Adequate property in amalgamated algebra along an ideal

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Abstract Let $f : A \to B$ be a ring homomorphism and let $J$ be an ideal of $B$. In this paper, we investigate the transfert of the notion of adequate rings to the amalgamation $A \bowtie_f J$. Our aim is to give new classes of commutative rings satisfying this property.

1 Introduction

All rings in this paper are commutative with unity. We denote by $U(R)$ the set of unit of a ring $R$. And, if $a, b \in R$, $a|b$ means $a$ divides $b$, that is $b = ac$ for some $c \in R$.

We know that an elementary divisor ring is a Hermite ring. Kaplansky showed that for the class of adequate domains being a Hermite ring was equivalent to being an elementary divisor ring. Gillman and Henriksen showed that this was also true for rings with zero-divisors. See for instance [11, 14, 18, 24].

Now, we give the definition of adequate ring. A ring $A$ is said an adequate ring if for all $a \in A - \{0\}$ and $b \in A$, there exists two non-zero elements $r, s$ of $A$ such that :

a) $a = rs$.

b) $rA + bA = A$.

c) For every $t \in A - U(A)$, $t$ divides $s$ implies $tA + bA \neq A$.

The notion of an adequate domain was originally defined by Helmer [14]. By definition, every adequate domain is a Prüfer domain. Also, every principal ideal domain is adequate. An example of an adequate ring which is not a principal ideal domain is furnished by the set of integral functions with coefficients in a field $F$. Also, it is clear to see that a local ring is adequate. For instance, see [14, 24].

Let $A$ and $B$ be two rings, let $J$ be an ideal of $B$ and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie_f J = \{(a, f(a) + j) : a \in A, j \in J\}$$

called the amalgamation of $A$ with $B$ along $J$ with respect to $f$ (introduced and studied by D’Anna, Finocchiaro, and Fontana in [6, 7]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D’Anna and Fontana in [8, 9, 10] and denoted by $A \bowtie I$). Moreover, other classical constructions (such as the $A + XB[[X]]$, $A + XA[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation [6, Examples 2.5 & 2.6] and other classical constructions, such as the Nagata’s idealization and the CPI extensions (in the sense of Boisen and Sheldon [3]) are strictly related to it (see [6, Example 2.7 & Remark 2.8]).

One of the key tools for studying $A \bowtie_f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [6, Section 4]. This point of view allows the authors in [6, 7] to provide an ample description of various properties of $A \bowtie_f J$, in connection with the properties of $A, J$ and $f$. Namely, in [6], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie_f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. For instance, see [5, 6, 7, 8, 9, 10, 17, 21, 22].
In this paper, we investigate the transfer of the notion of adequate rings to the amalgamation $A \triangleright J$. Our aim is to give new classes of commutative rings satisfying this property.

2 Main Results

Now, we investigate the transfer of the adequate property to amalgamation of rings $A \triangleright J$.

Theorem 2.1. Let $A$ be an integral domain, $B$ be a ring, $f : A \rightarrow B$ be a ring homomorphism and $J$ be an ideal of $B$. Then, the following statements hold:

(i) $f$ is injective.
(ii) $f(A) \cap J = (0)$.
(iii) For each $x \in (f(A) + J) - J$, $xJ = J$.

Then $A \triangleright J$ is an adequate ring if and only if the following statements hold:

a) $A$ is an adequate ring.
b) For each $a, b \in A - U(A)$, $aA + bA \neq A$.

The proof of this theorem requires the following lemmas.

Lemma 2.2. Let $A$ and $B$ be a pair of rings, $f : A \rightarrow B$ be a ring homomorphism, $J$ be an ideal of $B$, and let $(a, x) \in A \times B$. Then, $(a, x) \in A \triangleright J$ if and only if $x - f(a) \in J$.

Proof. Let $(a, x) \in A \times B$, $(a, x) \in A \triangleright J$ $\iff (a, x) = (a, f(a) + j)$. So, it follows that there exists $j \in J : x = f(a) + j$ and so $x - f(a) = j \in J$.

Lemma 2.3. Let $A$ and $B$ be two rings, $f : A \rightarrow B$ be a ring homomorphism and $J$ be an ideal of $B$.

a) Assume that

(i) $f(A) \cap J = (0)$.
(ii) For each $x \in (f(A) + J) - J$, $xJ = J$.
(iii) $A \triangleright J$ is an adequate ring.

Let $(a, x), (b, y) \in A \triangleright J$ such that $b \neq 0$, and let $c \in A$. Then, $a = bc$ if and only if there exists $z \in f(A) + J$ such that:

\[
\begin{align*}
(a, x) &= (b, y)(c, z) \\
(c, z) &\in A \triangleright J
\end{align*}
\]

b) Let $(a, x), (b, y) \in A \triangleright J$. Then, $(a, x)A \triangleright J + (b, y)A \triangleright J = A \triangleright J$ if and only if $aA + bA = A$.

c) Let $(a, x) \in A \triangleright J$. Then, $(a, x) \in U(A \triangleright J)$ if and only if $a \in U(A)$.

Proof. a) Assume that (1), (2) and (3) hold.

Let $(a, x)$ and $(b, y) \in A \triangleright J$ such that $b \neq 0$ and let $c \in A$. Assume that there exists $z \in f(A) + J$ such that $z = f(a) + J$ and $(a, x) = (b, y)(c, z)$. Then, it follows that $a = bc$.

Conversely, assume that $a = bc$. Let $x - f(a) = j$ and $y - f(b) = k$. Then $x = f(a) + j$ and $y = f(b) + k$. Since $(a, x)$ and $(b, y) \in A \triangleright J$, then by Lemma 2.2, $j, k \in J$. We claim that $y \notin J$.

Deny, $y - f(b) = k$. So, $f(b) = y - k \in J$. Therefore, $f(b) \in f(A) \cap J = (0)$. Consequently, $f(b) = 0$. Using the fact $f$ is injective, it follows that $b = 0$, which is a contradiction. Hence, $y \notin J$. Since $y \in (f(A) + J) - J$, then $yJ = J$. We have $y - kf(c)k \in J$ $\iff$ $yJ = yJ$, and so there exists $l \in J$ such that $yJ = l - kf(c)$. So, $y = l - j$ and $f(c) = c + l$. Then $z = f(c) + l$. Therefore, $f(z) \in f(A) \cap J$ and $x = f(a) + j = f(bc) + kf(c) + yl = (f(b) + k)f(c) + yl = yf(c) + yl = yf(c) + l)$. So, $x = yz$. Hence,}

\[
\begin{align*}
(a, x) &= (b, y)(c, z) \\
(c, z) &\in A \triangleright J
\end{align*}
\]
b) Let \((a, x), (b, y) \in A \bowtie J\). Then:

\[
(a, x) A \bowtie J + (b, y) A \bowtie J = A \bowtie J \iff 
\pi((a, x) A \bowtie J + (b, y) A \bowtie J) = A
\]

\[
\pi((a, x) A \bowtie J) + \pi((b, y) A \bowtie J) = A
\]

\[
\pi((a, x)) \pi(A \bowtie J) + \pi((b, y)) \pi(A \bowtie J) = A
\]

\[
aA + bA = A.
\]

c) Let \((a, x) \in A \bowtie J\). Then:

\[
(a, x) \in U(A \bowtie J) \iff \exists (a, x)|(1, 1) \text{ such that } (a, x) = (t, u)(b, y)
\]

\[
\exists a \neq 0 \text{ and } (a, x)|(1, 1)
\]

\[
\exists a \neq 0 \text{ and } a|1
\]

\[
a \in U(A)
\]

\[\square\]

Lemma 2.4. Let \(f : A \to B\) be a rings homomorphism and \(J\) be an ideal of \(B\) such that the following statements hold:

(i) \(f\) is injective.

(ii) \(f(A) \cap J = (0)\).

(iii) \(\forall x \in (f(A) + J) - J, xJ = J\).

(iv) \(A \bowtie J\) is an adequate ring.

Then \(A\) is an adequate ring.

Proof. Let \(a \in A - \{0\}, b \in A\). Then \((a, f(a)) \in (A \bowtie J) - \{0\}\) and \((b, f(b)) \in A \bowtie J\). Since \(A \bowtie J\) is an adequate ring, then there exists \((r, u), (s, v) \in A \bowtie J\) such that

\[
\begin{align*}
(a, f(a)) &= (r, u)(s, v) \\
(r, u)A \bowtie J + (b, f(b))(A \bowtie J) &= A \bowtie J \\
\forall (t, w) \in A \bowtie J - \{U(A \bowtie J)\} : (t, w)|(s, v) \Rightarrow (t, w)A \bowtie J + (b, f(b))A \bowtie J \neq A \bowtie J
\end{align*}
\]

- We have \((a, f(a)) = (r, u)(s, v) = (rs, uv)\). So, \(a = rs\). Let \(\pi\) be the canonical projection of \(A \bowtie J\) on \(A\). Since \((r, u)A \bowtie J + (b, f(b))(A \bowtie J) = A \bowtie J\), then:

\[
rA + bA = \pi((r, u)\pi(A \bowtie J) + \pi((s, v))\pi(A \bowtie J)
\]

\[
= \pi((r, u)A \bowtie J) + \pi((s, v)A \bowtie J)
\]

\[
= \pi((r, u)A \bowtie J + (s, v)A \bowtie J)
\]

\[
= \pi(A \bowtie J)
\]

\[
= A
\]

- Let \(t \in A - U(A)\) such that \(t|s\) and \(s|a\), then \(t|a\) since \(t \neq 0 (a \neq 0)\). By \(a\) of Lemma 2.3, \((t, f(t))(s, v)\). Since \(t \in A - U(A)\), then one can easily check that \((t, f(t)) \in A \bowtie J - U(A \bowtie J)\). Therefore, \((t, f(t))(s, v)\). So, \((t, f(t))A \bowtie J + (b, f(b))(A \bowtie J) \neq A \bowtie J\). Hence, it follows that \(tA + bA \neq A\). Thus, \(A\) is an adequate ring. \[\square\]
Proof of Theorem 2.1. Assume that $A$ is an integral domain, $f$ is injective, $f(A) \cap J = \{0\}$ and for each $x \in (f(A) + J) - J$, $xf = J$. If $A \bowtie f$ is an adequate ring, then $A$ is an adequate ring by Lemma 2.4. Now, let $t, p \in A - U(A)$ such that $ta + pA = A$. So, $t \neq 0$ since $p \notin U(A)$. Let $0 \neq j \in J$. Clearly, $(0, j)$ and $(p, f(p)) \in A \bowtie f$. Using the fact $A \bowtie f$ is an adequate ring, then there exists $(r, u), (s, v) \in A \bowtie f$ such that:

\[
\begin{align*}
(0, j) &= (r, u)(s, v) \\
(r, u)A \bowtie f J + (p, f(p))(A \bowtie f J) &= A \bowtie f J \\
\forall k \in A \bowtie f J - (U(A \bowtie f J)) : k(s, v) &= kA \bowtie f J + (p, f(p))A \bowtie f J 
eq A \bowtie f J
\end{align*}
\]

Then by $b)$ of Lemma 2.3, $rA + pA = A$ and we have $p \neq 0$ since $p \notin U(A)$. Since $(0, j) = (r, u)(s, v)$, then $s = 0$ (since $A$ is an integral domain, $rs = 0$ and $r \neq 0$). Since $t(s)$ and $s = 0$, then by assumption and by $a)$ of Lemma 2.3, $(t, f(t))(s, v)$. In fact of view $t \in A - U(A)$, by $c)$ of Lemma 2.3, $(t, f(t)) \in A \bowtie f J - U(A \bowtie f J)$. We have $t \in A - U(A)$ and $t(s)$ since $s = 0$. And so $(t, f(t))A \bowtie f J + (p, f(p))(A \bowtie f J) \neq A \bowtie f J$. Therefore, by $b)$ of Lemma 2.3, $tA + pA \neq A$, a contradiction. Hence, for each $a, b \in A - U(A)$, $aA + bA \neq A$.

Conversely, assume that $a)$ and $b)$ hold. Consider $(a, x) \in A \bowtie f J - \{0\}$ and $(b, y) \in A \bowtie f J$. Two cases are possible:

Case 1: $a \neq 0$. Since $A$ is an adequate ring, then $a \in A - \{0\}$ and $b \in A$, and so there exists $r, s \in A$ such that:

\[
\begin{align*}
a &= rs \\
rA + bA &= A \\
\forall t \in A - U(A) : t(s) &= tA + bA \neq A
\end{align*}
\]

Since $rs = a \neq 0$, then $r \neq 0$, and by $a)$ of Lemma 2.3, there exists $u \in f(A) + J$ such that:

\[
\begin{align*}
(a, x) &= (r, f(r))(s, u) \\
(s, u) &\in A \bowtie f J
\end{align*}
\]

Since $rA + bA = A$, then by $b)$ of Lemma 2.3, $(r, f(r))A \bowtie f J + (b, y)A \bowtie f J = A \bowtie f J$.

Let $(t, v) \in A \bowtie f J - U(A \bowtie f J)$ such that $(t, v)(s, u)$. By $c)$ of Lemma 2.3, $t \in A - U(A)$ since $(t, v) \in A \bowtie f J - U(A \bowtie f J)$. Using the fact $(t, v)(s, u)$, we obtain $t(s)$ and so $t(s)$ and $t \in A - U(A)$.

Consequently, $tA + bA \neq A$. By $b)$ of Lemma 2.3, $(t, v)A \bowtie f J + (b, y)A \bowtie f J \neq A \bowtie f J$.

Case 2: $a = 0$. $(a, x) = (0, x) \neq 0$ and so $x \neq 0$.

If $b \in U(A)$: then $(b, y) \in U(A \bowtie f J)$ and

\[
\begin{align*}
(a, x) &= (a, x)(1, 1) \\
(a, x)A \bowtie f J + (b, y)A \bowtie f J &= (a, x)A \bowtie f J + A \bowtie f J = A \bowtie f J \\
\forall k \in A \bowtie f J - U(A \bowtie f J) : k(1, 1) &= kA \bowtie f J + (b, y)A \bowtie f J \neq A \bowtie f J
\end{align*}
\]

Assume that $b \notin U(A)$: Then by $c)$ of Lemma 2.3, $(b, y) \notin U(A \bowtie f J)$. So

\[
\begin{align*}
(a, x) &= (1, 1)(a, x) \\
(1, 1)A \bowtie f J + (b, y)A \bowtie f J &= A \bowtie f J + (a, x)A \bowtie f J = A \bowtie f J \\
\forall (t, v) \in A \bowtie f J - U(A \bowtie f J) : (t, v)(a, x) &= t \in A - U(A) \text{ by } c)$ of Lemma 2.3.
\]

Since $t, b \in A - U(A)$, then by $b)$, $tA + bA \neq A$. Hence, by $b)$ of Lemma 2.3, it follows that $(t, v)A \bowtie f J + (b, y)A \bowtie f J \neq A \bowtie f J$.

Thus, $A \bowtie f J$ is an adequate ring.

\[\square\]

Corollary 2.5. Let $A$ be an integral domain, $f : A \to B$ be a rings homomorphism and $J$ be an ideal of $B$ such that:

(i) $f$ is injective.

(ii) $f(A) \subseteq U(B) \cup \{0\}$.

(iii) For every $x \in (f(A) + J) - J$, $xfJ = J$.

Then $A \bowtie f J$ is an adequate ring if and only if the following statements hold:

a) $A$ is an adequate ring.

b) For every $a, b \in A - U(A)$, $aA + bA \neq A$.
Proof. Assume that $A$ is an integral domain and (1), (2) and (3) hold. By Theorem 2.1, we need to show that $f(A) \cap J = \{0\}$. But $f(A) \cap J \subset (U(B) \cup \{0\}) \cap J = (U(B) \cap J) \cup (\{0\} \cap J) = \emptyset \cap 0 = 0$. Hence, we obtain desired result by Theorem 2.1.

Corollary 2.6. Let $A$ be an integral domain, $B$ be a ring, $f : A \to B$ be a ring homomorphism and $J$ be a proper ideal of $B$. Assume that the following statements hold:

(i) $f$ is injective.

(ii) $f(A) \subseteq U(B) \cup \{0\}$.

(iii) $B$ is local.

Then $A \rightsquigarrow J$ is an adequate ring if and only if the following statements hold:

a) $A$ is an adequate ring.

b) For every $a, b \in A - U(A)$, $aA + bA \neq A$.

Proof. Assume that $A$ is an integral domain and the statement (1), (2) and (3) hold. By assumption, $B$ is local, then $B$ has an unique maximal ideal. Since $J$ is a proper ideal of $B$, then $J \subset M$. For every $x \in f(A) + J$, $x \in f(A) + J$ imply that there exists $b \in f(A)$ and $j \in J$ such that $x = b + j$. Since $x = b + j \notin J$ and $j \in J$, then $b \neq 0$. We have $b \in f(A) \subseteq U(B) \cup \{0\}$ and using the fact $b \neq 0$, then $b \in U(B)$. We claim that $x \notin M$. Suppose that $x \in M$. So:

$$\begin{cases} b + j = x \in M \\ j \in J \subset M \end{cases}$$

Therefore, $b = b + j - j \in M$ and so $b \notin U(B)$, a contradiction. Hence, $x \notin M$. Since $(B, M)$ is local and $x \notin M$, then necessarily $x \in U(B)$. So $xJ = J$. We showed that the statement (3) of Corollary 2.5. Hence, by Corollary 2.5, we obtain the result desired.

Corollary 2.7. Let $A$ be an integral domain, $K := qf(A)$ the quotient field of $A$, $B := K[[x]]$ be the ring of power series with an indeterminate $x$ with coefficients in $K$, $f : A \to B$ be an injective ring homomorphism and $J := x^0K[[x]]$ be a proper ideal of $B$. Then, $A \rightsquigarrow J$ is an adequate ring if and only if the following statements hold:

a) $A$ is an adequate ring.

b) $\forall a, b \in A - U(A)$, $aA + bA \neq A$.

Proof. Assume that $A$ is an integral domain, $f$ is injective, $B := K[[x]]$, and $J := K[[x]]$. We have $f(A) \subseteq U(K[[x]]) \cup \{0\}$. Therefore, the statement (2) of Corollary 2.6. Since $B := K[[x]]$ is local, then we obtain the desired result by Corollary 2.6.

We end the first main result by the following characterization.

Theorem 2.8. Let $A$ be a principal ideal domain, $B$ be a ring, $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$ such that the following statements hold:

(i) $f$ is injective.

(ii) $f(A) \subseteq U(B) \cup \{0\}$.

(iii) $B$ is local.

Then $A \rightsquigarrow J$ is an adequate ring if and only if $A$ is local.

Before proving this Proposition, we need the following Lemma.

Lemma 2.9. Let $A$ be a principal ideal domain. Then $A$ is local if and only if for every $p, q \in A - U(A)$, $pA + qA \neq A$. 
Proof. Assume that $A$ is local and let $M$ be its maximal ideal. Then, for each $p, q \in A - U(A)$, $pA + qA \subset M$ and so $pA + qA \neq A$.

Conversely, assume that for each $p, q \in A - U(A)$, $pA + qA \neq A$. We claim that $A$ is not local. Deny. Then, $A$ has at least two maximal ideals denoted $M$ and $N$. Using the fact $A$ is a principal ideal domain, then there exists $p, q \in A$ such that $M = pA$ and $N = qA$. Therefore, $p$ and $q$ are irreducible since $A$ is not a field and $M$ and $N$ are maximal ideals of $A$. Hence, $p$ and $q$ are not associated (since $M \neq N$), so $p$ and $q$ are co-primes and hence $pA + qA = A$, (since $A$ is a principal ideal domain) a contradiction. Hence, for every $p, q \in A - U(A)$, $pA + qA \neq A$. \[\square\]

Proof of Theorem 2.8. Assume that $A$ is a principal ideal domain, $f$ is injective, $f(A) \subseteq U(B) \cup \{0\}$ and $B$ is local. If $A \not\twoheadrightarrow J$ is an adequate ring, then by Corollary 2.6, $aA + bA \neq A$ for every $a,b \in A - U(A)$. Hence, by Lemma 2.9, $A$ is local, as desired. Conversely, assume that $A$ is local. Hence, $A \not\twoheadrightarrow J$ is local (since $B$ is local and so $J \subset Rad(B)$) and so $A \not\twoheadrightarrow J$ is an adequate ring, as desired. \[\square\]

Next, we explore a different context, namely, when $J^2 = 0$. We need the following Lemma.

Lemma 2.10. Let $A$ be an integral domain, $B$ be a ring, $f : A \rightarrow B$ be a ring homomorphism and $J$ be an ideal of $B$ such that:

(i) $f$ is injective.

(ii) $J^2 = \{0\}$.

(iii) For every $t \in A - \{0\}$, $f(t)J = J$.

Then $A$ is an adequate ring provided $A \not\twoheadrightarrow J$ is an adequate ring.

Proof. Assume that $A \not\twoheadrightarrow J$ is an adequate ring. Let $a \in A - \{0\}$ and $b \in A$. Clearly, $(a, f(a))$ and $(b, f(b)) \in A \not\twoheadrightarrow J - \{0\}$. Since $A \not\twoheadrightarrow J$ is an adequate ring, then there exists $(r, u)$ and $(s, v) \in A \not\twoheadrightarrow J$ such that:

\[
\begin{align*}
(a, f(a)) &= (r, u)(s, v) \\
(r, u)A \not\twoheadrightarrow J + (b, f(b))A \not\twoheadrightarrow J &= A \not\twoheadrightarrow J \\
\forall (t, w) \in A \not\twoheadrightarrow J - U(A \not\twoheadrightarrow J) : (t, w)(s, v) \Rightarrow (t, w)A \not\twoheadrightarrow J + (b, f(b))A \not\twoheadrightarrow J &\neq A \not\twoheadrightarrow J
\end{align*}
\]

We have $(a, f(a)) = (r, u)(s, v) = (rs, uv)$. So, $a = rs$. Let $\pi$ be the surjection of $A \not\twoheadrightarrow J$ to $A$. Since $(r, u)A \not\twoheadrightarrow J + (b, f(b))A \not\twoheadrightarrow J = A \not\twoheadrightarrow J$, then:

\[
\begin{align*}
rA + bA &= \pi((r, u)A \not\twoheadrightarrow J) + \pi((s, v)A \not\twoheadrightarrow J) \\
&= \pi((r, u)A \not\twoheadrightarrow J) + \pi((s, v)A \not\twoheadrightarrow J) \\
&= \pi((r, u)A \not\twoheadrightarrow J) + \pi((s, v)A \not\twoheadrightarrow J) \\
&= \pi(A \not\twoheadrightarrow J) \\
&= A
\end{align*}
\]

Let $t \in A - U(A)$ such that $t|s$. Using the fact $t|s$ and $s|a (a = rs)$, then $t|a$ and so $t \neq 0$. Therefore, by $a$ of Lemma 2.3, $((t, f(t))(s, v)$, and so by $c$ of Lemma 2.3, $(t, f(t)) \in A \not\twoheadrightarrow J - U(A \not\twoheadrightarrow J)$ (since $t \in A - U(A)$). Consequently, $(t, f(t)A \not\twoheadrightarrow J + (b, f(b))A \not\twoheadrightarrow J \neq A \not\twoheadrightarrow J$. Hence, by $b$ of Lemma 2.3, $tA + bA \neq A$. Thus, $A$ is an adequate ring. \[\square\]

Now, to the second main result of this paper.

Theorem 2.11. Let $A$ be an integral domain, $B$ be a ring, $f : A \rightarrow B$ be a ring homomorphism and $J$ be an ideal of $B$ such that:

(i) $f$ is injective.
(ii) \( J^2 = \{0\} \).

(iii) For every \( t \in A - \{0\} \), \( f(t)J = J \).

Then \( A \bowtie J \) is an adequate ring if and only if the following statements hold:

a) \( A \bowtie J \) is a adequate ring.

b) For every \( a, b \in A - U(A) \), \( aA + bA \neq A \).

Proof. Assume that \( A \) is an integral domain and the statement (1), (2) and (3) hold. Assume that \( A \bowtie J \) is an adequate ring. Then:

a) By Lemma 2.10, \( A \) is an adequate ring.

b) We show that \( aA + bA \neq A \), for every \( a, b \in A - U(A) \). Suppose that there exists \( t, p \in A - U(A) \) such that \( tA + pA = A \). Necessarily \( t \neq 0 \) since \( p \in A - U(A) \). Let \( 0 \neq j \in J \). Clearly, \( (0, j) \) and \( (p, f(p)) \) are elements of \( A \bowtie J \) which is an adequate ring. So, there exists \( (r, u) \) and \( (s, v) \in A \bowtie J \) such that

\[
\begin{align*}
(0, j) & = (r, u)(s, v) \\
(r, u)A \bowtie J + (p, f(p))A \bowtie J = A \bowtie J \\
\forall k \in A \bowtie J - U(A \bowtie J) : k(s, v) \Rightarrow kA \bowtie J + (p, f(p))A \bowtie J \neq A \bowtie J
\end{align*}
\]

Since \( (r, u)A \bowtie J + (p, f(p))A \bowtie J = A \bowtie J \), then by b) of Lemma 2.3, \( rA + pA = A \). It is easy to see that \( r \neq 0 \) since \( p \notin U(A) \). We have \( (0, j) = (r, u)(s, v) \) and so \( rs = 0 \). Therefore, \( s = 0 \) since \( r \neq 0 \) and \( A \) is an integral domain. By a) of Lemma 2.3, we obtain \( (t, f(t))(s, v) \) since \( t|s \). By b) of Lemma 2.3, \( (t, f(t)) \in A \bowtie J - U(A \bowtie J) \) since \( t \in A - U(A) \). Using the fact \( t|s \) (since \( s = 0 \)) and \( t \in A - U(A) \), then \( (t, f(t))A \bowtie J + (p, f(p))A \bowtie J \neq A \bowtie J \). Hence By b) of Lemma 2.3, \( tA + pA \neq A \), a contradiction. Thus, \( \forall a, b \in A - U(A), aA + bA \neq A \).

Conversely, assume that \( A \) is an adequate ring and \( \forall a, b \in A - U(A), aA + bA \neq A \). Let \( (a, x) \in A \bowtie J - \{0\} \), and let \( (b, y) \in A \bowtie J \). Two cases are possible:

Case 1: \( a \neq 0 \). Since \( A \) is an adequate ring and \( a \in A - \{0\} \) and \( b \in A \), then there exists \( r, s \in A \) such that:

\[
\begin{align*}
a & = rs \\
rA + bA & = A \\
\forall t \in A - U(A) : t|s \Rightarrow tA + bA \neq A.
\end{align*}
\]

Since \( rs = a \), then \( r \neq 0 \) and by a) of Lemma 2.3, there exists \( u \in f(A) + J \) such that

\[
\begin{align*}
(a, x) & = (r, f(r))(s, u) \\
(s, u) & \in A \bowtie J
\end{align*}
\]

Using the fact \( rA + bA = A \), then by b) of Lemma 2.3, \( (r, f(r))A \bowtie J + (b, y)A \bowtie J = A \bowtie J \). Let \( (t, v) \in A \bowtie J - U(A \bowtie J) \) such that \( (t, v)|(s, u) \). By b) of Lemma 2.3, \( t \in A - U(A) \) since \( (t, v) \in A \bowtie J - U(A \bowtie J) \). Using the fact \( t \in A - U(A) \) and \( t|s \), then \( tA + bA \neq A \). Hence, by b) of Lemma 2.3, it follows that \( (t, v)A \bowtie J + (b, y)A \bowtie J \neq A \bowtie J \).

Case 2: \( a = 0 \).

\[
(0, x) = (0, x) \neq 0 \text{ and } x \neq 0 \text{. If } b \in U(A), \text{ then by } c) \text{ of Lemma 2.3, } (b, y) \in U(A \bowtie J). \text{ Then :}
\]

\[
\begin{align*}
(a, x) & = (x)(1, 1) \\
(a, x)A \bowtie J + (b, y)A \bowtie J & = (a, x)A \bowtie J + A \bowtie J = A \bowtie J \\
\forall k \in A \bowtie J - U(A \bowtie J) : k(1, 1) \Rightarrow kA \bowtie J + (b, y)A \bowtie J \neq A \bowtie J
\end{align*}
\]

Assume that \( b \notin U(A) \). Then \( (b, y) \notin U(A \bowtie J) \). Therefore,

\[
\begin{align*}
(a, x) & = (1, 1)(a, x) \\
(1, 1)A \bowtie J + (b, y)A \bowtie J & = (a, x)A \bowtie J + A \bowtie J = A \bowtie J \\
\forall (t, v) \in A \bowtie J - U(A \bowtie J) : (t, v)((a, x)) \Rightarrow kA \bowtie J + (b, y)A \bowtie J \neq A \bowtie J
\end{align*}
\]

Since \( (t, v) \in U(A \bowtie J) \), then \( t \in A - U(A) \). Moreover \( t, b \in A - U(A) \). Therefore, \( tA + bA \neq A \).

By b) of Lemma 2.3, we obtain

\[
(t, v)A \bowtie J + (b, y)A \bowtie J \neq A \bowtie J. \text{ Thus, } A \bowtie J \text{ is an adequate ring.} \]
Corollary 2.12. Let $A$ be a principal ideal domain, $B$ be a ring, $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$ such that $J \subseteq \text{Rad}(B)$ and:

(i) $f$ is injective.

(ii) $J^2 = (0)$.

(iii) For every $t \in A - \{0\}$, $f(t)J = J$.

Then $A \not\approx J$ is an adequate ring if and only if $A$ is local.

Proof. Assume that $A$ is a principal ideal domain, $f$ is injective, $J^2 = (0)$ and for all $t \in A - \{0\}$, $f(t)J = J$. If $A \not\approx J$ is an adequate ring, then by Theorem 2.11, $aA + bA \neq A$ for every $a, b \in A - U(A)$, and so $A$ is local by Lemma 2.9.

Conversely, assume that $A$ is local. Hence, $A \not\approx J$ is local since $J \subseteq \text{Rad}(B)$ (since $J^2 = (0)$), and so $A \not\approx J$ is an adequate ring.

Example 2.13. Let $A := \mathbb{Z}$, $B := \mathbb{R}[[X]]/(X^2 + 1)^2\mathbb{R}[[X]]$, $J = (X^2 + 1)^2\mathbb{R}[[X]]/(X^2 + 1)^4\mathbb{R}[[X]]$ be an ideal of $B$ and

$$f : A \to B$$

$$a \to f(a) = \bar{a}$$

be a ring homomorphism. Then $A \not\approx J$ is not an adequate ring.

Proof. $A$ is a principal ideal domain which is not local, it is clear that $f$ is injective and $J \subseteq \text{Rad}(B)$ (since $B := \mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]]$ is local). On the other hand, $J^2 = [(X^2+1)^2\mathbb{R}[[X]]/(X^2+1)^4\mathbb{R}[[X]]] = \{0\}$ and for $t \in A - \{0\}$, $f(t)J = f((X^2 + 1)^2\mathbb{R}[[X]]/(X^2 + 1)^4\mathbb{R}[[X]]) = ((X^2 + 1)^2\mathbb{R}[[X]]/(X^2 + 1)^4\mathbb{R}[[X]]) = (X^2 + 1)^2\mathbb{R}[[X]]/(X^2 + 1)^4\mathbb{R}[[X]] = J$. Hence, by Theorem 2.8, $A \not\approx J$ is not an adequate ring since $A := \mathbb{Z}$ is not local.

Example 2.14. Let $A := \mathbb{Z}_{2\mathbb{Z}}$, $B := \mathbb{R}[[X]]/(X^2 + 1)^4\mathbb{R}[[X]]$, $J = (X^2 + 1)^2\mathbb{R}[[X]]/(X^2 + 1)^4\mathbb{R}[[X]]$ be an ideal of $B$ and

$$f : A \to B$$

$$a \to f(a) = \bar{a}$$

be a ring homomorphism. Then $A \not\approx J$ is an adequate ring (since $A$ is a discrete valuation domain).

References


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