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Fully Invariant Multiplication Modules

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Abstract. Let R be a commutative ring with identity. A unital R-module M is called a fully invariant multiplication module provided for each fully invariant submodule L of M there exists an ideal A of R such that $L = AM$. It is proved that every direct sum of isomorphic copies of a fixed fully invariant multiplication module X is also a fully invariant multiplication module. In particular this implies that every free R-module is a fully invariant multiplication module. In case R is a domain then every fully invariant multiplication module X is either torsion-free or $BX = 0$ for some non-zero ideal B of R and every torsion-free fully invariant multiplication module is divisible or reduced. If R is a Dedekind domain then every finitely generated torsion-free R-module is a fully invariant multiplication module and a classification is given for all torsion fully invariant multiplication R-modules.

1 Introduction

All rings are commutative with an identity and all modules are unital. For any undefined terms see [4]. Let R be a ring and let M be an R-module. Recall that a submodule L of M is called *fully invariant* provided $\varphi(L) \subseteq L$ for every endomorphism φ of M. Clearly 0 and M are fully invariant submodules of M. Every submodule of the R-module R is fully invariant. In case M is the direct sum of isomorphic copies of a simple R-module U then it is easy to check that 0 and M are the only fully invariant submodules of M. It is clear that the sum and the intersection of any collection of fully invariant submodules are also fully invariant. Thus the collection of fully invariant submodules of M form a sublattice of the complete modular lattice of all submodules of M. Note that the submodule AM of M is fully invariant for every ideal A of R.

The module M is called a *multiplication module* provided for each submodule N of M there exists an ideal B of R such that $N = BM$. Note that if M is a multiplication module then every submodule of M is fully invariant. The study of multiplication modules dates back to [14]. For more information about multiplication modules see [1]-[3], [5]-[6], [14], [17]-[20] and [22]. In particular, [1] contains many references and [22] discusses multiplication modules over certain non-commutative rings.

Given an R-module M and submodules L, N of M then $(N :_R L)$ will denote the set of elements $r \in R$ such that $rL \subseteq N$. Note that $(N :_R L)$ is an ideal of R for all submodules L, N of M. We now define the R-module M to be a *fully invariant multiplication module* in case for each fully invariant submodule K of M there exists an ideal G of R such that $K = GM$. It is clear that the module M is a fully invariant multiplication module if and only if $K = (K :_R M)M$ for every fully invariant submodule K of M. Clearly a module M is a multiplication module if and only if M is a fully invariant multiplication module and every submodule of M is fully invariant. It is also clear that any isomorphic copy of a fully invariant multiplication module is also a fully invariant multiplication module. It is well known (and easily proved) that if X is any non-zero R-module then the R-module $X \oplus X$ is not a multiplication module. However, our above comments show that every direct sum of isomorphic copies of a fixed simple module is a fully invariant multiplication module. We shall prove that, for each fully invariant multiplication R-module Y, every direct sum of isomorphic copies of Y is a fully invariant multiplication module (see Theorem 2.8). In particular, this shows that every direct sum of isomorphic copies of a fixed multiplication module is a fully invariant multiplication module (Corollary 2.9). Thus for any ring R, every free R-module is a fully invariant multiplication module (Corollary 2.10).

In addition if M_i $(1 \le i \le n)$ is a collection of R-modules for some positive integer n such that $R = \text{ann}_R(M_i) + \text{ann}_R(M_j)$ for all $1 \le i \ne j \le n$ then the R-module $M = M_1 \oplus \cdots \oplus M_n$ is a fully invariant multiplication module if and only if M_i is a fully invariant multiplication module for all $1 \le i \le n$ (Corollary 2.12). Here, for any R-module X, $\text{ann}_R(X)$ denotes the *annihilator* of X in R, that is $\text{ann}_R(X) = \{r \in R : rX = 0\}$.

Let R be a domain. If M is a fully invariant multiplication module over the ring R then M is torsion-free or $BM = 0$ for some non-zero ideal B of R (Lemma 4.1). Moreover every torsion-free fully invariant multiplication R-module is divisible or reduced (Proposition 4.4). On the other hand, every torsion-free divisible R-module is a fully invariant multiplication module (Proposition 4.6).

Now suppose that R is a Dedekind domain. It is proved that a non-zero torsion R-module M is a fully invariant multiplication module if and only if there exist positive integers n, k_i $(1 \le i \le n)$, distinct maximal ideals P_i $(1 \le i \le n)$ and index sets I_j $(1 \le j \le n)$ such that

$$M \cong (R/P_1^{k_1})^{(I_1)} \oplus \cdots \oplus (R/P_n^{k_n})^{(I_n)}.$$

(Theorem 5.3). One consequence is that a finitely generated R-module M is a fully invariant multiplication module if and only if M is torsion-free or there exist positive integers n, k_i $(1 \le i \le n)$, distinct maximal ideals P_i $(1 \le i \le n)$ and index sets I_j $(1 \le j \le n)$ such that

$$M \cong (R/P_1^{k_1})^{(I_1)} \oplus \cdots \oplus (R/P_n^{k_n})^{(I_n)}.$$

(Corollary 5.4).

2 Fully invariant multiplication modules

Let R be a ring. Note the following elementary and (well known) fact.

Lemma 2.1. Let an R-module $M = \oplus_{i \in I} M_i$ be the direct sum of submodules M_i $(i \in I)$ and let L be a fully invariant submodule of M. Then $L = \oplus_{i \in I} (L \cap M_i)$.

Corollary 2.2. Let an R-module $M = \oplus_{i \in I} M_i$ be the direct sum of R-modules M_i $(i \in I)$ and let L be a fully invariant submodule of M. Then $L = \oplus_{i \in I} L_i$ for some fully invariant submodule L_i of M_i for all $i \in I$.

Proof. By Lemma 2.1. □

Let M be any R-module. If M is a multiplication module then so too is any homomorphic image of M. This is not true for fully invariant multiplication modules. Let $\mathbb{Z}$ denote the ring of rational integers, let p be any prime in $\mathbb{Z}$ and let U and V be cyclic $\mathbb{Z}$-modules of order p^2. We claim that the $\mathbb{Z}$-module $M = U \oplus V$ is a fully invariant multiplication module. For, let L be any fully invariant submodule of M with $L \ne 0, M$. By Corollary 2.2, $L = pU \oplus 0$ or $0 \oplus pV$ or $pU \oplus pV$. If θ is the endomorphism of M defined by $\theta(u, v) = (v, u)$ for all $u \in U, v \in V$ then $\theta(pU \oplus 0) = 0 \oplus pV$ and $\theta(0 \oplus pV) = pU \oplus 0$. Thus $L = pU \oplus pV = pM$. It follows that M is a fully invariant multiplication module. Now let N denote the submodule $U \oplus pV$ of M. Note that N is a homomorphic image (and also a maximal submodule) of M. The socle L of N is $pU \oplus pV$ which is a fully invariant submodule of N. However $L \ne a(U \oplus pV) = aN$ for any $a \in \mathbb{Z}$. Thus the module N is not a fully invariant multiplication module. However we have the following result.

Proposition 2.3. Let K be a fully invariant submodule of a fully invariant multiplication module M. Then the module M/K is also a fully invariant multiplication module.

Proof. Let L be a submodule of M containing K such that L/K is a fully invariant submodule of M/K. Let φ be any endomorphism of M. Since $\varphi(K) \subseteq K$, φ induces a mapping $\bar{\varphi} : M/K \to M/K$ defined by $\bar{\varphi}(m + K) = \varphi(m) + K$ for all $m \in M$. It is easy to check that $\bar{\varphi}$ is an endomorphism of M/K. Hence $\bar{\varphi}(L/K) \subseteq L/K$ and it follows that $\varphi(L) \subseteq L + K = L$. Hence L is a fully invariant submodule of M. By hypothesis, there exists an ideal B of R such that $L = BM$ and this implies that $L/K = B(M/K)$. It follows that M/K is a fully invariant multiplication module. □

Corollary 2.4. Let R be any ring and let M be a fully invariant multiplication R-module. Then the R-module M/AM is a fully invariant multiplication R-module for any ideal A of R.

Proof. By Proposition 2.3. □

Recall the following elementary facts.

Lemma 2.5. Let R be any ring and let $L \subseteq K$ be submodules of an R-module M such that L is a fully invariant submodule of K and K is a fully invariant submodule of M, Then L is a fully invariant submodule of M.

Lemma 2.6. Let R be any ring and let an R-module $M = K \oplus K'$ be the direct sum of submodules K, K'. Then K is a fully invariant submodule of M if and only if $Hom(K, K') = 0$.

Proposition 2.7. Let R be any ring and let a fully invariant multiplication R-module $M = K \oplus K'$ be a direct sum of submodules K, K' such that $Hom(K, K') = 0$ or K' is fully invariant in M. Then K is a fully invariant multiplication module.

Proof. Suppose first that $\mathrm{Hom}(K, K') = 0$. Let L be any fully invariant submodule of K. By Lemmas 2.5 and 2.6, L is a fully invariant submodule of M and hence $L = BM = (BK) \oplus (BK') = BK$ for some ideal B of R. It follows that K is a fully invariant multiplication module. Now suppose that K' is a fully invariant submodule of M. Apply Proposition 2.3. □

Given any index set I, $M^{(I)}$ will denote (as usual) the module $\oplus_{i \in I} M_i$ where $M_i = M$ for all $i \in I$.

Theorem 2.8. Let R be any ring and let M be any fully invariant multiplication module over the ring R. Then the R-module $M^{(I)}$ is a fully invariant multiplication module for every index set I.

Proof. Let $M_i = M$ for each $i \in I$ and let $X = \oplus_{i \in I} M_i$. Let Y be any fully invariant submodule of X. Then $Y = \oplus_{i \in I} N_i$ where N_i is a submodule of M_i for all $i \in I$ (Corollary 2.2). Let j and k be distinct elements of I. Let $\varphi : X \to X$ be the mapping defined by $\varphi(\{m_i\}) = \{m'_i\}$ where $m_i \in M_i \, (i \in I)$ and $m'_j = m_k, m'_k = m_j$ and $m'_i = 0$ for all $i \in I \setminus \{j, k\}$. It is clear that φ is an endomorphism of M. Let $u = \{u_i\} \in L$ where $u_i \in N_i \, (i \in I)$. Then $\varphi(u) \in L$ implies that $u_j \in N_k$ and $u_k \in N_j$. It follows that $N_j = N_k$ for all $j, k \in I$. Thus $Y = \oplus_{i \in I} N_i$ where $N_i = N \, (i \in I)$ for some submodule N of M. Because Y is a fully invariant submodule of X it is easy to see that N is a fully invariant submodule of M and hence $N = BM$ for some ideal B of R. Thus $Y = BX$. It follows that X is a fully invariant multiplication module. □

Corollary 2.9. Let R be any ring and let M be any multiplication module over the ring R. Then the R-module $M^{(I)}$ is a fully invariant multiplication module for every index set I.

Proof. By Theorem 2.8. □

Corollary 2.10. Let R be any ring. Then every free R-module is a fully invariant multiplication R-module.

Proof. By Corollary 2.9 because the R-module R is a multiplication module. □

Corollary 2.10 raises the question whether every projective module over an arbitrary ring is a fully invariant multiplication module. We shall return to this question in § 5. Next we give another result concerning direct sums. It raises the question when the direct sum $M_1 \oplus M_2$ of fully invariant multiplication modules M_1, M_2 is a fully invariant multiplication module.

Theorem 2.11. Let R be any ring and let M_1 and M_2 be R-modules such that $R = \mathrm{ann}_R(M_1) + \mathrm{ann}_R(M_2)$. Then the R-module $M = M_1 \oplus M_2$ is a fully invariant multiplication module if and only if both M_1 and M_2 are fully invariant multiplication modules.

Proof. Let $A_i = \mathrm{ann}_R(M_i) \, (i = 1, 2)$ and note that $R = A_1 + A_2$. Suppose first that M is a fully invariant multiplication module. Let $\varphi : M_1 \to M_2$ be any homomorphism. Then

$$\begin{aligned}\varphi(M_1) &= \varphi(A_1 M_1 + A_2 M_1) = \varphi(A_1 M_1) + \varphi(A_2 M_1) \\ &= \varphi(0) + A_2 \varphi(M_1) \subseteq A_2 M_2 = 0.\end{aligned}$$

It follows that $\mathrm{Hom}(M_1, M_2) = 0$. By Proposition 2.7, M_1 is a fully invariant multiplication module. Similarly, M_2 is a fully invariant multiplication module.

Conversely, suppose that M_1 and M_2 are both fully invariant multiplication modules. Let N be any fully invariant submodule of M. Then $N = N_1 \oplus N_2$ for some fully invariant submodule N_1 of M_1 and some fully invariant submodule N_2 of M_2 (Corollary 2.2). By hypothesis, there exist ideals $B_i \, (i = 1, 2)$ such that $N_i = B_i M_i \, (i = 1, 2)$. Now we have $M_1 = (A_1 + A_2) M_1 = A_2 M_1$ and similarly $M_2 = A_1 M_2$, so that

$$\begin{aligned}(A_2 B_1 + A_1 B_2) M &= A_2 B_1 M_1 + A_2 B_1 M_2 + A_1 B_2 M_1 + A_1 B_2 M_2 \\ &= A_2 B_1 M_1 + A_1 B_2 M_2 = B_1 M_1 + B_2 M_2 = N_1 + N_2 = N.\end{aligned}$$

It follows that M is a fully invariant multiplication module. □

Let R be any domain which is not a field and let U be a simple R-module. Let M denote the R-module $R \oplus U$. The modules R and U are both multiplication modules and hence also fully invariant multiplication modules. However M is not a fully invariant multiplication module because $\text{Soc}({}_RM) = 0 \oplus U$ which is a fully invariant submodule of M but $\text{Soc}({}_RM) \neq BM$ for any ideal B of R. Thus the arbitrary direct sum of fully invariant multiplication modules need not be itself a fully invariant multiplication module. Compare Theorem 2.11.

Corollary 2.12. Let R be any ring, let n be a positive integer and let $M_i\,(i \in I)$ be R-modules such that $R = \text{ann}_R(M_i) + \text{ann}_R(M_j)$ for all $1 \leq i < j \leq n$. Then the R-module $M = M_1 \oplus \cdots \oplus M_n$ is a fully invariant multiplication module if and only if M_i is a fully invariant multiplication module for all $1 \leq i \leq n$.

Proof. By Theorem 2.11 and induction on n. □

Corollary 2.12 is not true for infinite direct sums. For, let Π denote any infinite set of primes in $\mathbb{Z}$. Let M denote the semisimple $\mathbb{Z}$-module $\oplus_{p\in\Pi}(\mathbb{Z}/\mathbb{Z}p)$. Clearly $R = \text{ann}_R(\mathbb{Z}/\mathbb{Z}p) + \text{ann}_R(\mathbb{Z}/\mathbb{Z}q)$ for any distinct primes p, q in Π. There exist disjoint infinite subsets Π_1 and Π_2 of Π such that $\Pi = \Pi_1 \cup \Pi_2$. Let $L = \oplus_{p\in\Pi_1}(\mathbb{Z}/\mathbb{Z}p)$. Then L is a full invariant submodule of M but $L \neq AM$ for any ideal A of R. Thus M is not a fully invariant multiplication module.

3 Special submodules

If R is a ring and M a faithful multiplication module then there is an easy description of various submodules of M, in particular the socle, the singular submodule, the radical and the prime radical of M (see, for example, [6]).

Let R be any ring and let M be an R-module. Recall that the *socle* of M is the sum of all simple submodules of M and is zero in case M has no simple submodule. The *radical* of M is the intersection of all maximal submodules of M and is M in case M has no maximal submodule. A non-zero submodule L of M is called *essential* provided $L \cap N \neq 0$ for every non-zero submodule N of M. The *singular submodule* of M is the submodule consisting of all elements $m \in M$ such that $Em = 0$ for some essential ideal E of R. The socle, radical and singular submodule of the R-module M will be denoted by Soc$({}_RM)$, Rad$({}_RM)$ and Z$({}_RM)$, respectively, and of the R-module R simply by Soc(R), Rad(R) and Z(R), respectively. For a faithful multiplication module M, Soc$({}_RM)$ = Soc$(R)M$, Rad$({}_RM)$ = Rad$(R)M$ and Z$({}_RM)$ = Z$(R)M$ (see [6, Theorem 2.7 and Corollary 2.14]).

Let $\mathcal{A}$ be any non-empty collection of ideals of R. For any R-module M, let $T_{\mathcal{A}}(M)$ denote the set of elements $m \in M$ such that $(A_1 \cap \cdots \cap A_n)m = 0$ for some positive integer n and ideals $A_i \in \mathcal{A}\,(1 \leq i \leq n)$. It is easy to check that $T_{\mathcal{A}}(M)$ is a submodule of M. Note that if $\mathcal{A}$ consists of all the maximal ideals of R then $T_{\mathcal{A}}(M) = \text{Soc}({}_RM)$ and if $\mathcal{A}$ is the set of essential ideals of R then $T_{\mathcal{A}}(M) = \text{Z}({}_RM)$. We denote $T_{\mathcal{A}}({}_RR)$ simply by $T_{\mathcal{A}}(R)$. Now we prove:

Theorem 3.1. Let R be any ring and let M be a faithful fully invariant multiplication module. Then $T_{\mathcal{A}}({}_RM) = T_{\mathcal{A}}(R)M$ for any non-empty collection $\mathcal{A}$ of ideals of R.

Proof. Let $\mathcal{B}$ denote the collection of finite intersections of ideals in $\mathcal{A}$. Let $a \in T_{\mathcal{A}}(R)$. Then $Ba = 0$ for some $B \in \mathcal{B}$ and hence $BaM = 0$. This implies that $aM \subseteq T_{\mathcal{A}}({}_RM)$. It follows that $T_{\mathcal{A}}(R)M \subseteq T_{\mathcal{A}}({}_RM)$. On the other hand, let $m \in T_{\mathcal{A}}({}_RM)$. There exists $C \in \mathcal{B}$ such that $Cm = 0$. Let $L = \{x \in M : Cx = 0\}$. Then L is a fully invariant submodule of M. By hypothesis, there exists an ideal G in R such that $L = GM$ and hence $CGM = 0$. Because M is faithful, $CG = 0$ and $G \subseteq T_{\mathcal{A}}(R)$. Now $m \in GM$ and we have proved that $T_{\mathcal{A}}({}_RM) \subseteq T_{\mathcal{A}}(R)M$. The result follows. □

The next result generalizes [6, Corollary 2.14].

Corollary 3.2. Let R be any ring and let M be a faithful fully invariant multiplication module. Then

(a) $Soc({}_RM) = Soc(R)M$, and

(b) $Z({}_RM) = Z(R)M$.

Proof. (a) Apply Theorem 3.1 with $\mathcal{A}$ the collection of maximal ideals of R.

(b) Apply Theorem 3.1 with $\mathcal{A}$ the collection of essential ideals of R. □

The corresponding result for the radical is the following one. It generalizes [6, Theorem 2.7].

Theorem 3.3. Let R be any ring and let M be a fully invariant multiplication module. Then $Rad(_RM) = CM$ where C is the intersection of all maximal ideals P of R such that $M \neq PM$.

Proof. If Rad$(_RM) = M$ then $CM \subseteq$ Rad$(_RM)$. Now suppose that M contains a maximal submodule L. There exists a maximal ideal Q of R such that $Q(M/L) = 0$ and hence $QM \subseteq L$. Note that $QM \neq M$ and hence $C \subseteq Q$ and $CM \subseteq QM \subseteq L$. Thus $CM \subseteq$ Rad$(_RM)$. Next note that because Rad$(_RM)$ is a fully invariant submodule of M, there exists an ideal B of R such that Rad$(_RM) = BM$. Let G be any maximal ideal of R such that $M \neq GM$. Note that M/GM is a semisimple R-module so that $BM =$ Rad$(_RM) \subseteq GM$. If $B \nsubseteq G$ then $R = B+G$ and hence $M = BM + GM = GM$, a contradiction. Thus $B \subseteq G$. It follows that $B \subseteq C$. We conclude that Rad$(_RM) = BM \subseteq CM$ and the result follows. □

Corollary 3.4. Let R be any ring and let M be a finitely generated faithful fully invariant multiplication module. Then $Rad(_RM) = Rad(R)M$.

Proof. Suppose that $M = PM$ for some maximal ideal P of R. By the usual determinant argument, M being finitely generated implies that there exists $p \in P$ such that $(1 - p)M = 0$. But M being faithful gives that $1 - p = 0$ and hence $P = R$, a contradiction. Thus $M \neq PM$ for every maximal ideal P of R. Now apply Theorem 3.3. □

Let R be any ring and let M be any non-zero R-module. A proper submodule L of M is called *prime* in case whenever $r \in R$ and $m \in M$ such that $rm \in L$ then $m \in L$ or $rM \subseteq L$. It is well known and easy to prove that a submodule N of M is prime if and only if $P = (N :_R M)$ is a prime ideal of R and the (R/P)-module M/N is torsion-free. Given any prime ideal Q of R, a submodule K of M will be called Q*-prime* if K is a prime submodule of M such that $Q = (K :_R M)$. We define the *prime radical*, denoted by rad$(_RM)$, to be the intersection of all prime submodules of M and to be M in case M has no prime submodule. There is an extensive literature on prime submodules of a module M and attempts to describe rad$(_RM)$ stretching back to the early 1970s (see, for example, [6], [8] - [13], [15] and [21]).

Let P be any prime ideal of a ring R. Given an R-module M we define $K_P(M)$ to be the set of all elements $m \in M$ such that $cm \in PM$ for some $c \in R \setminus P$. Note that $K_P(M)$ is a submodule of M containing PM such that $K_P(M)/PM$ is the torsion submodule of the (R/P)-module M/PM and hence $K_P(M) = M$ or $K_P(M)$ is a P-prime submodule of M. We include the next result for completeness.

Lemma 3.5. Let P be a prime ideal of a ring R and let M be an R-module such that $M \neq K_P(M)$. Then $K_P(M)$ is the intersection of all P-prime submodules of M. Moreover $K_P(M)$ is a fully invariant submodule of M.

Proof. Let L be any P-prime submodule of M. Then $PM \subseteq L$ and M/L is a torsion-free (R/P)-module. It follows that $K_P(M) \subseteq L$. The first part of the result follows. Let φ be any endomorphism of M. Let $m \in K_P(M)$. There exists $c \in R \setminus P$ such that $cm \in PM$ and hence

$$c\varphi(m) = \varphi(cm) \in \varphi(PM) = P\varphi(M) \subseteq PM.$$

It follows that $\varphi(m) \in K_P(M)$ for every endomorphism φ of M. Thus $K_P(M)$ is a fully invariant submodule of M. □

Corollary 3.6. Let R be a ring and M an R-module. Then rad$(_RM) = \cap K_P(M)$ where the intersection is taken over all prime ideals P of R. Moreover rad$(_RM)$ is a fully invariant submodule of M.

Proof. By Lemma 3.5. □

Given a ring R and a non-zero R-module M, let $\Pi(M)$ denote the collection, possibly empty, of prime ideals P of R such that $M \neq K_P(M)$. Note the $M \neq K_P(M)$ if and only if the (R/P)-module M/PM is not torsion. Compare the next result with [6, Theorem 2.12].

Theorem 3.7. Let R be a ring and let M be an R-module. Then rad$(_RM) = BM$ where B is the intersection of all prime ideals P of R such that $M \neq K_P(M)$.

Proof. If M does not contain any prime submodules then rad$(_RM) = M$ and $B = R$ so the result is true in this case. Now suppose that M does contain a prime submodule so that the collection $\Pi(M)$ is non-empty (Lemma 3.5). By Corollary 3.6, rad$(_RM) = \cap_{P \in \Pi(M)} K_P(M)$ and rad$(_RM) = CM$ where $C =$ (rad$(_RM) :_R M)$. Let $P \in \Pi(M)$. Then $CM =$ rad$(_RM) \subseteq K_P(M)$ and $M \neq K_P(M)$. Because the submodule $K_P(M)$ is P-prime, $C \subseteq P$. It follows that $C \subseteq B$. On the other hand, $BM \subseteq PM \subseteq K_P(M)$. It follows that $BM \subseteq$ rad$(_RM)$ and hence $B \subseteq$ (rad$(_RM) :_R M) = C$. It follows that $B = C$ as required. □

Corollary 3.8. Let R be a ring and let M be a finitely generated faithful R-module. Then rad$({}_RM)$ = rad$({}_RR)M$.

Proof. Let $G = \text{rad}({}_RR)$. It is well known that G is the set of all nilpotent elements of R and also the intersection of all prime ideals for R. Let P be any prime ideal of R. Suppose that $M = K_P(M)$. There exist a positive integer n and elements $m_i\,(1 \leq i \leq n)$ in M such that $M = Rm_1 + \cdots + Rm_n$. For each $1 \leq i \leq n$ there exists $c_i \in R \setminus P$ such that $c_im_i \in PM$. Using the usual determinant argument, it follows that $dM = 0$ for some $d \in R \setminus P$. But M is faithful, so that $d = 0$, a contradiction. Thus $M \neq K_P(M)$ for every prime ideal P of R. Now apply Theorem 3.7. □

4 Modules over domains

In this section we shall look at modules over domains. If R is a domain then every fully invariant multiplication module is torsion or torsion-free, as we show next.

Lemma 4.1. Let R be a domain and let M be any fully invariant multiplication module over R. Then M is torsion-free or there exists a non-zero ideal B of R such that $BM = 0$.

Proof. Suppose that the R-module M is not torsion-free. Then there exists a non-zero element $m \in M$ and a non-zero ideal G of R such that $Gm = 0$. Let $L = \{x \in M\,:\,Gx = 0\}$. Note that L is non-zero because $m \in L$. It can easily be checked that L is a fully invariant submodule of M and hence $L = HM$ for some ideal H of R. Note that $H \neq 0$. Next $GHM = GL = 0$ and GH is a non-zero ideal of R. □

Note the following simple fact.

Lemma 4.2. Let R be a domain and let M be a torsion-free R-module. Then AM is an essential submodule of M for every non-zero ideal A of R.

Proof. Let N be any submodule of M such that $AM \cap N = 0$. Then $AN \subseteq AM \cap N$ gives that $AN = 0$ and hence $N = 0$. □

Corollary 4.3. Let R be a domain and let M be a torsion-free fully invariant multiplication R-module. Then every non-zero fully invariant submodule of M is essential in M.

Proof. By Lemma 4.2 . □

Let R be any domain. An R-module M is called *divisible* in case $M = aM$ for every non-zero element a of R. Injective modules are divisible (see for example [16, Proposition 2.6]) and every torsion-free divisible R-module is injective (see, for example, [16, Proposition 2.7]). An R-module X is called *reduced* in case it does not contain a non-zero divisible submodule.

Proposition 4.4. Let R be a domain. Then every torsion-free fully invariant multiplication R-module is divisible or reduced.

Proof. Let M be a torsion-free fully invariant multiplication module which is not reduced. Let L be the sum of all divisible submodules of M. Then $L \neq 0$ and it is easy to check that L is divisible. In this case L is injective and hence a direct summand of M. If N is a divisible submodule of M then so too is $\varphi(N)$ for every endomorphism φ of M. It follows that L is a fully invariant submodule of M and hence L is essential in M by Corollary 4.3. Thus $M = L$, as required. □

Recall that an R-module M is called *uniform* in case $L \cap N \neq 0$ for all non-zero submodules L, N. Presumably the next result is well known.

Lemma 4.5. Let R be a domain and let M be a non-zero torsion-free R-module. If L is a fully invariant submodule of M then $aL \subseteq bL$ for all elements a, b in R such that $aM \subseteq bM$. Moreover, the converse holds if M is uniform.

Proof. Suppose first that L is fully invariant in M. If $M = 0$ then there is nothing to prove. Suppose that $M \neq 0$. Let a be a non-zero element of R such that $aM \subseteq bM$ for some $b \in R$. Clearly $b \neq 0$. Let $m \in M$. Then $am = bm'$ for some element m' in M. If $am = b\bar{m}$ for some $\bar{m} \in M$ then $b(m' - \bar{m}) = 0$ and hence $\bar{m} = m'$. We can define a mapping $\varphi : M \to M$ by $\varphi(m) = m'$ for all $m \in M$. It is easy to check that φ is an endomorphism of M. It follows that $\varphi(L) \subseteq L$ and hence $aL \subseteq bL$. If $a = 0$ then $aL \subseteq bL$ for all $b \in R$.

Conversely, suppose that M is uniform and that L has the stated property. Let θ be any non-zero endomorphism of M. There exists a non-zero element $m \in M$ such that $\theta(m) \neq 0$. Because M is uniform, we have $Rm \cap R\theta(m) \neq 0$ and hence $am = b\theta(m) \neq 0$ for some non-zero elements a, b of R. Let $0 \neq x \in M$. Then $Rx \cap Rm \neq 0$ gives that $rx = sm$ for some non-zero elements r, s in R. Thus

$$br\theta(x) = b\theta(rx) = b\theta(sm) = bs\theta(m) = sam = arx,$$

so that $r(b\theta(x) - ax) = 0$. Because M is torsion-free, we conclude that $b\theta(x) = ax$ for all $x \in M$. In particular this implies that $aM \subseteq bM$. By hypothesis, $aL \subseteq bL$. Let $y \in L$. Then $ay = bz$ for some $z \in L$ and hence $b\theta(y) = ay = bz$. This implies that $b(\theta(y) - z) = 0$ and hence $\theta(y) = z \in L$. We have proved that $\theta(L) \subseteq L$ so that L is a fully invariant submodule of M. □

Proposition 4.6. Let R be a domain. Then every torsion-free divisible R-module is a fully invariant multiplication module.

Proof. Let F denote the field of fractions of R. Let M be any non-zero torsion-free divisible R-module. It is well known that M is a vector space over F and hence the R-module $M \cong F^{(I)}$ for some index set I. By Theorem 2.8 it is sufficient to prove that the R-module F is a fully invariant multiplication module and thus we can suppose without loss of generality that M is uniform. Let L be a non-zero fully invariant submodule of M. For each non-zero element $a \in R$, $M = aM$ and hence $L = aL$ by Lemma 4.5. Thus $L = aL$ for each $0 \neq a \in R$, so that L is divisible, hence injective and a direct summand of the uniform module M. We conclude that $L = M = RM$. Thus M is a fully invariant multiplication module, as required. □

Recall that if R is a domain then the zero R-module is the only divisible R-module which is a multiplication module. Combining Propositions 4.4 and 4.6 we see that if R is a domain then a torsion-free R-module M is a fully invariant multiplication module if and only if M is divisible or a reduced fully invariant multiplication module.

5 Modules over Dedekind Domains

Let R be a (commutative) domain with field of fractions F. Given any ideal A of R, A^* will denote the set of elements $f \in F$ such that $fA \subseteq R$. Note that A^* is an R-submodule of F, $R \subseteq A^*$, A^*A is an ideal of R and $A \subseteq A^*A$. The ideal A is called *invertible* provided $A^*A = R$. The ring R is a *Dedekind domain* if every non-zero ideal is invertible. For more information about Dedekind domains see [7, p. 442 §37]. In this section we shall consider modules over a Dedekind domain R. First we deal with finitely generated torsion-free modules. It is well known that a finitely generated module M over a Dedekind domain is projective if and only if it is torsion-free and in this case $M \cong H \oplus A$ for some (possibly zero) free module H and ideal A of R.

Theorem 5.1. Let R be a Dedekind domain. Then every finitely generated torsion-free R-module is a fully invariant multiplication module.

Proof. Let M be any finitely generated torsion-free R-module. If $M \cong A$ for some non-zero ideal A of R then M is a multiplication module. On the other hand, if M is a free R-module than M is a fully invariant multiplication module by Corollary 2.10. Thus without loss of generality we can suppose that $M = R^n \oplus A$ for some positive integer n and non-zero ideal A of R. Let L be any fully invariant submodule of M. By Corollary 2.2, $L = B_1 \oplus \cdots \oplus B_n \oplus C$ for some ideals $B_i\,(1 \leq i \leq n), C$ of R with $C \subseteq A$. Let π be any permutation of the set $\{1, \ldots, n\}$ and let φ_π denote the endomorphism of M defined by

$$\varphi_\pi(r_1, \ldots, r_n, a) = (r_{\pi(1)}, \ldots, r_{\pi(n)}, a),$$

for all $r_i \in R\,(1 \leq i \leq n), a \in A$. It is clear that φ_π is an endomorphism of M for each permutation π of $\{1, \ldots, n\}$. Because $\varphi_\pi(L) \subseteq L$ for every permutation π of $\{1, \ldots, n\}$, we have $B_1 = \cdots = B_n = B$(say).

Let $a \in A, f \in A^*$. Define a mapping $\theta : M \to M$ by

$$\theta(s_1, \ldots, s_n, d) = (fd, 0, \ldots, 0, s_1 a),$$

for all $s_i \in R\,(1 \leq i \leq n)$, $d \in A$. It is easy to check that θ is an endomorphism of M. The fact that $\theta(L) \subseteq L$ implies that $fd \in B$ and $s_1 a \in C$ for all $d \in C, s_1 \in B$. Thus $fC \subseteq B$ and $aB \subseteq C$. We have proved that $A^*C \subseteq B$ and $AB \subseteq C$. But $C = RC = AA^*C \subseteq AB$ so that $C = AB$ and $L = BM$. It follows that M is a fully invariant multiplication module. □

Corollary 5.2. Let R be a Dedekind domain. Then every projective R-module is a fully invariant multiplication module.

Proof. Let M be any projective R-module. Then M is finitely generated or free. The result follows by Corollary 2.10 and Theorem 5.1. □

Let R be an arbitrary domain and let P be a maximal ideal of R. We shall call an R-module M *P-torsion* provided for each $m \in M$ there exists a positive integer n such that $P^n m = 0$.

Theorem 5.3. Let R be a Dedekind domain and let M be a non-zero torsion R-module. Then M is a fully invariant multiplication module if and only if there exist positive integers $n, k_i\,(1 \leq i \leq n)$, distinct maximal ideals $P_i\,(1 \leq i \leq n)$ and index sets $I_j\,(1 \leq j \leq n)$ such that

$$M \cong (R/P_1^{k_1})^{(I_1)} \oplus \cdots \oplus (R/P_n^{k_n})^{(I_n)}.$$

Proof. Suppose first that M is a fully invariant multiplication module. By Lemma 4.1 there exists a non-zero ideal B of R such that $BM = 0$. The ideal B is a (finite) product of maximal ideals and therefore $M = M_1 \oplus \cdots \oplus M_n$ for some positive integer n and submodules $M_i\,(1 \leq i \leq n)$ of M such that for each $1 \leq i \leq n$ there exist a maximal ideal P_i containing B and a positive integer k_i with $P_i^{k_i} M_i = 0$. Clearly we can assume that the maximal ideals $P_i\,(1 \leq i \leq n)$ are distinct and each of the positive integers $k_i\,(1 \leq i \leq n)$ is as small as possible.

Let $1 \leq i \leq n$, let $N = M_i$, let $P = P_i$ and let $k = k_i$. By Corollary 2.12 N is a fully invariant multiplication R-module. It is well known that in this situation there exist an index set Λ and cyclic submodules $N_\lambda\,(\lambda \in \Lambda)$ such that $N = \oplus_{\lambda \in \Lambda} N_\lambda$. For each $\lambda \in \Lambda$ let k_λ be the least positive integer such that $P^{k_\lambda} N_\lambda = 0$. Clearly $k = k_\mu$ for some $\mu \in \Lambda$. For each $\lambda \in \Lambda$, let $U_\lambda = \text{Soc}(_R N_\lambda)$. Then $U_\lambda = P^{k_\lambda - 1} N_\lambda$ and is simple. Now $\text{Soc}(_R N) = \oplus_{\lambda \in \Lambda} U_\lambda$ and $\text{Soc}(_R N)$ is a fully invariant submodule of N and hence also of M. By hypothesis,

$$\oplus_{\lambda \in \Lambda} P^{k_\lambda - 1} N_\lambda = CN = \oplus_{\lambda \in \Lambda} CN_\lambda,$$

for some ideal C of R. Without loss of generality, we can choose C to be maximal with this property and in this case $C = P^h$ for some non-negative integer h. This implies that $h = k_\lambda - 1\,(\lambda \in \Lambda)$ and this in turn implies that $N_\lambda \cong R/P^{h+1}\,(\lambda \in \Lambda)$. It follows that there exist positive integers $n, k_i\,(1 \leq i \leq n)$, distinct maximal ideals $P_i\,(1 \leq i \leq n)$ and index sets $I_j\,(1 \leq j \leq n)$ such that

$$M \cong (R/P_1^{k_1})^{(I_1)} \oplus \cdots \oplus (R/P_n^{k_n})^{(I_n)}.$$

This proves the necessity.

Conversely, suppose that M has the stated decomposition. By Corollary 2.9 the module $(R/P_j^{k_j})^{(I_j)}$ is a fully invariant multiplication module for each $1 \leq j \leq n$. Now apply Corollary 2.12 to deduce that M is a fully invariant multiplication module. □

Corollary 5.4. Let R be a Dedekind domain. Then a finitely generated R-module M is a fully invariant multiplication module if and only if M is torsion-free or there exist positive integers $n, k_i\,(1 \leq i \leq n)$, distinct maximal ideals $P_i\,(1 \leq i \leq n)$ and index sets $I_j\,(1 \leq j \leq n)$ such that

$$M \cong (R/P_1^{k_1})^{(I_1)} \oplus \cdots \oplus (R/P_n^{k_n})^{(I_n)}.$$

Proof. The necessity follows by Lemma 4.1 and Theorem 5.3. The sufficiency follows by Theorems 5.1 and 5.3. □

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