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Adequate property in amalgamated algebra along an ideal

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Abstract Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. In this paper, we investigate the transfert of the notion of adequate rings to the amalgamation $A \bowtie^f J$. Our aim is to give new classes of commutative rings satisfying this property.

1 Introduction

All rings in this paper are commutative with unity. We denote by $U(R)$ the set of unit of a ring R. And, if $a, b \in R$, $a|b$ means a divides b, that is $b = ac$ for some $c \in R$.

We know that an elementary divisor ring is a Hermite ring. Kaplansky showed that for the class of adequate domains being a Hermite ring was equivalent to being an elementary divisor ring. Gillman and Henriksen showed that this was also true for rings with zero-divisors. See for instance [11, 14, 18, 24].

Now, we give the definition of adequare ring. A ring A is said an adequate ring if for all $a \in A - \{0\}$ and $b \in A$, there exists two non-zero elements r, s of A such that :
a) $a = rs$.
b) $rA + bA = A$.
c) For every $t \in A - U(A)$, t divides s implies $tA + bA \neq A$.

The notion of an adequate domain was originally defined by Helmer [14]. By definition, every adequate domain is a Prüfer domain. Also, every principal ideal domain is adequate. An example of an adequate ring which is not a principal ideal domain is furnished by the set of integral functions with coefficients in a field F. Also, it is clear to see that a local ring is adequate. For instance, see [14, 24].

Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j)/a \in A, j \in J\}$$

called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana in [6, 7]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [8, 9, 10] and denoted by $A \bowtie I$). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation [6, Examples 2.5 & 2.6] and other classical constructions, such as the Nagata's idealization and the CPI extensions (in the sense of Boisen and Sheldon [3]) are strictly related to it (see [6, Example 2.7 & Remark 2.8]).

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [6, Section 4]. This point of view allows the authors in [6, 7] to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of A, J and f. Namely, in [6], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. For instance, see [5, 6, 7, 8, 9, 10, 17, 21, 22].

In this paper, we investigate the transfert of the notion of adequate rings to the amalgamation $A \bowtie^f J$. Our aim is to give new classes of commutative rings satisfying this property.

2 Main Results

Now, we investigate the transfer of the adequate property to amalgamation of rings $A \bowtie^f J$.

Theorem 2.1. *Let A be an integral domain, B be a ring, $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then, the following statements hold:*

(i) f *is injective.*

(ii) $f(A) \cap J = (0)$.

(iii) For each $x \in (f(A) + J) - J$, $xJ = J$.

Then $A \bowtie^f J$ *is an adequate ring if and only if the following statements hold :*
a) A is an adequate ring.
b) For each $a, b \in A - U(A)$, $aA + bA \neq A$.

The proof of this theorem requires the following lemmas.

Lemma 2.2. *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism, J be an ideal of B, and let $(a, x) \in A \times B$. Then, $(a, x) \in A \bowtie^f J$ if and only if $x - f(a) \in J$.*

Proof. Let $(a, x) \in A \times B$, $(a, x) \in A \bowtie^f J \Leftrightarrow (a, x) = (a, f(a) + j)$. So, it follows that there exists $j \in J : x = f(a) + j$ and so $x - f(a) = j \in J$. □

Lemma 2.3. *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B.*
a) Assume that

(i) $f(A) \cap J = (0)$.

(ii) For each $x \in (f(A) + J) - J$, $xJ = J$.

(iii) $A \bowtie^f J$ *is an adequate ring.*

Let (a, x), $(b, y) \in A \bowtie^f J$ such that $b \neq 0$, and let $c \in A$. Then, $a = bc$ if and only if there exists $z \in f(A) + J$ such that :

$$\begin{cases} (a, x) = (b, y)(c, z) \\ (c, z) \in A \bowtie^f J \end{cases}$$

b) Let (a, x), $(b, y) \in A \bowtie^f J$. Then, $(a, x)A \bowtie^f J + (b, y)A \bowtie^f J = A \bowtie^f J$ if and only if $aA + bA = A$.
c) Let $(a, x) \in A \bowtie^f J$. Then, $(a, x) \in U(A \bowtie^f J)$ if and only if $a \in U(A)$.

Proof. a) Assume that (1), (2) and (3) hold.
Let (a, x) and $(b, y) \in A \bowtie^f J$ such that $b \neq 0$ and let $c \in A$. Assume that there exists $z \in f(A) + J$ such that $(c, z) \in A \bowtie^f J$ and $(a, x) = (b, y)(c, z)$. Then, it follows that $a = bc$.
Conversely, assume that $a = bc$. Let $x - f(a) = j$ and $y - f(b) = k$. Then $x = f(a) + j$ and $y = f(b) + k$. Since (a, x) and $(b, y) \in A \bowtie^f J$, then by Lemma 2.2, $j, k \in J$. We claim that $y \notin J$. Deny, $y - f(b) = k$. So, $f(b) = y - k \in J$. Therefore, $f(b) \in f(A) \cap J = (0)$. Consequently, $f(b) = 0$. Using the fact f is injective, it follows that $b = 0$, which is a contradiction. Hence, $y \notin J$. Since $y \in (f(A) + J) - J$, then $yJ = J$. We have $y - kf(c)k \in J\,J = yJ$, and so there exists $l \in J$ such that $yl = j - kf(c)$. So, $j = yl + kf(c)$. Let $z = f(c) + l$. Then $z \in f(A) + J$ and $x = f(a) + j = f(bc) + kf(c) + yl = (f(b) + k)f(c) + yl = yf(c) + yl = y(f(c) + l)$. So, $x = yz$. Hence,

$$\begin{cases} (a, x) = (b, y)(c, z) \\ (c, z) \in A \bowtie^f J \end{cases}$$

b) Let $(a, x), (b, y) \in A \bowtie^f J$. Then:

$$\begin{aligned}
(a, x)A \bowtie^f J + (b, y)A \bowtie^f J = A \bowtie^f J &\Leftrightarrow \pi((a, x)A \bowtie^f J + (b, y)A \bowtie^f J) = A \\
&\Leftrightarrow \pi((a, x)A \bowtie^f J) + \pi((b, y)A \bowtie^f J) = A \\
&\Leftrightarrow \pi((a, x))\pi(A \bowtie^f J) + \pi((b, y))\pi(A \bowtie^f J) = A \\
&\Leftrightarrow aA + bA = A.
\end{aligned}$$

c) Let $(a, x) \in A \bowtie^f J$. Then:

$$\begin{aligned}
(a, x) \in U(A \bowtie^f J) &\Leftrightarrow \exists (a, x)|(1, 1) \text{ such that } (a, x) = (t, u)(b, y) \\
&\Leftrightarrow \exists a \neq 0 \text{ and } (a, x)|(1, 1) \\
&\Leftrightarrow \exists a \neq 0 \text{ and } a|1 \\
&\Leftrightarrow a \in U(A)
\end{aligned}$$

□

Lemma 2.4. *Let $f : A \to B$ be a rings homomorphism and J be an ideal of B such that the following statements hold:*

(i) f *is injective.*

(ii) $f(A) \cap J = (0)$.

(iii) $\forall\, x \in (f(A) + J) - J, xJ = J$.

(iv) $A \bowtie^f J$ *is an adequate ring.*

Then A is an adequate ring.

Proof. Let $a \in A - \{0\}$, $b \in A$. Then $(a, f(a)) \in (A \bowtie^f J) - \{0\}$ and $(b, f(b)) \in A \bowtie^f J$. Since $A \bowtie^f J$ is an adequate ring, then there exists $(r, u), (s, v) \in A \bowtie^f J$ such that

$$\begin{cases}
(a, f(a)) = (r, u)(s, v) \\
(r, u)A \bowtie^f J + (b, f(b))(A \bowtie^f J) = A \bowtie^f J \\
\forall (t, w) \in A \bowtie^f J - \{U(A \bowtie^f J)\} : (t, w)|(s, v) \Rightarrow (t, w)A \bowtie^f J + (b, f(b))A \bowtie^f J \neq A \bowtie^f J
\end{cases}$$

- We have $(a, f(a)) = (r, u)(s, v) = (rs, uv)$. So, $a = rs$. Let π be the canonical projection of $A \bowtie^f J$ on A. Since $(r, u)A \bowtie^f J + (b, f(b))(A \bowtie^f J) = A \bowtie^f J$, then:

$$\begin{aligned}
rA + bA &= \pi((r, u))\pi(A \bowtie^f J) + \pi((s, v))\pi(A \bowtie^f J) \\
&= \pi((r, u)A \bowtie^f J) + \pi((s, v)A \bowtie^f J) \\
&= \pi((r, u)A \bowtie^f J + (s, v)A \bowtie^f J) \\
&= \pi(A \bowtie^f J) \\
&= A
\end{aligned}$$

- Let $t \in A - U(A)$ such that $t|s$: Using the fact $t|s$ and $s|a$, then $t|a$ since $t \neq 0$ $(a \neq 0)$. By *a*) of Lemma 2.3,$(t, f(t))|(s, v)$. Since $t \in A - U(A)$, then one can easily check that $(t, f(t)) \in A \bowtie^f J - U(A \bowtie^f J)$. Therefore, $(t, f(t))|(s, v)$. So, $(t, f(t))A \bowtie^f J + (b, f(b))(A \bowtie^f J) \neq A \bowtie^f J$. Hence, it follows that $tA + bA \neq A$. Thus, A is an adequate ring. □

Proof of Theorem 2.1. Assume that A is an integral domain, f is injective, $f(A) \cap J = (0)$ and for each $x \in (f(A) + J) - J$, $xJ = J$. If $A \bowtie^f J$ is an adequate ring, then A is an adequate ring by Lemma 2.4. Now, let $t, p \in A - U(A)$ such that $tA + pA = A$. So, $t \neq 0$ since $p \notin U(A)$. Let $0 \neq j \in J$. Clearly, $(0, j)$ and $(p, f(p)) \in A \bowtie^f J$. Using the fact $A \bowtie^f J$ is an adequate ring, then there exists $(r, u), (s, v) \in A \bowtie^f J$ such that :

$$\begin{cases} (0, j) = (r, u)(s, v) \\ (r, u)A \bowtie^f J + (p, f(p))(A \bowtie^f J) = A \bowtie^f J \\ \forall k \in A \bowtie^f J - \{U(A \bowtie^f J)\} : k|(s, v) \Rightarrow kA \bowtie^f J + (p, f(p))A \bowtie^f J \neq A \bowtie^f J \end{cases}$$

Then by *b*) of Lemma 2.3, $rA + pA = A$ and we have $p \neq 0$ since $p \notin U(A)$. Since $(0, j) = (r, u)(s, v)$, then $s = 0$ (since A is an integral domain, $rs = 0$ and $r \neq 0$). Since $t|s$ and $s = 0$, then by assumption and by *a*) of Lemma 2.3, $(t, f(t))|(s, v)$. In fact of view $t \in A - U(A)$, by *c*) of Lemma 2.3, $(t, f(t)) \in A \bowtie^f J - U(A \bowtie^f J)$. We have $t \in A - U(A)$ and $t|s$ since $s = 0$. And so $(t, f(t))A \bowtie^f J + (p, f(p))(A \bowtie^f J) \neq A \bowtie^f J$. Therefore, by *b*) of Lemma 2.3, $tA + pA \neq A$, a contradiction. Hence, for each $a, b \in A - U(A)$, $aA + bA \neq A$.
Conversely, assume that *a*) and *b*) hold. Consider $(a, x) \in A \bowtie^f J - \{0\}$ and $(b, y) \in A \bowtie^f J$. Two cases are possible :
Case 1 : $a \neq 0$. Since A is an adequate ring, then $a \in A - \{0\}$ and $b \in A$, and so there exists $r, s \in A$ such that

$$\begin{cases} a = rs \\ rA + bA = A \\ \forall t \in A - U(A) : \quad t|s \Rightarrow tA + bA \neq A \end{cases}$$

Since $rs = a \neq 0$, then $r \neq 0$, and by *a*) of Lemma 2.3, there exists $u \in f(A) + J$ such that :

$$\begin{cases} (a, x) = (r, f(r))(s, u) \\ (s, u) \in A \bowtie^f J \end{cases}$$

Since $rA + bA = A$, then by *b*) of Lemma 2.3, $(r, f(r))A \bowtie^f J + (b, y)A \bowtie^f J = A \bowtie^f J$. Let $(t, v) \in A \bowtie^f J - U(A \bowtie^f J)$ such that $(t, v)|(s, u)$. By *c*) of Lemma 2.3, $t \in A - U(A)$ since $(t, v) \in A \bowtie^f J - U(A \bowtie^f J)$. Using the fact $(t, v)|(s, u)$, we obtain $t|s$ and so $t|s$ and $t \in A - U(A)$. Consequently, $tA + bA \neq A$. By *b*) of Lemma 2.3, $(t, v)A \bowtie^f J + (b, y)A \bowtie^f J \neq A \bowtie^f J$.
Case 2 : $a = 0$. $(a, x) = (0, x) \neq 0$ and so $x \neq 0$.
If $b \in U(A)$: then $(b, y) \in U(A \bowtie^f J)$ and

$$\begin{cases} (a, x) = (a, x)(1, 1) \\ (a, x)A \bowtie^f J + (b, y)A \bowtie^f J = (a, x)A \bowtie^f J + A \bowtie^f J = A \bowtie^f J \\ \forall k \in A \bowtie^f J - U(A \bowtie^f J) : k|(1, 1) \Rightarrow kA \bowtie^f J + (b, y)A \bowtie^f J \neq A \bowtie^f J \end{cases}$$

Assume that $b \notin U(A)$: Then by *c*) of Lemma 2.3, $(b, y) \notin U(A \bowtie^f J)$. So

$$\begin{cases} (a, x) = (1, 1)(a, x) \\ (1, 1)A \bowtie^f J + (b, y)A \bowtie^f J = A \bowtie^f J + (a, x)A \bowtie^f J = A \bowtie^f J \\ \forall (t, v) \in A \bowtie^f J - U(A \bowtie^f J) : (t, v)|(a, x) \Rightarrow t \in A - U(A) \text{ by } c) of Lemma 2.3. \end{cases}$$

Since $t, b \in A - U(A)$, then by *b*), $tA + bA \neq A$. Hence, by *b*) of Lemma 2.3, it follows that $(t, v)A \bowtie^f J + (b, y)A \bowtie^f J \neq A \bowtie^f J$.
Thus, $A \bowtie^f J$ is an adequate ring. □

Corollary 2.5. *Let A be an integral domain, $f : A \to B$ be a rings homomorphism and J be an ideal of B such that:*

(i) f is injective.

(ii) $f(A) \subseteq U(B) \cup \{0\}$.

(iii) For every $x \in (f(A) + J) - J$, $xJ = J$.

Then $A \bowtie^f J$ is an adequate ring if and only if the following statements hold:
a) A is an adequate ring.
b) For every $a, b \in A - U(A)$, $aA + bA \neq A$.

Proof. Assume that A is an integral domain and (1), (2) and (3) hold. By Theorem 2.1, we need to show that $f(A) \cap J = (0)$. But $f(A) \cap J \subset (U(B) \cup \{0\}) \cap J = (U(B) \cap J) \cup (\{0\} \cap J) = \cap 0 = 0$. Hence, we obtain desired result by Theorem 2.1.

□

Corollary 2.6. *Let A be an integral domain, B be a ring, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B. Assume that the following statements hold:*

(i) f is injective.

(ii) $f(A) \subseteq U(B) \cup \{0\}$.

(iii) B is local.

Then $A \bowtie^f J$ is an adequate ring if and only if the following statements hold :
a) A is an adequate ring.
b) For every $a, b \in A - U(A)$, $aA + bA \neq A$.

Proof. Assume that A is an integral domain and the statement (1), (2) and (3) hold. By assumption, B is local, then B has an unique maximal ideal. Since J is a proper ideal of B, then $J \subset M$. For every $x \in [f(A) + J] - J$, $x \in f(A) + J$ imply that there exists $b \in f(A)$ and $j \in J$ such that $x = b + j$. Since $x = b + j \notin J$ and $j \in J$, then $b \neq 0$. We have $b \in f(A) \subseteq U(B) \cup \{0\}$ and using the fact $b \neq 0$, then $b \in U(B)$. We claim that $x \notin M$. Suppose that $x \in M$. So :

$$\begin{cases} b + j = x \in M \\ j \in J \subset M \end{cases}$$

Therefore, $b = b + j - j \in M$ and so $b \notin U(B)$, a contradiction. Hence, $x \notin M$. Since (B, M) is local and $x \notin M$, then necessarily $x \in U(B)$. So $xJ = J$. We showed that the statement (3) of Corollary 2.5. Hence, by Corollary 2.5, we obtain the result desired.

□

Corollary 2.7. *Let A be an integral domain, $K := qf(A)$ the quotient field of A, $B := K[[x]]$ be the ring of power series with an indeterminate x with coefficients in K, $f : A \to B$ be an injective ring homomorphism and $J := x^n K[[x]]$ be a proper ideal of B. Then, $A \bowtie^f J$ is an adequate ring if and only if the following statements hold :*
a) A is an adequate ring.
b) $\forall\ a, b \in A - U(A)$, $aA + bA \neq A$.

Proof. Assume that A is an integral domain, f is injective, $B := K[[x]]$, and $J := K[[x]]$. We have $f(A) \subseteq U(K[[x]]) \cup \{0\}$. Therefore, the statement (2) of Corollary 2.6. Since $B := K[[x]]$ is local, then we obtain the desired result by Corollary 2.6. □

We end the first main result by the following characterization.

Theorem 2.8. *Let A be a principal ideal domain, B be a ring, $f : A \to B$ be a ring homomorphism and J be an ideal of B such that the following statements hold:*

(i) f is injective.

(ii) $f(A) \subseteq U(B) \cup \{0\}$.

(iii) B is local.

Then $A \bowtie^f J$ is an adequate ring if and only if A is local.

Before proving this Proposition, we need the following Lemma.

Lemma 2.9. *Let A be a principal ideal domain. Then A is local if and only if for every $p, q \in A - U(A)$, $pA + qA \neq A$.*

Proof. Assume that A is local and let M be its maximal ideal. Then, for each $p, q \in A - U(A)$, $pA + qA \subset M$ and so $pA + qA \neq A$.
Conversely, assume that for each $p, q \in A - U(A)$, $pA + qA \neq A$. We claim that A is not local. Deny. Then, A has at least two maximal ideals denoted M and N. Using the fact A is a principal ideal domain, then there exists $p, q \in A$ such that $M = pA$ and $N = qA$. Therefore, p and q are irreducible since A is not a field and M and N are maximal ideals of A. Hence, p and q are not associated (since $M \neq N$), so p and q are co-primes and hence $pA + qA = A$, (since A is a principal ideal domain) a contradiction. Hence, for every $p, q \in A - U(A)$, $pA + qA \neq A$. □

Proof of Theorem 2.8. Assume that A is a principal ideal domain, f is injective, $f(A) \subseteq U(B) \cup \{0\}$ and B is local. If $A \bowtie^f J$ is an adequate ring, then by Corollary 2.6, $aA + bA \neq A$ for every $a, b \in A - U(A)$. Hence, by Lemma 2.9, A is local, as desired.
Conversely, assume that A is local. Hence, $A \bowtie^f J$ is local (since B is local and so $J \subset Rad(B)$) and so $A \bowtie^f J$ is an adequate ring, as desired. □

Next, we explore a different context, namely, when $J^2 = 0$. We need the following Lemma.

Lemma 2.10. *Let A be an integral domain, B be a ring, $f : A \to B$ be a ring homomorphism and J be an ideal of B such that:*

(i) f is injective.

(ii) $J^2 = (0)$.

(iii) For every $t \in A - \{0\}$, $f(t)J = J$.

Then A is an adequate ring provided $A \bowtie^f J$ is an adequate ring.

Proof. Assume that $A \bowtie^f J$ is an adequate ring. Let $a \in A - \{0\}$ and $b \in A$. Clearly, $(a, f(a))$ and $(b, f(b)) \in A \bowtie^f J - \{0\}$. Since $A \bowtie^f J$ is an adequate ring, then there exists (r, u) and $(s, v) \in A \bowtie^f J$ such that:

$$\begin{cases} (a, f(a)) = (r, u)(s, v) \\ (r, u)A \bowtie^f J + (b, f(b))A \bowtie^f J = A \bowtie^f J \\ \forall (t, w) \in A \bowtie^f J - U(A \bowtie^f J) : (t, w)|(s, v) \Rightarrow (t, w)A \bowtie^f J + (b, f(b))A \bowtie^f J \neq A \bowtie^f J \end{cases}$$

We have $(a, f(a)) = (r, u)(s, v) = (rs, uv)$. So, $a = rs$. Let π be the surjection of $A \bowtie^f J$ to A. Since $(r, u)A \bowtie^f J + (b, f(b))A \bowtie^f J = A \bowtie^f J$, then :

$$\begin{aligned} rA + bA &= \pi((r, u))\pi(A \bowtie^f J) + \pi((s, v))\pi(A \bowtie^f J) \\ &= \pi((r, u)A \bowtie^f J) + \pi((s, v)A \bowtie^f J) \\ &= \pi((r, u)A \bowtie^f J + (s, v)A \bowtie^f J) \\ &= \pi(A \bowtie^f J) \\ &= A \end{aligned}$$

Let $t \in A - U(A)$ such that $t|s$. Using the fact $t|s$ and $s|a$ ($a = rs$), then $t|a$ and so $t \neq 0$. Therefore, by a) of Lemma 2.3, $(t, f(t))|(s, v)$, and so by c) of Lemma 2.3, $(t, f(t)) \in A \bowtie^f J - U(A \bowtie^f J)$ (since $t \in A - U(A)$). Consequently, $(t, f(t)A \bowtie^f J + (b, f(b))A \bowtie^f J \neq A \bowtie^f J$. Hence, by b) of Lemma 2.3, $tA + bA \neq A$. Thus, A is an adequate ring. □

Now, to the second main result of this paper.

Theorem 2.11. *Let A be an integral domain, B be a ring, $f : A \to B$ be a ring homomorphism and J be an ideal of B such that:*

(i) f is injective.

(ii) $J^2 = (0)$.

(iii) *For every* $t \in A - \{0\}$, $f(t)J = J$.

Then $A \bowtie^f J$ *is an adequate ring if and only if the following statements hold :*
a) *A is an adequate ring.*
b) *For every* $a, b \in A - U(A)$, $aA + bA \neq A$.

Proof. Assume that A is an integral domain and the statement (1), (2) and (3) hold. Assume that $A \bowtie^f J$ is an adequate ring. Then :
a) By Lemma 2.10, A is an adequate ring.
b) We show that $aA+bA \neq A$, for every $a, b \in A-U(A)$. Suppose that there exists $t, p \in A-U(A)$ such that $tA + pA = A$. Necessarily $t \neq 0$ since $p \in A - U(A)$. Let $0 \neq j \in J$. Clearly, $(0, j)$ and $(p, f(p))$ are elements of $A \bowtie^f J$ which is an adequate ring. So, there exists (r, u) and $(s, v) \in A \bowtie^f J$ such that

$$\begin{cases} (0, j) = (r, u)(s, v) \\ (r, u)A \bowtie^f J + (p, f(p))A \bowtie^f J = A \bowtie^f J \\ \forall k \in A \bowtie^f J - U(A \bowtie^f J) : k|(s, v) \Rightarrow kA \bowtie^f J + (p, f(p))A \bowtie^f J \neq A \bowtie^f J \end{cases}$$

Since $(r, u)A \bowtie^f J + (p, f(p))A \bowtie^f J = A \bowtie^f J$, then by *b*) of Lemma 2.3, $rA + pA = A$. It is easy to see that $r \neq 0$ since $p \notin U(A)$. We have (0,j)=(r,u)(s,v) and so $rs = 0$. Therefore, $s = 0$ since $r \neq 0$ and A is an integral domain. By *a*) of Lemma 2.3, we obtain $(t, f(t))|(s, v)$ since $t|s$. By *c*) of Lemma 2.3, $(t, f(t)) \in A \bowtie^f J - U(A \bowtie^f J)$ since $t \in A - U(A)$. Using the fact $t|s$ (since $s = 0$) and $t \in A - U(A)$, then $(t, f(t))A \bowtie^f J + (p, f(p))A \bowtie^f J \neq A \bowtie^f J$. Hence By *b*) of Lemma 2.3, $tA + pA \neq A$, a contradiction. Thus, $\forall\ a, b \in A - U(A)$, $aA + bA \neq A$.

Conversely, assume that A is an adequate ring and $\forall\ a, b \in A - U(A)$, $aA + bA \neq A$. Let $(a, x) \in A \bowtie^f J - \{0\}$, and let $(b, y) \in A \bowtie^f J$. Two cases are possible:
Case 1 : $a \neq 0$. Since A is an adequate ring and $a \in A - \{0\}$ and $b \in A$, then there exists $r, s \in A$ such that :

$$\begin{cases} a = rs \\ rA + bA = A \\ \forall t \in A - U(A) : t|s \Rightarrow tA + bA \neq A. \end{cases}$$

Since $rs = a$, then $r \neq 0$ and by *a*) of Lemma 2.3, there exists $u \in f(A) + J$ such that

$$\begin{cases} (a, x) = (r, f(r))(s, u) \\ (s, u) \in A \bowtie^f J \end{cases}$$

Using the fact $rA+bA = A$, then by *b*) of Lemma 2.3, $(r, f(r))A \bowtie^f J+(b, y)A \bowtie^f J = A \bowtie^f J$. Let $(t, v) \in A \bowtie^f J - U(A \bowtie^f J)$ such that $(t, v)|(s, u)$. By *c*) of Lemma 2.3, $t \in A - U(A)$ since $(t, v) \in\in A \bowtie^f J - U(A \bowtie^f J)$. Using the fact $t \in A - U(A)$ and $t|s$, then $tA + bA \neq A$. Hence, by *b*) of Lemma 2.3, it follows that $(t, v)A \bowtie^f J + (b, y)A \bowtie^f J \neq A \bowtie^f J$.
Case 2 : $a = 0$.
$(a, x) = (0, x) \neq 0$ and so $x \neq 0$. If $b \in U(A)$, then by *c*) of Lemma 2.3, $(b, y) \in U(A \bowtie^f J)$. Then :

$$\begin{cases} (a, x) = (a, x)(1, 1) \\ (a, x)A \bowtie^f J + (b, y)A \bowtie^f J = (a, x)A \bowtie^f J + A \bowtie^f J = A \bowtie^f J \\ \forall k \in A \bowtie^f J - U(A \bowtie^f J) : k|(1, 1) \Rightarrow kA \bowtie^f J + (b, y)A \bowtie^f J \neq A \bowtie^f J \end{cases}$$

Assume that $b \notin U(A)$. Then $(b, y) \notin U(A \bowtie^f J)$. Therefore,

$$\begin{cases} (a, x) = (1, 1)(a, x) \\ (1, 1)A \bowtie^f J + (b, y)A \bowtie^f J = (a, x)A \bowtie^f J + A \bowtie^f J = A \bowtie^f J \\ \forall (t, v) \in A \bowtie^f J - U(A \bowtie^f J) : (t, v)|(a, x) \Rightarrow kA \bowtie^f J + (b, y)A \bowtie^f J \neq A \bowtie^f J \end{cases}$$

Since $(t, v) \in U(A \bowtie^f J)$, then $t \in A - U(A)$. Moreover $t, b \in A - U(A)$. Therefore, $tA + bA \neq A$. By *b*) of Lemma 2.3, we obtain
$(t, v)A \bowtie^f J + (b, y)A \bowtie^f J \neq A \bowtie^f J$. Thus, $A \bowtie^f J$ is an adequate ring. □

Corollary 2.12. *Let A be a principal ideal domain, B be a ring, $f : A \to B$ be a ring homomorphism and J be an ideal of B such that $J \subset Rad(B)$ and:*

(i) f *is injective.*

(ii) $J^2 = (0)$.

(iii) For every $t \in A - \{0\}$, $f(t)J = J$.

Then $A \bowtie^f J$ is an adequate ring if and only if A is local.

Proof. Assume that A is a principal ideal domain, f is injective, $J^2 = (0)$and $forall\ t \in A - \{0\}$, $f(t)J = J$. If $A \bowtie^f J$ is an adequate ring, then by Theorem 2.11, $aA + bA \neq A$ for every $a, b \in A - U(A)$, and so A is local by Lemma 2.9.
Conversely, assume that A is local. Hence, $A \bowtie^f J$ is local since $J \subseteq Rad(B)$ (since $J^2 = (0)$), and so $A \bowtie^f J$ is an adequate ring. □

Example 2.13. Let $A := \mathbb{Z}$, $B := \mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]]$, $J = (X^2+1)^2\mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]]$ be an ideal of B and

$$\begin{array}{rccc} f: & A & \to & B \\ & a & \to & f(a) = \bar{a} \end{array}$$

be a ring homomorphism. Then $A \bowtie^f J$ is not an adequate ring.

Proof. A is a principal ideal domain which is not local, it is clear that f is injective and $J \subset Rad(B)$ (since $B := \mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]]$ is local). On the other hand, $J^2 = [(X^2+1)^2\mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]]]^2 = (X^2+1)^4\mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]] = 0$ and for $t \in A - \{0\}$, $f(t)J = \bar{t}((X^2+1)^2\mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]]) = ((X^2+1)^2t\mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]]) = (X^2+1)^2\mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]] = J$. Hence, by Theorem 2.8, $A \bowtie^f J$ is not an adequate ring since $A := \mathbb{Z}$ is not local. □

Example 2.14. Let $A := \mathbb{Z}_{2\mathbb{Z}}$, $B := \mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]]$, $J = (X^2+1)^2\mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]]$ be an ideal of B and

$$\begin{array}{rccc} f: & A & \to & B \\ & a & \to & f(a) = \bar{a} \end{array}$$

be a ring homomorphism. Then $A \bowtie^f J$ is an adequate ring (since A is a discrete valuation domain).

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