d, "B \otimes m"'] & A \otimes B \otimes A \arrow[r, "A \otimes \lambda"] & A \otimes A \otimes B \arrow[d, "m \otimes B"] \\ B \otimes A \arrow[rr, "\lambda"] & & A \otimes B \\ B \otimes B \otimes A \arrow[r, "B \otimes \lambda"] \arrow[u, "m' \otimes A"] & B \otimes A \otimes B \arrow[r, "\lambda \otimes B"] & A \otimes B \otimes B, \arrow[u, "A \otimes m'"'] \end{tikzcd}$$

$$\begin{tikzcd} B \otimes A \arrow[rr, "\lambda"] & & A \otimes B \\ & B \arrow[ul, "B \otimes \eta"] \arrow[ur, "\eta \otimes B"'] & \end{tikzcd}, \qquad \begin{tikzcd} B \otimes A \arrow[rr, "\lambda"] & & A \otimes B \\ & A \arrow[ul, "\eta' \otimes A"] \arrow[ur, "A \otimes \eta'"'] & \end{tikzcd}.$$

Modules M over an A-algebra are defined by R-bilinear maps $\overline{\varrho} : A \times M \to M$. Using the tensor product this can again be expressed by referring only to objects and morphisms from the category $\mathbb{M}_R$.

1.6. A-modules. Let (A, m, η) be an R-algebra. A (unital) left A-module is a pair (M, ϱ_M), where M is an R-module and $\varrho_M : A \otimes M \to A$ (written as $(a \otimes m) \mapsto am$) is an R-linear map leading to commutativity of the diagrams

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "m \otimes I_M"] \arrow[d, "I_A \otimes \varrho_M"'] & A \otimes M \arrow[d, "\varrho_M"] \\ A \otimes M \arrow[r, "\varrho_M"] & A, \end{tikzcd} \qquad \begin{tikzcd} R \otimes M \arrow[r, "\eta \otimes I_M"] \arrow[dr, "="'] & A \otimes M \arrow[d, "\varrho_M"] \\ & M. \end{tikzcd}$$

An A-*module morphism* between two A-modules (M, ϱ_M) and (N, ϱ_N) is an R-linear map $f : M \to N$ with commutative diagram

$$\begin{tikzcd} A \otimes M \arrow[r, "I_A \otimes f"] \arrow[d, "\varrho_M"'] & A \otimes N \arrow[d, "\varrho_N"] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

The category of left A-modules is denoted by ${}_A\mathbb{M}$. Similar to $\mathbb{M}_R$, it also has products, coproducts, kernels and cokernels and a projective generator (= A) but it need not allow for a tensor product.

For any R-module X, $A \otimes X$ is a left A-module by multiplication of A and this induces the *free* and *forgetful functors*,

$$\phi_A : \mathbb{M}_R \to {}_A\mathbb{M}, \quad X \mapsto (A \otimes X, m \otimes I_X),$$
$$U_A : {}_A\mathbb{M} \to \mathbb{M}_R, \quad (M, \varrho_M) \mapsto M.$$

(ϕ_A, U_A) form an adjoint pair by the bijections, for $X \in \mathbb{M}_R$, $(M, \varrho_M) \in {}_A\mathbb{M}$,

$$\mathrm{Hom}_A(A \otimes X, M) \to \mathrm{Hom}_R(X, M), \quad A \otimes X \xrightarrow{f} M \mapsto X \xrightarrow{\eta \otimes I_X} A \otimes X \xrightarrow{f} M,$$
$$\mathrm{Hom}_R(X, M) \to \mathrm{Hom}_A(A \otimes X, M), \quad X \xrightarrow{g} M \mapsto A \otimes X \xrightarrow{I_A \otimes g} A \otimes M \xrightarrow{\varrho} M,$$

and unit $\overline{\eta}$ and counit $\overline{\varepsilon}$ for this adjunction come out as

$$\overline{\eta}_X : X \xrightarrow{\eta \otimes I_X} A \otimes X, \quad \overline{\varepsilon}_M : A \otimes M \xrightarrow{\varrho_M} M.$$

The algebra structure on the tensor product $A \otimes B$ of two algebras may also be seen as a *lifting of functors*, investigated in a general categorical setting by P. Johnstone [8], which here comes out as follows.

1.7. Lifting of functors. Let (A, m, η) and (B, m', η') be R-algebras and consider the diagram

$$\begin{tikzcd} {}_B\mathbb{M} \arrow[r, dashed, "\widehat{A}"] \arrow[d, "U_B"'] & {}_B\mathbb{M} \arrow[d, "U_B"] \\ {}_R\mathbb{M} \arrow[r, "A \otimes -"] & {}_R\mathbb{M}. \end{tikzcd}$$

The following are equivalent:

(a) there exists a functor $\widehat{A}$ making the diagram commutative;

(b) there is a distributive law $\lambda : B \otimes A \to A \otimes B$ (see 1.5);

(c) $A \otimes B$ has an algebra structure induced by some R-linear map $\lambda : B \otimes A \to A \otimes B$.

Hereby, for a B-module (M, ρ), $\widehat{A}(M)$ is the object $A \otimes M$ with the B-module structure

$$B \otimes A \otimes M \xrightarrow{\lambda \otimes I} A \otimes B \otimes M \xrightarrow{I \otimes \rho} A \otimes M,$$

and one may write

$$\widehat{A}(-) = (A \otimes B) \otimes_B - : {}_B\mathbb{M} \to {}_B\mathbb{M}.$$

2 Coalgebras over commutative rings

In the paper [7] (1941), H. Hopf pointed out the rich structure of the homology of manifolds which admit a product operation: it allows for a coproduct and a product satisfying certain compatibility properties. In [18] (1965), J.W. Milnor and J.C. Moore analysed the algebraic parts of this structure and provided an introduction to the theory of coalgebras and comodules. The framework we built up for algebras and modules in the preceding section is suitable for a natural transition to coalgebras and comodules. This will be described in the subsequent section. Again R will denote a commutative ring.

2.1. Coalgebras. A *coalgebra* over R is a triple (C, Δ, ε) where C is an R-module with coassociative product and counit, that is, there are R-linear maps

$$\Delta : C \to C \otimes C, \quad \varepsilon : C \to R,$$

inducing commutativity of the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "I\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes I"] & C \otimes C \otimes C, \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[d, "\Delta"] & \\ R \otimes C \arrow[ur, "="] & C \otimes C \arrow[l, "\varepsilon\otimes I_C"] \arrow[r, "I_C\otimes\varepsilon"'] & C \otimes R \arrow[ul, "="'] \end{tikzcd}$$

Similar to the situation for algebras, the product of two R-coalgebras (C, Δ, ε) and $(D, \Delta', \varepsilon')$ can be defined using the twist map $\tau_{C,D} : C \otimes D \to D \otimes C$; the latter can be replaced by a distributive law $\varphi : C \otimes D \to D \otimes C$ with commutative diagrams (e.g. [24, 4.11])

$$\begin{tikzcd} C \otimes C \otimes D \arrow[r, "C\otimes\varphi"] & C \otimes D \otimes C \arrow[r, "\varphi\otimes C"] & D \otimes C \otimes C \\ C \otimes D \arrow[u, "\Delta\otimes D"] \arrow[rr, "\varphi"] \arrow[d, "C\otimes\Delta'"'] & & D \otimes C \arrow[u, "D\otimes\Delta"'] \arrow[d, "\Delta'\otimes C"] \\ C \otimes D \otimes D \arrow[r, "\varphi\otimes D"] & D \otimes C \otimes D \arrow[r, "D\otimes\varphi"] & D \otimes D \otimes C, \end{tikzcd}$$

$$\begin{tikzcd} C \otimes D \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon\otimes D"'] & & D \otimes C \arrow[dl, "D\otimes\varepsilon"] \\ & D & \end{tikzcd} , \qquad \begin{tikzcd} C \otimes D \arrow[rr, "\varphi"] \arrow[dr, "C\otimes\varepsilon'"'] & & D \otimes C \arrow[dl, "\varepsilon'\otimes C"] \\ & C & \end{tikzcd} .$$

2.2. C-comodules. Let (C, Δ, ε) be an R-coalgebra. A left *C-comodule* is a pair (M, ρ^M) where M is an R-module and $\rho^M : M \to C \otimes M$ is an R-linear map with commutative diagrams

$$\begin{tikzcd} M \arrow[r, "\rho^M"] \arrow[d, "\rho^M"'] & C \otimes M \arrow[d, "\Delta\otimes I"] \\ C \otimes M \arrow[r, "I\otimes\rho^M"] & C \otimes C \otimes M, \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\rho^M"'] \arrow[dr, "="] & \\ C \otimes M \arrow[r, "\varepsilon\otimes I_M"'] & M. \end{tikzcd}$$

A *C-comodule morphism* between two C-comodules (M, ρ^M) and (N, ρ^N) is an R-linear map $f : M \to N$ with commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho^M"'] & N \arrow[d, "\rho^N"] \\ C \otimes M \arrow[r, "I\otimes f"] & C \otimes N. \end{tikzcd}$$

These data form the category of left C-comodules, denoted by ${}^C\mathbb{M}$. There are the forgetful and the cofree functors,

$$U^C : {}^C\mathbb{M} \to \mathbb{M}_R, \quad (M, \rho^M) \mapsto M,$$
$$\phi^C : {}_R\mathbb{M} \to {}^C\mathbb{M}, \quad X \mapsto (C \otimes_R X, \Delta \otimes I_X,)$$

and (U^C, ϕ^C) is an adjoint pair by the bijection, for $(M, \rho^M) \in {}^C\mathbb{M}$, $X \in \mathbb{M}_R$,

$$\begin{array}{ll} \mathrm{Hom}^C(M, C \otimes X) \to \mathrm{Hom}_R(M, X), & M \xrightarrow{f} C \otimes X \mapsto M \xrightarrow{f} C \otimes X \xrightarrow{\varepsilon \otimes I_X} X, \\ \mathrm{Hom}_R(M, X) \to \mathrm{Hom}^C(M, C \otimes X), & M \xrightarrow{g} X \mapsto M \xrightarrow{\rho^M} C \otimes M \xrightarrow{I_C \otimes g} C \otimes X, \end{array}$$

and unit $\widetilde{\eta}$ and counit $\widetilde{\varepsilon}$ of this adjunction come out as

$$\widetilde{\eta}_M : M \xrightarrow{\rho^M} C \otimes M, \quad \widetilde{\varepsilon}_X : C \otimes X \xrightarrow{\varepsilon \otimes I_X} X.$$

From this adjunction a number of properties of comodules and their categories can be derived. For example, choosing $X = R$ and $M = C$, we obtain the isomorphisms

$$\mathrm{Hom}^C(M, C) \simeq \mathrm{Hom}_R(M, R), \quad \mathrm{End}^C(C) = \mathrm{Hom}^C(C, C) \simeq \mathrm{Hom}_R(C, R),$$

showing that the R-dual modules play a significant role here.

3 Frobenius and separable algebras

In [6] (1903), F. Frobenius investigated finite dimensional K-algebras A over a field K with the property that $A \simeq A^* := \mathrm{Hom}_K(A, K)$ as left A-modules. They can also be characterised by the existence of a non-degenerate bilinear form $\sigma : A \times A \to K$ with $\sigma(ab, c) = \sigma(a, bc)$ for all $a, b, c \in A$.

Such algebras A were named *Frobenius algebras* by Brauer and Nesbitt (1937); their duality properties were pointed out by Nakayama (1939); Eilenberg and Nakayama observed (1955) that the notion makes sense over commutative rings, provided A is finitely generated and projective as an R-module.

Frobenius algebras are of considerable interest in representation theory of finite groups, number theory, combinatorics, coding theory, etc. Their relation with coalgebras were mentioned by Lawvere (1967), Quinn (1991), Abrams (1999) e.a. As pointed out by Dijkgraaf (1989), Abrams (1996), and others, they show up in the framework of topological quantum field theory. An outline of their categorical formulation, the Frobenius monads, is given by Street in [20].

3.1. Coalgebra structure of A^*. Let (A, m, η) be an R-algebra and assume A to be finitely generated and projective as an R-module. Then there is an R-linear isomorphism $\lambda : A \to A^*$ and $(A \otimes_R A)^* \simeq A^* \otimes_R A^*$ as R-modules.

Applying $(-)^* := \mathrm{Hom}_R(-, R)$ to $m : A \otimes_K A \to A$ and $\eta : R \to A$ yields comultiplication and counit on A^*,

$$A^* \xrightarrow{m^*} (A \otimes_R A)^* \simeq A^* \otimes_R A^*, \quad A^* \xrightarrow{\eta^*} R.$$

Applying λ, the coproduct and counit of A^* can be transferred to A:

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\lambda"'] & A \otimes_K A \\ A^* \arrow[r, "m^*"] & A^* \otimes_K A^* \arrow[u, "\lambda^{-1} \otimes \lambda^{-1}"'] , \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "\lambda"'] \arrow[dr, "\varepsilon"] & \\ A^* \arrow[r, "\eta^*"] & R, \end{tikzcd}$$

making (A, δ, ε) a counital coalgebra.

Now, if we assume $\lambda : A \to A^*$ to be left A-linear, a little computation shows that δ is also left A-linear and - by symmetry - also right A-linear and this means that product and coproduct on A are related by the *Frobenius conditions*, that is, commutativity of the diagrams

$$\begin{array}{ccc} A\otimes A & \xrightarrow{m} & A \\ {\scriptstyle I\otimes\delta}\downarrow & & \downarrow{\scriptstyle \delta} \\ A\otimes A\otimes A & \xrightarrow{m\otimes I} & A\otimes A\,, \end{array} \qquad \begin{array}{ccc} A\otimes A & \xrightarrow{m} & A \\ {\scriptstyle \delta\otimes I}\downarrow & & \downarrow{\scriptstyle \delta} \\ A\otimes A\otimes A & \xrightarrow{I\otimes m} & A\otimes A. \end{array} \tag{3.1}$$

It follows from general category theory (also shown in [1]) that the category ${}_A\mathbb{M}$ of left A-modules is isomorphic to the category ${}^A\mathbb{M}$ of left A-comodules: ${}_A\mathbb{M} \simeq {}^A\mathbb{M}$. This isomorphism can be seen as a characterising property of Frobenius algebras (e.g. [15, Theorem 3.13]).

The commutativity of the diagrams can be read in different ways.

3.2. Reformulation of the Frobenius conditions. Let (A, m, δ) be given as above.

(1) The following are equivalent:
   - (a) $\delta \circ m = (m \otimes I_A) \circ (I_A \otimes \delta)$ (left hand diagram);
   - (b) δ is a left A-module morphism;
   - (c) m is a right A-comodule morphism.

(2) The following are equivalent:
   - (a) $\delta \circ m = (I_A \otimes m) \circ (\delta \otimes I_A)$ (right hand diagram);
   - (b) δ is a right A-module morphism;
   - (c) m is a left A-comodule morphism.

By these observations one obtains:

3.3. Characterisation of Frobenius algebras. For an R-module A, let (A, m, η) be an R-algebra and (A, δ, ε) a coalgebra. Then the following are equivalent:

(a) (A, m, δ) satisfies the Frobenius conditions;
(b) δ is a left A-module morphism and m is a left A-comodule morphism;
(c) δ is a left A-module morphism and a right A-module morphism;
(d) m is a left A-comodule and a right A-comodule morphism;
(e) $A \otimes_R -$ (equivalently $- \otimes_R A$) is adjoint to itself by the unit and counit

$$I_{\mathbb{A}} \xrightarrow{\eta} A \xrightarrow{\delta} A\otimes A, \quad A\otimes A \xrightarrow{m} A \xrightarrow{\varepsilon} I_{\mathbb{A}}.$$

Notice that in (b) the conditions only refer to one side, no twist map is needed for this property. From (c) it follows that $\delta(a) = a\delta(1_A) = \delta(1_A)a$, for all $a \in \mathbb{A}$.

3.4. Frobenius bimodules. Let $(A, m, \eta, \delta, \varepsilon)$ be a Frobenius algebra. Then an R-module M is called a *Frobenius bimodule* provided it has an A-module and also an A-comodule structure, $\varrho : A \otimes M \to M$ and $\nu : M \to A \otimes M$, inducing commutativity of the diagrams

$$\begin{array}{ccc} A\otimes M & \xrightarrow{\varrho} & M \\ {\scriptstyle I\otimes\nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ A\otimes A\otimes M & \xrightarrow{m\otimes I} & A\otimes M\,, \end{array} \qquad \begin{array}{ccc} A\otimes M & \xrightarrow{\varrho} & M \\ {\scriptstyle \delta\otimes I}\downarrow & & \downarrow{\scriptstyle \nu} \\ A\otimes A\otimes M & \xrightarrow{I\otimes\varrho} & A\otimes M. \end{array}$$

Taking for objects the Frobenius bimodules and for morphisms the R-linear maps which are A-module and A-comodule morphisms, one obtains the *category of Frobenius bimodules* which we denote by ${}^A_A\mathbb{M}$.

Obviously, (A, m, δ) itself is a Frobenius bimodule and there is a pair of functors

$$A \otimes_R - : {}_RM \to {}^A_A\mathbb{M}, \quad X \mapsto (A \otimes X, m_A \otimes X, \delta_A \otimes X),$$
$$\mathrm{Hom}^A_A(A, -) : {}^A_A\mathbb{M} \to {}_RM, \quad M \mapsto \mathrm{Hom}^A_A(A, M),$$

which form an adjoint pair by the bijection, for $X \in \mathbb{M}_R$, $(M, \varrho, \nu) \in {}^A_A\mathbb{M}$,

$$\mathrm{Hom}^A_A(A \otimes X, M) \xrightarrow{\simeq} \mathrm{Hom}_R(X, \mathrm{Hom}^A_A(A, M)), \quad \varphi \mapsto [x \mapsto \varphi(- \otimes x)],$$

and unit $\widehat{\eta}$ and counit $\widehat{\varepsilon}$ are given by

$$\widehat{\eta}_X : X \to \mathrm{Hom}^A_A(A, A \otimes X), \quad x \mapsto [a \mapsto a \otimes x],$$
$$\widehat{\varepsilon}_M : A \otimes \mathrm{Hom}^A_A(A, M) \to M, \quad a \otimes f \mapsto f(a).$$

Coinvariants of a Frobenius module M are defined as the image of

$$\mathrm{Hom}^A_A(A, M) \to M, \quad f \mapsto f(1_A),$$

and this map is indeed surjective, in particular one has $\mathrm{End}^A_A(A) \simeq A$. Thus the pair of functors $(A \otimes_R -, \mathrm{Hom}^A_A(A, -))$ induces an equivalence between ${}_A\mathbb{M}$ and ${}^A_A\mathbb{M}$.

This can be expressed by showing that, for any left A-module (M, ϱ), there is an A-comodule structure on M,

$$\nu : \; M \xrightarrow{\eta \otimes I} A \otimes M \xrightarrow{\delta \otimes I} A \otimes A \otimes M \xrightarrow{I \otimes \varrho} A \otimes M,$$

making (M, ϱ, ν) a Frobenius bimodule, and

$$\boldsymbol{\Psi} : {}_A\mathbb{M} \to {}^A_A\mathbb{M}, \quad (M, \varrho) \mapsto (M \varrho, \nu),$$

is an isomorphism of categories.

Similarly, any left A-comodule (M, ν) allows for a right comodule structure

$$\varrho : A \otimes M \xrightarrow{A \otimes \nu} A \otimes A \otimes M \xrightarrow{m \otimes A} A \otimes M \xrightarrow{\varepsilon \otimes M} M,$$

leading to the isomorphism of categories

$$\boldsymbol{\Phi} : {}^A\mathbb{M} \to {}^A_A\mathbb{M}, \quad (M, \nu) \mapsto (M \varrho, \nu).$$

Combining these functors, we obtain the isomorphisms of the A-module and the A-comodule categories mentioned before,

$${}_A\mathbb{M} \xrightarrow{\Psi} {}^A_A\mathbb{M} \xrightarrow{U_A} {}^A\mathbb{M}, \quad {}^A\mathbb{M} \xrightarrow{\Phi} {}^A_A\mathbb{M} \xrightarrow{U^A} {}_A\mathbb{M}.$$

Because of these isomorphisms, the category of Frobenius bimodules may seem to be of little interest for Frobenius algebras $(A, m, \eta, \delta, \varepsilon)$. However, the approach sketched above also allows to deal with more general situations, for example, when no counit (or unit) is at hand (see [26]).

3.5. Separable algebras. An R-algebra (A, m, η) is called *separable* if there is some A-bimodule map $\delta : A \to A \otimes A$ with $m \circ \delta = I_A$. This implies that (A, m, δ) satisfies the Frobenius condition (3.1) and yields a (comparison) functor

$$K_A : {}_R\mathbb{M} \to {}_A\mathbb{M}_A, \quad X \mapsto (A \otimes X, m_A \otimes X),$$

which is right adjoint to the functor ${}_A Hom_A(A, -) : {}_A\mathbb{M}_A \to {}_R M$ by the bijection (derived from (1.1))

$${}_A\mathrm{Hom}_A(A \otimes X, M) \xrightarrow{\simeq} \mathrm{Hom}_R(X, {}_A\mathrm{Hom}_A(A, M)).$$

Here the coinvariants of any $M \in {}_A\mathbb{M}_A$ are defined as the image of

$${}_A\mathrm{Hom}_A(A, M) \to M, \quad f \mapsto f(1_A),$$

and $Z(A) := {}_A\mathrm{End}_A(A)$ is the center of A leading to the equivalence

$$A \otimes_{Z(A)} - : {}_{Z(A)}\mathbb{M} \to {}_A\mathbb{M}_A, \quad N \mapsto (A \otimes_{Z(A)} N, m \otimes I_N).$$

The R-algebra A is called *central* if the map $R \to A$, $r \mapsto r1_A$, induces an isomorphism $R \simeq Z(A)$ and a central separable algebra is called *Azumaya algebra*. More about this kind of algebras can be found, for example, in [22] and [16].

In general categories, separable functors are considered in [19]; for Azumaya monads and comonads we refer to [17] for a recent account.

4 Bialgebras and Hopf algebras

In this section, we will again consider R-modules endowed with an algebra and a coalgebra structure but with different compatibility conditions.

4.1. Bialgebras. Let B be an R-module with an algebra structure $\underline{B} = (B, m, \eta)$ and a coalgebra structure $\overline{B} = (B, \Delta, \varepsilon)$. Then $(B, m, \eta, \Delta, \varepsilon)$ is called a *bialgebra* if

Δ and ε are algebra morphisms, or, equivalently,
μ and η are coalgebra morphisms.

To make Δ an algebra morphism one needs commutativity of the outer path in the diagram

$$\begin{tikzcd}
B \otimes B \arrow[r, "m"] \arrow[d, "B\otimes\Delta"'] & B \arrow[r, "\Delta"] & B \otimes B \\
B \otimes B \otimes B \arrow[rr, dashed, "\omega\otimes B"] \arrow[d, "\Delta\otimes B\otimes B"'] & & B \otimes B \otimes B \arrow[u, "B\otimes m"'] \\
B \otimes B \otimes B \otimes B \arrow[rr, "B\otimes\tau\otimes B"] & & B \otimes B \otimes B \otimes B. \arrow[u, "m\otimes B\otimes B"']
\end{tikzcd}$$

Defining an R-linear map

$$\omega : B \otimes B \xrightarrow{\Delta\otimes B} B \otimes B \otimes B \xrightarrow{B\otimes\tau} B \otimes B \xrightarrow{m\otimes B} B \otimes B,$$

the condition reduces to commutativity of the upper rectangle. With the map

$$\overline{\omega} : B \otimes B \xrightarrow{B\otimes\Delta} B \otimes B \otimes B \xrightarrow{\tau\otimes B} B \otimes B \xrightarrow{B\otimes m} B \otimes B$$

one obtains a similar rectangle (sides interchanged). These morphisms may be considered as *entwinings* between algebras and coalgebras (see Section 5),

$$\omega : \underline{B} \otimes \overline{B} \to \overline{B} \otimes \underline{B}, \quad \overline{\omega} : \overline{B} \otimes \underline{B} \to \underline{B} \otimes \overline{B}.$$

They can be applied to define *bimodules* which fit into the setting.

4.2. Hopf modules and algebras. Given a bialgebra $(\underline{B}, \overline{B}, \omega)$, an R-module M is called a *Hopf module* provided it is a B-module $\rho : B \otimes M \to M$ and a B-comodule $\nu : M \to B \otimes M$ inducing commutativity of the diagram

$$\begin{tikzcd}
B \otimes M \arrow[r, "\rho"] \arrow[d, "B\otimes\nu"'] & M \arrow[r, "\nu"] & B \otimes M \\
B \otimes B \otimes M \arrow[rr, "\omega\otimes M"] & & B \otimes B \otimes M. \arrow[u, "B\otimes\rho"']
\end{tikzcd}$$

The *category of Hopf modules*, denoted by ${}^B_B\underline{\mathbb{M}}$, has the Hopf modules as objects and as morphisms those R-linear maps, which are B-module as well as B-comodule morphisms. As can be shown easily, for any R-module X, $B \otimes_R X$ is a Hopf module and this observation leads to the functor

$$B \otimes_R - : {}_R\mathbb{M} \to {}^B_B\underline{\mathbb{M}}, \quad X \mapsto (B \otimes X, m \otimes X, \Delta \otimes X),$$

which is left adjoint to $\mathrm{Hom}^B_B(B, -) : {}^B_B\underline{\mathbb{M}} \to {}_R\mathbb{M}$ by the bijection (derived from (1.1))

$$\mathrm{Hom}^B_B(B \otimes X, M) \xrightarrow{\simeq} \mathrm{Hom}_R(X, \mathrm{Hom}^B_B(B, M)).$$

For any $M \in {}^B_B\underline{\mathbb{M}}$, the *coinvariants* are the image of $\mathrm{Hom}^B_B(B, M) \to M$, $f \mapsto f(1_B)$, and the coinvariants of B come out as $\mathrm{End}^B_B(B) \simeq R$.

A bialgebra $(\underline{B}, \overline{B}, \omega)$ is called a *Hopf algebra* provided $B \otimes_R - : {}_R\mathbb{M} \to {}^B_B\underline{\mathbb{M}}$ is an equivalence (known as *Fundamental Theorem*). This can be characterised by the existence of an *antipode*, and is also equivalent to require that the (fusion) morphism

$$(m \otimes I_B) \cdot (I_B \otimes \Delta) : B \otimes B \to B \otimes B$$

is an isomorphism (e.g. [5]). The corresponding constructions for monads and comonads on categories can be found in [12].

5 Entwining algebras and coalgebras

Suitable distributive laws (e.g. the twist map) allow for giving the tensor product of two algebras an algebra structure and the tensor product of two coalgebras a coalgebra structure. The question arises: which structure can be given to the tensor product of an algebra and a coalgebra? This leads to the notions of mixed distributive laws and corings over non-commutative rings (e.g. [4], [5]).

Let (A, m, η) be an R-algebra and (C, δ, ε) an R-coalgebra.

5.1. Entwining from A to C. An R-linear map $\omega : A \otimes C \to C \otimes A$ is called an *entwining from the algebra A to the coalgebra C* provided it induces commutativity of the diagrams

$$\begin{tikzcd}
A \otimes A \otimes C \arrow[r, "A\otimes\omega"] \arrow[d, "m\otimes C"'] & A \otimes C \otimes A \arrow[r, "\omega\otimes A"] & C \otimes A \otimes A \arrow[d, "C\otimes m"] \\
A \otimes C \arrow[rr, "\omega"] \arrow[d, "A\otimes\delta"'] & & C \otimes A \arrow[d, "\delta\otimes A"] \\
A \otimes C \otimes C \arrow[r, "\omega\otimes C"] & C \otimes A \otimes C \arrow[r, "C\otimes\omega"] & C \otimes C \otimes A,
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes C \arrow[rr, "\omega"] & & C \otimes A \\
& C \arrow[ul, "\eta\otimes C"] \arrow[ur, "C\otimes\eta"'] &
\end{tikzcd}, \qquad
\begin{tikzcd}
A \otimes C \arrow[rr, "\omega"] \arrow[dr, "A\otimes\varepsilon"'] & & C \otimes A \arrow[dl, "\varepsilon\otimes A"] \\
& A &
\end{tikzcd}.$$

The Hopf modules for bialgebras can be generalised to bimodules for entwined structures.

5.2. Bimodules for entwinings from A to C. For an entwining $\omega : A \otimes C \to C \otimes A$, an R-module M with an A-module structure $\varrho_M : A \otimes M \to M$ and a C-comodule structure $\varrho^M : M \to C \otimes M$ is called an *entwined module* if one gets commutativity of the diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "\varrho_M"] \arrow[d, "I_A\otimes\varrho^M"'] & M \arrow[r, "\varrho^M"] & C \otimes M \\
A \otimes C \otimes M \arrow[rr, "\omega\otimes I_M"] & & C \otimes A \otimes M. \arrow[u, "I_C\otimes\varrho_M"']
\end{tikzcd}$$

Taking as morphisms the R-linear maps which are A-module as well as C-comodule morphisms defines the category ${}^C_A\mathbb{M}$ of entwined modules. There is an (induction) functor (e.g. [5, 32.7])

$$C \otimes_R - : {}_A\mathbb{M} \to {}^C_A\mathbb{M}, \quad M \mapsto C \otimes_R M,$$

that is right adjoint to the forgetful functor ${}^CU : {}^C_A\mathbb{M} \to {}_A\mathbb{M}$.

Now assume A belongs to ${}^C_A\mathbb{M}$, that is, A is a C-comodule $\varrho : A \to C \otimes_R A$ with grouplike element $\varrho(1_A)$, and put $S := \mathrm{End}^C_A(A)$ (a subalgebra of A). Then there is a (comparison) functor

$${}_S\mathbb{M} \to {}^C_A\mathbb{M}, \quad X \mapsto (A \otimes_S X, m \otimes_S I_X, \varrho \otimes_S I_X),$$

and this is an equivalence provided C_R is flat and $C \otimes A$ is a *Galois coring* (see e.g. [5], [13], [23]). In case $A = C = B$, we get $S = R$ and this brings us back to the Hopf modules and the Fundamental Theorem (see 4.2).

5.3. Entwining from C to A. An R-linear map $\varpi : C \otimes A \to A \otimes C$ is an *entwining from the coalgebra C to the algebra A* if it induces commutativity of the diagrams

$$\begin{tikzcd}
C \otimes A \otimes A \arrow[r, "\varpi \otimes A"] \arrow[d, "C \otimes m"'] & A \otimes C \otimes A \arrow[r, "A \otimes \varpi"] & A \otimes A \otimes C \arrow[d, "m \otimes C"] \\
C \otimes A \arrow[rr, "\varpi"] \arrow[d, "\delta \otimes A"'] & & A \otimes C \arrow[d, "A \otimes \delta"] \\
C \otimes C \otimes A \arrow[r, "C \otimes \varpi"] & C \otimes A \otimes C \arrow[r, "\varpi \otimes C"] & A \otimes C \otimes C,
\end{tikzcd}$$

$$\begin{tikzcd}
C \otimes A \arrow[rr, "\varpi"] & & A \otimes C \\
& C \arrow[ul, "C \otimes \eta"] \arrow[ur, "\eta \otimes C"'] &
\end{tikzcd} , \qquad
\begin{tikzcd}
C \otimes A \arrow[rr, "\varpi"] \arrow[dr, "\varepsilon \otimes A"'] & & A \otimes C \arrow[dl, "A \otimes \varepsilon"] \\
& A &
\end{tikzcd} .$$

It was observed in Section 1.7 that the distributive laws between two algebras may be understood as liftings of functors to module categories. The situation for entwinings between algebras and coalgebras is quite similar.

5.4. Liftings and entwinings from A to C. An entwining $\omega : A \otimes C \to C \otimes A$ from A to C corresponds to a lifting $\widehat{C}$ of $C \otimes_R -$ to ${}_A\mathbb{M}$ and also to a lifting $\widehat{A}$ of $A \otimes_R -$ to ${}^C\mathbb{M}$, that is, there are commutative diagrams

$$\begin{tikzcd}
{}_A\mathbb{M} \arrow[r, "\widehat{C}"] \arrow[d, "U_A"'] & {}_A\mathbb{M} \arrow[d, "U_A"] \\
{}_R\mathbb{M} \arrow[r, "C \otimes -"] & {}_R\mathbb{M},
\end{tikzcd} \qquad
\begin{tikzcd}
{}^C\mathbb{M} \arrow[r, "\widehat{A}"] \arrow[d, "U^C"'] & {}^C\mathbb{M} \arrow[d, "U^C"] \\
{}_R\mathbb{M} \arrow[r, "A \otimes -"] & {}_R\mathbb{M},
\end{tikzcd}$$

where the U's denote the forgetful functors. The reader can find a more detailed description of liftings for tensor functors in [25].

For entwinings from a coalgebra to an algebra the situation is slightly different: they do not correspond to liftings to the (Eilenberg-Moore) categories ${}_A\mathbb{M}$ and ${}^C\mathbb{M}$ but to extensions to the *Kleisli categories* ${}_A\widetilde{\mathbb{M}}$ and ${}^C\widetilde{\mathbb{M}}$ (which may be seen as subcategories determined by the (co)free objects of the Eilenberg-Moore categories, e.g. [3]).

5.5. Liftings and entwining from C to A. An entwining $\varpi : C \otimes A \to A \otimes C$ from C to A corresponds to an extension $\widetilde{C}$ of $C \otimes_R -$ to ${}_A\widetilde{\mathbb{M}}$ and also to an extension $\widetilde{A}$ of $A \otimes_R -$ to ${}^C\widetilde{\mathbb{M}}$, that is, there are commutative diagrams

$$\begin{tikzcd}
{}_R\mathbb{M} \arrow[r, "C \otimes -"] \arrow[d, "\phi_A"'] & {}_R\mathbb{M} \arrow[d, "\phi_A"] \\
{}_A\widetilde{\mathbb{M}} \arrow[r, "\widetilde{C}"] & {}_A\widetilde{\mathbb{M}},
\end{tikzcd} \qquad
\begin{tikzcd}
{}_R\mathbb{M} \arrow[r, "A \otimes -"] \arrow[d, "\phi^C"'] & {}_R\mathbb{M} \arrow[d, "\phi^C"] \\
{}^C\widetilde{\mathbb{M}} \arrow[r, "\widetilde{A}"] & {}^C\widetilde{\mathbb{M}},
\end{tikzcd}$$

where the ϕ's denote the (co)free functors.

The notions in the preceding section can be readily transferred from the category $\mathbb{M}_R$ of R-modules to arbitrary categories $\mathbb{A}$. Hereby $A \otimes_R - : \mathbb{M}_R \to \mathbb{M}_R$ is to be replaced by any *monad* $F : \mathbb{A} \to \mathbb{A}$ and $C \otimes_R -$ is to be replaced by any *comonad* $G : \mathbb{A} \to \mathbb{A}$. The role of an entwining $\omega : A \otimes_R C \to C \otimes_R A$ is taken by a natural transformation $\omega : FG \to GF$ requiring commutativity of the corresponding diagrams and the definition of entwined modules is obvious. This allows to apply the basic theory in fairly general situations (e.g. [12], [13]).

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