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BCH-Semigroup Ideals in BCH-Semigroups

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Abstract We consider BCH-semigroup ideals and their properties in BCH-semigroups. Some results about BCH-semigroup homomorphisms related to BCH-semigroup ideals are investigated.

1 Introduction

In 1966, two classes of abstract algebras, BCK-algebras and BCI-algebras, were introduced by Imai and Iseki [11, 12, 14]. Since then far-reaching investigations on these algebras have been made by many researchers (see [5, 17, 18] and references therein). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras [11].
In [9, 10], a class of abstract algebras, BCH-algebras was introduced by Ping Hu and Li. Both these researchers studied some properties of these algebras which are more general than BCK-algebras and BCI-algebras. Besides this, Chaudhry and Iseki also studied these algebras [4, 13]. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. For more information regarding BCH-algebras, we refer the reader to [3, 4].
In this paper, we describe the concept of a BCH-semigroup ideal of a BCH-semigroup. Certain conditions for a semi-subgroup to be a BCH-semigroup ideal have been investigated. Some results about BCH-semigroup homomorphisms related to BCH-semigroup ideals are also given.

2 Notations and preliminary results

This section presents some useful definitions and some basic results.

Definition 2.1. (BCH-Algebra) [10] A BCH-algebra is an algebra $(B; *, 0)$ of type $(2, 0)$ fulfilling the following conditions:
(1) $b * b = 0$ for all $b \in B$,
(2) $b * t = t * b = 0$ $implies$ $b = t$,
(3) $(b * t) * d = (b * d) * t$ for all $b, t, d \in B$.

Remark 2.2. [8, 10] In any BCH-algebra a partial order $\leq$ is defined by $b \leq t$ if and only if $b * t = 0$. It has been noted that the following identities hold for every two elements x, y of any BCH-algebra B.
(1) $x * (x * y) \leq y$,
(2) $x * 0 = 0$ $implies$ $x = 0$,
(3) $0 * (x * y) = (0 * x) * (0 * y)$,
(4) $x * 0 = x$,
(5) $(x * y) * x = 0 * y$.

Example 2.3. It is a routine exercise to see that $(B = \mathbb{Z}_4;\ *,\ 0)$ is a BCH-algebra with $*$ defined as:

$*$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Definition 2.4. (BCH-Homomorphisms) [6] A mapping ζ from a BCH-algebra B to a BCH-algebra $\acute{B}$ is said to be a homomorphism if $\zeta(t * d) = \zeta(t) * \zeta(d),$ for all $t, d \in B.$

Note that, if $\zeta : B \longrightarrow \acute{B}$ is a BCH-homomorphism, then $\zeta(0) = \acute{0}$ and $t \leq d$ implies $\zeta(t) \leq \zeta(d)$ for any $t, d \in B$.

Definition 2.5. (Sub-algebra) [4] A subset $I \neq \emptyset$ of a BCH-algebra $(B; *, 0)$ is called a sub-algebra if $q, r \in I \; implies \; q * r \in I.$

Definition 2.6. (Closed ideal) [4] A non-empty subset D of a BCH-algebra B is said to be a closed ideal in B if
(i) $0 * t \in D \quad for \; all \; t \in D,$
(ii) $m * t \in D, \;\; t \in D \; implies \; m \in D.$

Every closed ideal is a sub-algebra but converse is not true in general because in Example 2.3 if we assume $I = \{0, 4\},$ then I is a sub-algebra of B but it is not a closed ideal.

Definition 2.7. (BCH-Semigroup) An algebraic system $(B; \odot, *, 0)$ with two binary operations $*$ and $\odot$ is said to be a BCH-semigroup if it satisfies the following conditions:
(1) $(B; \odot)$ is a semigroup,
(2) $(B; *, 0)$ is a BCH-algebra,
(3) The binary operation $\odot$ is distributive (left as well as right) over the operation $*$. That is, $d \odot (c_1 * m) = (d \odot c_1) * (d \odot m)$ and
$(c_1 * m) \odot d = (c_1 \odot d) * (m \odot d)$ for all $d, c_1, m \in B.$

Example 2.8. Consider the set $B = \{0, 1, 2\}$ with two operations $\odot$ and $*$ defined below:

$\odot$	0	1	2
0	0	0	0
1	0	2	1
2	0	1	2

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Routine calculations give that $(B; \odot, *, 0)$ is a BCH-semigroup.

Proposition 2.9. *If* $(B; \odot, *, 0)$ *is a BCH-semigroup.Then*
(i) $0 \odot m = m \odot 0 = 0,$ *for all* $m \in B.$
(ii) $m \leq s_1 \; implies \; m \odot b \leq s_1 \odot b \; and \; b \odot m \leq b \odot s_1$, *for all* $m, s_1, b \in B.$

Proof. (i) For all $m \in B$, we have

$$\begin{aligned}
0 \odot m &= (0 * 0) \odot m && (\because 0 * 0 = 0) \\
&= (0 \odot m) * (0 \odot m) && (\odot \; is \; distributive \; over \; *) \\
&= 0, && (\because \; m * m = 0, \;\; for \; all \; m \in B). \\
m \odot 0 &= m \odot (0 * 0) \\
&= (m \odot 0) * (m \odot 0) && (By \; distribution \; law) \\
&= 0 && (By \; m * m = 0, \;\; for \; all \; m \in B)
\end{aligned}$$

(ii) Let $m, s_1, b \in B$ with $m \leq s_1$. Then

$$m * s_1 = 0.$$

We claim that $m \odot b \leq s_1 \odot b$. This is true if $(m \odot b) * (s_1 \odot b) = 0.$ Consider

$$\begin{aligned}
(m \odot b) * (s_1 \odot b) &= (m * s_1) \odot b \\
&= 0 \odot b \\
&= 0. \\
(b \odot m) * (b \odot s_1) &= b \odot (m * s_1) \\
&= b \odot 0 \\
&= 0.
\end{aligned}$$

Thus $m \odot b \leq s_1 \odot b.$ □

Definition 2.10. (BCH-Semigroup Homomorphism) A BCH-semigroup homomorphism of a BCH-semigroup $(B; \odot, *, 0)$ to a BCH-semigroup $(\acute{B}; \acute{\odot}, \acute{*}, \acute{0})$ is a mapping $\zeta : B \longrightarrow \acute{B}$ satisfying:

$$\zeta\,(p \odot q) = \zeta\,(p)\,\acute{\odot}\,\zeta\,(q) \quad and \; \zeta\,(p * q) = \zeta\,(p)\,\acute{*}\,\zeta\,(q).$$

A surjective homomorphism is said to be an epimorphism. The set of those elements of B which are mapped onto the element $\acute{0}$ of $\acute{B}$ is said to be the kernel of ζ, denoted by Ker ζ, and written as Ker $\zeta = \{ k \in B : \zeta(k) = \acute{0} \}$.

Definition 2.11. (Semi-subgroup) A subset $H \neq \emptyset$ of a BCH-semigroup $(B; \odot, *, 0)$ is said to be a semi-subgroup if

$r, t \in H$ *implies* $r * t \in H$ *and* $r \odot t \in H$.

Theorem 2.12. *Let* $(B; \odot, *, 0)$ *and* $(\acute{B}; \acute{\odot}, \acute{*}, \acute{0})$ *be two BCH-semigroups. Let* $\xi : B \longrightarrow \acute{B}$ *be an epimorphism. Then*
(a) *The* $Ker\xi = K$ *is a semi-subgroup of* B.
(b) *The homomorphic image* $\xi(B) = im\xi$ *is a semi-subgroup of* $\acute{B}$.

Proof. (a) Let $s_1, s_2 \in K$. Then $\xi(s_1) = \xi(s_2) = \acute{0}$, which implies $\xi(s_1 \odot s_2) = \xi(s_1) \acute{\odot} \xi(s_2) = \acute{0} \acute{\odot} \acute{0} = \acute{0}$. Also $\xi(s_1 * s_2) = \xi(s_1) \acute{*} \xi(s_2) = \acute{0} \acute{*} \acute{0} = \acute{0}$. So $s_1 \odot s_2$ and $s_1 * s_2 \in K$. Thus K is a semi-subgroup of B.
(b) Let $\acute{s}, \acute{p} \in \xi(B)$ such that $\acute{s} = \xi(s)$ and $\acute{p} = \xi(p)$, for $s, p \in B$. Consider $\acute{s} \acute{*} \acute{p} = \xi(s) \acute{*} \xi(p)$. As ξ is a homomorphism so $\acute{s} \acute{*} \acute{p} = \xi(s * p) \in \xi(B)$. Similarly, $\acute{s} \acute{\odot} \acute{p} = \xi(s) \acute{\odot} \xi(p) = \xi(s \odot p) \in \xi(B)$. Hence $im\xi = \xi(B)$ is a semi-subgroup of $\acute{B}$. □

3 BCH-Semigroup Ideals in BCH-Semigroups

Definition 3.1. (BCH-Semigroup Ideal) Let $(B; \odot, *, 0)$ be a BCH-semigroup and S be any subset of B. Then
(1) S is said to be a left BCH-semigroup ideal in B if
(i) $r \in S,\ t \in B$ *imply* $t \odot r \in S$,
(ii) $0 * r \in S$ *for all* $r \in S$,
(iii) $t * r \in S,\ r \in S$ *imply* $t \in S$.
(2) S is said to be a right BCH-semigroup ideal in B if
(i) $r \in S,\ t \in B$ *imply* $r \odot t \in S$,
(ii) $0 * r \in S$ *for all* $r \in S$,
(iii) $t * r \in S,\ r \in S$ *imply* $t \in S$.
(3) S is said to be a BCH-semigroup ideal in B if it is both left and right BCH-semigroup ideal.

Remark 3.2. *[7]* It is known that a left(right) BCH-semigroup ideal S is closed under $*$ as well as under $\odot$. Moreover $0 \in S$ and a BCH-semigroup ideal is closed under $\odot$.

Theorem 3.3. *Let* $(B; \odot, *, 0)$ *be a BCH-semigroup satisfying the condition* $0 * t = 0$ *for all* $t \in B$. *Let* S *be a semi-subgroup of* B *satisfying the condition* $i * h \in S$ *imply* $h * i \in S,\ i, h \in B$. *Then* S *is a left BCH-semigroup ideal in* B.

Proof. It is given that S is a semi-subgroup of a BCH-semigroup B. Let $i \in S$. Since S is a semi-subgroup so $0 \in S$. Thus $0 * i \in S$, for all $i \in S$. Let $i \in S,\ t \in B$, then $t \odot i \in B$. So by hypothesis $0 * (t \odot i) = 0 \in S$, which implies $(t \odot i) * 0 = t \odot i \in S$. Next, suppose that $i, h * i \in S$. Then $i, i * h \in S$. Since S is a semi-subgroup, so $(i * h) * i \in S$, which gives $0 * h \in S$. Hence $h * 0 = h \in S$. Hence S is a left BCH-semigroup ideal in B. □

Theorem 3.4. *Let* $(B; \odot, *, 0)$ *be a BCH-semigroup satisfying the condition* $0 * q = 0$ *for all* $q \in B$. *Let* S *be a semi-subgroup of* B *satisfying the condition* $s * h \in S$ *imply* $h * s \in S$, *for all* $s, h \in B$. *Then* S *is a right BCH-semigroup ideal in* B.

Proof. It is given that S is a semi-subgroup of a BCH-semigroup B. Let $s \in S$. Since S is a semi-subgroup so $0 \in S$. Thus $0 * s \in S$, for all $s \in S$. Let $s \in S,\ q \in B$ then $s \odot q \in B$. So by hypothesis $0 * (s \odot q) = 0 \in S$, which implies $(s \odot q) * 0 = s \odot q \in S$. Next, suppose that $s, h * s \in S$. Then $s, s * h \in S$. Since S is a semi-subgroup, so $(s * h) * s \in S$, which gives $0 * h \in S$. Hence $h * 0 = h \in S$. Hence S is a right BCH-semigroup ideal in B. □

Proposition 3.5. *Let* $(B; \odot, *, 0)$ *be a BCH-semigroup and let* I *be a BCH-semigroup ideal of* B. *If* $v \leq u$ *and* $u \in I$, *for any* $v \in B$, *then* $v \in I$.

Proof. Let $u \in I$ and $v \leq u$. Then $v * u = 0 \in I$. So by definition of a BCH-semigroup ideal, $v \in I$. □

Theorem 3.6. *Let $(B; \odot, *, 0)$ and $(\acute{B}; \acute{\odot}, \acute{*}, \acute{0})$ be BCH-semigroups. Let $f : B \longrightarrow \acute{B}$ be a BCH-semigroup homomorphism, then $Kerf$ is a BCH-semigroup ideal in the BCH-semigroup B.*

Proof. Let $u \in Kerf$. So $f(u) = \acute{0}$. Further $f(0 * u) = f(0) \acute{*} f(u) = \acute{0} \acute{*} \acute{0} = \acute{0}$. Thus $0 * u \in Kerf$. Now, for a left BCH-semigroup ideal, let $i \in H$ and $r \in Kerf$. We claim that $i \odot r \in Kerf$. Consider $f(i \odot r) = f(i) \acute{\odot} f(r) = f(i) \acute{\odot} \acute{0} = \acute{0}$. This implies $i \odot r \in Kerf$. Similarly, we can show that $r \odot i \in Kerf$. Let $u \in Kerf$, $v * u \in Kerf$. This implies $f(u) = \acute{0}$ and $f(v * u) = \acute{0}$. Thus $f(v) \acute{*} f(u) = \acute{0}$, which implies $f(v) \acute{*} \acute{0} = \acute{0}$. That is $f(v) = \acute{0}$. So $v \in Kerf$. So $Kerf$ is a BCH-semigroup ideal of B. □

Theorem 3.7. *Let $(B; \odot, *, 0)$ and $(\acute{B}; \acute{\odot}, \acute{*}, \acute{0})$ be BCH-semigroups. Let $\Psi : B \longrightarrow \acute{B}$ be an epimorphism. If S is a BCH-semigroup ideal of a BCH-semigroup B, then $\Psi(S)$ is a BCH-semigroup ideal of $\acute{B}$.*

Proof. It is given that $\Psi : B \longrightarrow \acute{B}$ be an epimorphism and S is a BCH-semigroup ideal of B. We have to prove that $\Psi(S)$ is a BCH-semigroup ideal of $\acute{B}$. From Theorem 2.12 (b), $\Psi(S)$ is a semi-subgroup of $\acute{B}$. Let $\acute{r} \in \Psi(S)$. Since $\Psi(S)$ is a semi-subgroup so $\acute{0} \acute{*} \acute{r} \in \Psi(S)$, for all $\acute{r} \in \Psi(S)$. Let $\acute{c} \in \acute{B}$ and $\acute{r} \in \Psi(S)$, since Ψ is an epimorphism, so there exists $c \in B$ and $r \in S$ such that $\Psi(c) = \acute{c}$ and $\Psi(r) = \acute{r}$. Consider, $\acute{c} \acute{\odot} \acute{r} = \Psi(c) \acute{\odot} \Psi(r) = \Psi(c \odot r)$. As S is a BCH-semigroup ideal, $c \odot r \in S$, $c \in B$, $r \in S$. This implies $\Psi(c \odot r) \in \Psi(S)$. This gives $\acute{c} \acute{\odot} \acute{r} \in \Psi(S)$. Also, consider, $\acute{r} \acute{\odot} \acute{c} = \Psi(r) \acute{\odot} \Psi(c) = \Psi(r \odot c)$. As S is a BCH-semigroup ideal, $r \odot c \in S$, $r \in S$, $c \in B$. This implies $\Psi(r \odot c) \in \Psi(S)$. This gives $\acute{r} \acute{\odot} \acute{c} \in \Psi(S)$. Now, we assume $\acute{x} \in \Psi(S)$, $\acute{v} \in \acute{B}$, so there exist $x \in S$ and $v \in B$ *such that* $\Psi(x) = \acute{x}$ *and* $\Psi(v) = \acute{v}$. Let $\acute{v} \acute{*} \acute{x} \in \Psi(S)$, $\acute{x} \in \Psi(S)$. We have to show that $\acute{v} \in \Psi(S)$. Consider, $\acute{v} \acute{*} \acute{x} = \Psi(v) \acute{*} \Psi(x) = \Psi(v * x)$. As S is a BCH-semigroup ideal so $v * x \in S$, $x \in S$ implies $v \in S$. Thus $\Psi(v) = \acute{v} \in \Psi(S)$. Hence $\Psi(S)$ is a BCH-semigroup ideal in $\acute{B}$. □

Theorem 3.8. *Let $(B; \odot, *, 0)$ and $(\acute{B}; \acute{\odot}, \acute{*}, \acute{0})$ be BCH-semigroups. Let $\Psi : B \longrightarrow \acute{B}$ be an epimorphism. Then a semi-subgroup $\acute{I}$ of $\acute{B}$ is a BCH-semigroup ideal in $\acute{B}$ if and only if its inverse image $I = \Psi^{-1}(\acute{I})$ is a BCH-semigroup ideal in B.*

Proof. Suppose that $\acute{I}$ is a BCH-semigroup ideal of $\acute{B}$ and $I = \Psi^{-1}(\acute{I}) = \{a \in B : \Psi(a) = \acute{a} \in \acute{I}\}$. To show that I is a BCH-semigroup ideal in B, let $a \in I$. Since I is a semi-subgroup so $0 * a \in I$, for all $a \in I$. Let $a \in I$ and $x \in B$. We Consider $x \odot a$ and $a \odot x$. These belongs to I if and only if $\Psi(x \odot a)$ and $\Psi(a \odot x)$ belong to $\acute{I}$. But $\Psi(x \odot a) = \Psi(x) \acute{\odot} \Psi(a)$ is an ingredient of $\acute{I}$, because $\acute{I}$ is a BCH-semigroup ideal in $\acute{B}$. Also $\Psi(a \odot x) = \Psi(a) \acute{\odot} \Psi(x)$ is an ingredient of $\acute{I}$, because $\acute{I}$ is a BCH-semigroup ideal in $\acute{B}$. Hence $x \odot a$, $a \odot x \in I$. Next, suppose that a, $x * a \in I$. Now, we have to prove that $x \in I$. Further, $x \in I$ if and only if $\Psi(x) \in \acute{I}$. It is obvious $\Psi(a)$, $\Psi(x * a) \in \acute{I}$. Consider, $\Psi(x * a) = \Psi(x) \acute{*} \Psi(a)$, which implies $\Psi(x) \in \acute{I}$, because $\acute{I}$ is a BCH-semigroup ideal in $\acute{B}$. Hence $x \in I$. Thus I is a BCH-semigroup ideal in B. Converse follows from Theorem 3.9. □

Theorem 3.9. *Let $(B; \odot, *, 0)$ and $(\acute{B}; \acute{\odot}, \acute{*}, \acute{0})$ be BCH-semigroups. Let $\Psi : B \longrightarrow \acute{B}$ be an epimorphism. Then there is one-one correspondence between the BCH-semigroup ideals of $\acute{B}$ and those BCH-semigroup ideals of B which contain the kernel K.*

Proof. Let β be a mapping from the collection Γ of all BCH-semigroup ideals of B containing K to the collection $\acute{\Gamma}$ of all BCH-semigroup ideals of $\acute{B}$ given by $\beta(I) = \acute{I} = \Psi(I)$, $I \in \Gamma$. Now $\acute{I} = \Psi(I) \in \acute{\Gamma}$ by Theorem 3.9. If I_1, $I_2 \in \Gamma$ *and* $\beta(I_1) = \beta(I_2) = \acute{I}$ (say) then we show that $I_1 = I_2$. Let $I_1 = \Psi^{-1}(\acute{I})$. Then certainly $I_1 \subseteq I$. Next, let $a \in I$. Then $\Psi(a) = \acute{a} = \Psi(a_1)$, from $\beta(I) = \acute{I} = \Psi(I_1)$, $(\because\ \acute{a} \in \acute{I}, a_1 \in I_1)$. Thus $\Psi(a) \acute{*} \Psi(a_1) = \acute{0}$, which implies $\Psi(a * a_1) = \acute{0}$, and hence $a * a_1 \in K \subseteq I_1$. It follows that $a * a_1 \in I_1$. As I_1 is a BCH-semigroup ideal so $a \in I_1$. It yields that $I \subseteq I_1$. So, we have $I = I_1$. Similarly $I = I_2$. Hence β is injective. Also, each $\acute{I} \in \acute{\Gamma}$ is the image of an $I = \Psi^{-1}(\acute{I})$. Hence β is surjective and therefore bijective. Consequently, β is one-one correspondence between the BCH-semi-group ideals of $\acute{B}$ and those BCH-semigroup ideals of B which contain K. □

4 Conclusion

In this paper, we have studied the BCH-semigroup ideal of a BCH-semigroup. Certain conditions for a semi-subgroup to be a BCH-semigroup ideal have been investigated. Some results about

BCH-semigroup homomorphisms related to BCH-semigroup ideals are also proved.

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