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Exponential Stability of the Wave Equation on a Star Shaped Network with Indefinite Sign Damping

Farah Abdallah, Denis Mercier and Serge Nicaise

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Abstract. In this paper, we study the stability of the wave equation on a star shaped network in the presence of indefinite sign damping terms. Extending the results of Freitas and Zuazua in [13], we search for sufficient conditions on the damping coefficients for which the problem becomes exponentially stable.

1 Introduction

For $N \geq 2$, we consider the following wave equation on a star shaped network:
(S_1)

$$\begin{cases} u^i_{tt}(x,t) - u^i_{xx}(x,t) + 2a_i(x)u^i_t(x,t) &= 0, \qquad x \in (0,L_i),\ t>0, i \in \{1,\cdots,N\},\\ u^i(L_i,t) &= 0,\\ u^i(0,t) &= u^j(0,t), \quad \forall i \neq j,\\ \displaystyle\sum_{i=1}^{N} u^i_x(0,t) &= 0,\\ u^i(x,0) &= u^i_0(x), \qquad x \in (0,L_i),\\ u^i_t(x,0) &= u^i_1(x), \qquad x \in (0,L_i). \end{cases}$$

where $L_i \in \mathbb{R}^+_*$, and $a_i \in W^{1,\infty}(0,L_i)$. This system models the vibrations of a group of strings attached at one extremity. The Kirchoff law $\sum_{i=1}^{N} u^i_x(0,t) = 0$ follows from the principle of stationnary action [1, 29].

The main goal of this work is to study the stability of system (S_1) but also to give more precise results when we replace in the system the damping coefficients $a_i(x)$ by $\epsilon a_i(x)$, where the parameter ϵ is positive and small enough. In this case we will denote this modified system by (S_ϵ) and we only need that $a_i \in L^\infty(0,L_i)$ for all $i \in \{1,\cdots,N\}$.

Using observability inequalities, the stability of the wave equations over a network with positive damping coefficients has been studied in [30]. In the case of one interval, the stability of a wave equation with an indefinite sign damping coefficient has been studied in [2, 12, 13, 22, 23, 28], where it was found that the stability of the wave equation is related to the mean of the damping coefficient. In this paper, using spectral analysis, we find (sufficient) conditions on the damping coefficients to get the exponential stability of (S_1) and (S_ϵ). In fact, we find necessary and sufficient conditions for which (S_1) is exponentially stable up to a finite dimensional space. The idea is inspired from [25] where the characteristic equation of (S_1) is approximated by another one using the shooting method. This approximation allows us to determine the behavior of the high frequencies and hence to deduce the conditions on the damping coefficients $\{a_i\}_{i=1}^N$

for which the high frequencies are situated to the left of the imaginary axis. In a second step, we prove that the generalized root vectors form a Riesz basis with parentheses and then deduce the exponential stability of (S_1) up to a finite dimensional space generated by the roots vectors corresponding to the low frequencies. Note that the shooting method in [10] based on the ansatz of Horn in [18] and used to analyze the high frequencies cannot be easily adapted to our problem as long as the solution in [10] is written in a power series form with unknown coefficients. On the other hand, when $\{a_i\}_{i=1}^N$ is replaced by $\{\epsilon a_i\}_{i=1}^N$ with the parameter ϵ small enough, we search for sufficient conditions for which (S_ϵ) is exponentially stable in the whole energy space. In this case, we note that the positivity of the mean of the damping coefficients in addition to another condition are required (see Theorem 1.4 below). In fact, for $\epsilon > 0$ small enough, unlike [10], we deal with multiple eigenvalues. Note that the study of the exponential stability of (S_ϵ) enters in the framework of the abstract theory done in [22]. Using the concepts introduced in [19] about the behavior of the spectrum, we shall interpret the hypothesis imposed in [22] to find explicit conditions on the damping coefficients for which (S_ϵ) is exponentially stable.

Throughout this paper, we make the following hypothesis on the geometry of the domain:

(H) There exists $q \in \mathbb{N}^*$ such that for all $i = 2, \cdots, N$, there exists $p_i \in \mathbb{N}^*$ for which $L_i = \dfrac{p_i}{q} L_1$.

In applications, the above hypothesis is more realistic. From a mathematical point of view, this above hypothesis is considered since otherwise when some of the lengths take irrational valued, then we can find examples for which numerically we see that the spectrum is not structured (for instance there is no asymptotes) and an infinite number of eigenvalues are situated to the right of the imaginary axis (see Figure 1). Moreover, hypothesis (H) allows us to find an equivalent and algebraic form of the approximated characteristic equation (see Lemma 3.7).

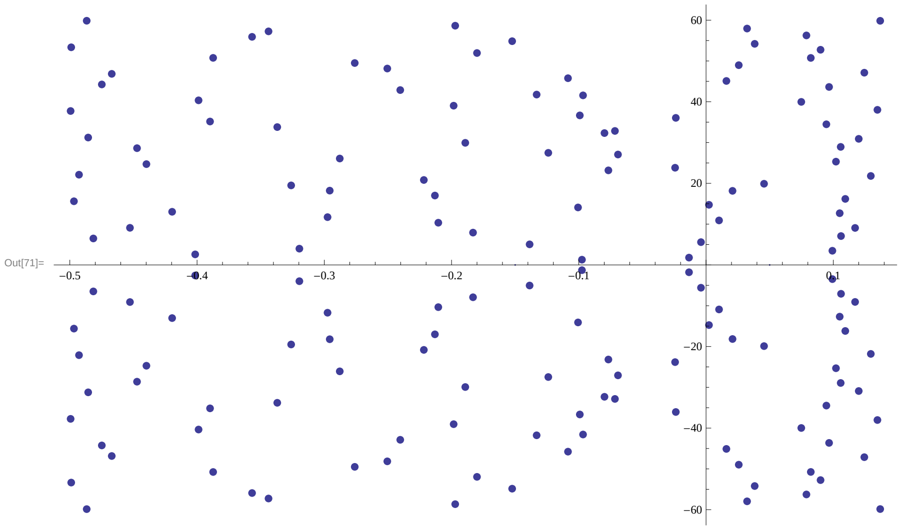

Figure 1. $a_1 = a_2 = \dfrac{1}{2},\ a_3 = -\dfrac{1}{3},\ L_i = \sqrt{i},\ i = 1, 2, 3.$

This paper is divided into three main parts. In the first part, we prove the following theorem:

Theorem 1.1. *Under the hypothesis (H), system (S_1) is exponentially stable up to a finite dimen-*

sional space if and only if the roots of the polynomial G defined by

$$(1.1)\quad G(z)=\sum_{i=1}^{N}\left(e^{\int_0^{L_i} a_i(x)dx}z^{p_i}+e^{-\int_0^{L_i} a_i(x)dx}\right)\prod_{k\neq i,k=1}^{N}\left(e^{\int_0^{L_k} a_k(x)dx}z^{p_k}-e^{-\int_0^{L_k} a_k(x)dx}\right)$$

are inside the unitary open disk.

If $N=2$, then according to Theorem 1.1, system (S_1) is exponentially stable up to a finite dimensional space if and only if

$$\int_0^{L_1} a_1(x)dx+\int_0^{L_2} a_2(x)dx>0.$$

Clearly this condition depends only on $\int_0^{L_i} a_i(x)dx$, hence for $N\geq 3$, we may state the following conjecture:

Conjecture 1.2. Although the degree of the polynomial G depends on the lengths L_i, and the coefficients are functions of the parameters a_i and L_i for all $i=1,\cdots,N$, the fact that the roots of G are inside the open unitary disk depends only on the values of $\displaystyle\int_0^{L_i} a_i(x)dx$ for all $i=1,\cdots,N$ (see the examples of Section 7).

Remark 1.3. If $N=2$, $a_1=1$, $a_2=\alpha\in\mathbb{R}$, and $L_1=L_2=1$, then we recover the result of Theorem 1.1 of [2] which states that (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha>-1$. Indeed, in this case, $G(z)=2e^{1+\alpha}z^2-2e^{-1-\alpha}$ and hence $G(z)=0$ yields $|z|=e^{-(1+\alpha)}$. Therefore, by Theorem 1.1 above, (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha>-1$.

In the second part, we consider system (S_ϵ) with $\epsilon>0$ and prove the following theorem:

Theorem 1.4. *Under the hypothesis (H), when $a_i(x)=a_i\in\mathbb{R}$ and $L_i=1$ for all $i=1,\cdots,N$, there exists $\epsilon_0>0$ such that, for all $\epsilon\in(0,\epsilon_0)$, (S_ϵ) is exponentially stable if one of the following two conditions holds:*

(i) There exists at most one $j_0\in\{1,\cdots,N\}$ such that $a_{j_0}=0$ and $a_i>0$ for all $i\neq j_0$.

(ii) There exists only one negative damping coefficient a_{i_0} such that $a_i>0$ for all $i\neq i_0$, $\displaystyle\sum_{i=1}^{N}a_i>0$, and $\displaystyle\sum_{i=1}^{N}\frac{1}{a_i}<0$.

Remark 1.5. If $N=2$, then we recover the result of Theorem 2.1 of [13] when the damping coefficient is piecewise constant. However, in this case, Theorem 1.4 yields the result of [13] without the assumption on the integrals I_k defined in [13].

Finally, in the third part, we look at some concrete examples of networks and specific values of a_i.

In the whole paper, we shall use the notation $A\lesssim B$ (resp. $A=O(B)$) for the existence of a positive constant $c>0$ independent of A and B such that $A\leq cB$ (resp. $|A|\leq c|B|$) and for shortness we will write $\|.\|_\infty$ for $\|.\|_{L^\infty(0,L_i)}$.

2 Formulation of the problem

We start by determining the suitable functional setting of system (S_1). If u is a regular solution of (S_1), then the energy of (S_1) is formally given by

$$E(t) = \frac{1}{2}\sum_{i=1}^{N}\int_0^{L_i}(|u_t^i|^2 + |u_x^i|^2)dx,$$

and

$$\frac{d}{dt}E(t) = -\sum_{i=1}^{N}\int_0^{L_i} a_i(x)|u_t^i|^2dx.$$

Since the sign of the a_i are not specified, the decay of the energy is not guaranteed.

As an energy space, let $\mathcal{H} = V \times H$ where $H = \prod_{i=1}^{N} L^2(0, L_i)$ and

$$V = \left\{ u = (u^1, \cdots, u^N)^\top \in \prod_{i=1}^{N} H^1(0, L_i);\ u^i(0) = u^j(0)\ \forall i \neq j, \text{ and } u^i(L_i) = 0,\ \forall i = 1, \cdots, N \right\}.$$

The Hilbert space $\mathcal{H}$ is endowed with the inner product

$$< (u, v)^\top, (f, g)^\top >= \sum_{i=1}^{N}\int_0^{L_i}(u_x^i\overline{f_x^i} + v^i\overline{g^i})dx, \forall (u, v)^\top, (f, g)^\top \in \mathcal{H}.$$

Define the operator $A : D(A) \to \mathcal{H}$ by

$$D(A) = \left\{ (u, v)^\top \in V \times V;\ u \in \prod_{i=1}^{N} H^2(0, L_i) \text{ and } \sum_{i=1}^{N} u_x^i(0) = 0 \right\},$$

and for all $(u, v)^\top \in D(A)$

$$A(u, v)^\top = \begin{pmatrix} 0 & A_1^0 \\ A_1^2 & A_{-2a}^0 \end{pmatrix}(u, v)^\top$$

with $A_\alpha^k w = (\alpha_i \partial_x^k w^i)_{i=1}^N$ for $\alpha = (\alpha_i)_{i=1}^N \in \prod_{i=1}^N L^\infty(0, L_i)$ and $w = (w^i)_{i=1}^N \in \prod_{i=1}^N H^k(0, L_i)$, for $k = 0$ or 2.

If u is a sufficiently smooth solution of (S_1), then $U = (u, u_t)^\top \in \mathcal{H}$ satisfies the first order evolution equation

$$\text{(2.1)} \qquad \begin{cases} U_t &= AU, \\ U(0) &= (u_0, u_1)^\top. \end{cases}$$

Using standard semigroup theory, we get the following theorem on the existence, uniqueness, and regularity of the solution of (S_1).

Theorem 2.1. *The operator A generates a C_0 semigroup on $\mathcal{H}$ and hence problem (2.1) admits a unique solution which implies that (S_1) is well-posed. Moreover, if $U(0) \in \mathcal{H}$, then $U \in C^0([0, +\infty); \mathcal{H})$ and if $U(0) \in D(A)$, then $U \in C^1([0, +\infty); \mathcal{H}) \cap C^0([0, +\infty); D(A))$.*

Proof. The well-posedness of (2.1) follows from the fact that the operator A is a bounded perturbation of a skew adjoint operator (see Theorem III.1.1 of [26]), hence it generates a strongly continuous semigroup on $\mathcal{H}$. The regularity results are then a direct consequence of Theorem I.2.4 of [26]. □

Remark 2.2. Since $D(A)$ is compactly embedded in the energy space $\mathcal{H}$, the spectrum $\sigma(A)$ is discrete and the eigenvalues of A have a finite algebraic multiplicity.

3 High frequencies

In this section, we shall determine the asymptotic behavior of the eigenvalues of the operator A. For this aim, we will adapt the shooting method to our system.

Let λ be an eigenvalue of A and $U = (y, z)$ be an associated eigenfunction. Then, $z = \lambda y$ and, for all $i = 1, \cdots, N$, we have

$$
\left\{
\begin{array}{rcll}
y^i_{xx} - 2a_i(x)\lambda y^i - \lambda^2 y^i &=& 0, & x \in (0, L_i), \\
y^i(L_i) &=& 0, & \\
y^i(0) &=& y^j(0), & \forall i \neq j, \\
\displaystyle\sum_{i=1}^{N} y^i_x(0) &=& 0. &
\end{array}
\right. \tag{3.1}
$$

It is easy to see that $\lambda = 0$ is not an eigenvalue of A.

Remark 3.1. We have

$$
|\Re\lambda| \leq 2 \max_{i\in\{1,\cdots,N\}} \{\|a_i\|_\infty\}. \tag{3.2}
$$

Indeed, if we multiply the first identity of (3.1) by y^i and then integrate by parts, we get

$$
\lambda^2 \sum_{i=1}^{N} \int_0^{L_i} |y^i|^2 dx + 2\lambda \sum_{i=1}^{N} \int_0^{L_i} a_i(x)|y^i|^2 dx + \sum_{i=1}^{N} \int_0^{L_i} |y^i_x|^2 dx = 0.
$$

Hence, we have

$$
\lambda = \frac{\displaystyle\sum_{i=1}^{N} \int_0^{L_i} a_i(x)|y^i|^2 dx \pm r(y)^{\frac{1}{2}}}{\displaystyle\sum_{i=1}^{N} \int_0^{L_i} |y^i|^2 dx},
$$

with

$$
r(y) := \left(\sum_{i=1}^{N} \int_0^{L_i} a_i(x)|y^i|^2 dx \right)^2 - \left(\sum_{i=1}^{N} \int_0^{L_i} |y^i_x|^2 dx \right) \left(\sum_{i=1}^{N} \int_0^{L_i} |y^i|^2 dx \right)
$$

and deduce the estimate (3.2) by distinguishing the case $r(y) \geq 0$ or not.

Now, we start by searching for the characteristic equation using the shooting method. In order to adapt the shooting method to problem (3.1), we first consider the the following separated initial value problems: for all $i = 1, \cdots, N$, let y^i_1 and y^i_2 be the solution of

$$
\left\{
\begin{array}{rcl}
y^i_{1xx} - 2a_i(x)\lambda y^i_1 - \lambda^2 y^i_1 &=& 0, \\
y^i_1(0) &=& \dfrac{1}{\lambda}, \\
y^i_{1x}(0) &=& 0.
\end{array}
\right. \tag{3.3}
$$

$$
\left\{
\begin{array}{rcl}
y^i_{2xx} - 2a_i(x)\lambda y^i_2 - \lambda^2 y^i_2 &=& 0, \\
y^i_2(0) &=& 0, \\
y^i_{2x}(0) &=& 1.
\end{array}
\right. \tag{3.4}
$$

The initial conditions are chosen such that the solutions y_1^i and y_2^i are linearly independent. Hence, y^i, the solution of (3.1), can be written as $y^i = c_i y_1^i + \alpha_i y_2^i$, where $\alpha_i, c_i \in \mathbb{C}$. By the continuity condition at zero, we get $c_i = c$ for all $i = 1, \cdots, N$, hence

$$y^i(x) = c y_1^i(x) + \alpha_i y_2^i(x). \tag{3.5}$$

Moreover, from the transmission condition, $\sum_{i=1}^{N} y_x^i(0) = 0$, we have $\sum_{i=1}^{N} \alpha_i = 0$ and from the boundary condition, $y^i(L_i) = 0$, we get

$$\begin{pmatrix} y_1^1(L_1) & y_2^1(L_1) & 0 & 0 & \cdots & 0 \\ y_1^2(L_2) & 0 & y_2^2(L_2) & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & 0 & \cdots & 0 & \ddots & 0 \\ y_1^N(L_N) & 0 & \cdots & & 0 & y_2^N(L_N) \\ 0 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} c \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_N \end{pmatrix} = 0.$$

Hence a non-zero eigenvector exists if and only if the determinant of the above matrix vanishes, or after some elementary calculations if and only if

$$Y(\lambda) = \sum_{k=1}^{N} y_1^k(L_k) \prod_{l \neq k, l=1}^{N} y_2^l(L_l) = 0. \tag{3.6}$$

Recall that G is defined by (1.1) and set $d := degree\ G$, the degree of G. Then let $r_j e^{i\varphi_j}, 1 \leq j \leq d$ be the roots of G repeated according to their multiplicity. Without loss of generality we can suppose that the φ_j are non decreasing, namely

$$0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_d \leq 2\pi.$$

Now we can state the following main result:

Theorem 3.2. *There exists $k_0 \in \mathbb{N}$ such that for all $j = 1, \cdots, d$ and all $k \in \mathbb{Z}$ such that $|k| > k_0$, the operator A has an eigenvalue $\lambda_{j,k}$ such that*

$$\lambda_{j,k} = \frac{q}{2L_1} \log r_j + \imath \frac{q}{2L_1} \varphi_j + \imath k\pi \frac{q}{L_1} + o_k(1), \tag{3.7}$$

where $o_k(1) \to 0$ as $|k| \to \infty$. Moreover the set $\sigma(A) \setminus \cup_{|k|>k_0} \cup_{j=1}^{d} \lambda_{j,k}$ is compact. Therefore, if $r_j < 1$, for all $j = 1, \cdots, d$, then the large eigenvalues of A are situated to the left of the imaginary axis.

Corollary 3.3. *There exists $\ell \in \mathbb{N}$ and $\alpha_0 > 0$ such that for all $k \in \mathbb{N}$ with $k > k_0$, we have*

$$\Im(\lambda_{1,k+\ell} - \lambda_{d,k}) \geq \alpha_0,$$
$$\Im(\lambda_{1,-k} - \lambda_{d,-k-\ell}) \geq \alpha_0.$$

This corollary shows that we can group the eigenvalues of A by packets made of a finite number of eigenvalues and in such a way that the packets remain at a positive distance to each other (see section 4 below). Namely for any $r > 0$, we can introduce the sets $G_p(r), p \in \mathbb{Z}$ as the connected components of the set $\cup_{\lambda \in \sigma(A)} D_\lambda(r)$ (where $D_\lambda(r)$ is the disc with center λ and radius r), as well as the packets of eigenvalues $\Lambda_p(r) = G_p(r) \cap \sigma(A)$.

Before we prove Theorem 3.2, we search for an approximation of the characteristic equation (3.6) for all λ large enough. For this aim, the next lemma gives an estimation of y_1^i and y_2^i for all $i = 1, \cdots, N$.

Lemma 3.4. *For $i = 1, \cdots, N$ and $\lambda \in \sigma(A)$ large enough, we have*

$$\|y_1^i\|_\infty \lesssim \frac{1}{|\lambda|}, \text{ and } \|y_2^i\|_\infty \lesssim \frac{1}{|\lambda|}.$$

Proof. First, for $i = 1, \cdots, N$ and $\lambda \in \sigma(A)$, we consider the homogenous equation

$$\begin{cases} z_{1xx}^i(x) - \lambda^2 z_1^i(x) &= 0, \quad x \in (0, L_i), \\ z_1^i(0) &= \dfrac{1}{\lambda}, \\ z_{1x}^i(0) &= 0, \end{cases}$$

which yields $z_1^i(x) = \frac{1}{\lambda}\cosh(\lambda x)$. Hence, for large enough λ, Remark 3.1 yields $\|z_1^i\|_\infty \lesssim \frac{1}{|\lambda|}$. Now, by the variation of constants formula, we find that

$$y_1^i(x) = z_1^i(x) + 2\int_0^x \sinh(\lambda(x-s))\, a_i(s) y_1^i(s) ds, \forall x \in (0, L_i).$$

Therefore, by the integral form of Gronwall's Lemma, we get

$$|y_1^i(x)| \le |z_1^i(x)| + 2\int_0^x \left[|z_1^i(s)| |\sinh(\lambda(x-s))\, a_i(s)| \exp\left(2\int_s^x |\sinh(\lambda(x-r)) a_i(r)| dr\right)\right] ds.$$

The above inequality and Remark 3.1 imply that, for λ large enough, $\|y_1^i\|_\infty \lesssim \frac{1}{|\lambda|}$.

A similar estimate for y_2^i is obtained by introducing $z_2^i = \frac{1}{\lambda}\sinh(\lambda x)$, the solution of

$$\begin{cases} z_{2xx}^i(x) - \lambda^2 z_2^i(x) &= 0, \quad x \in (0, L_i), \quad i = 1, 2, \\ z_2^i(0) &= 0, \\ z_{2x}^i &= 1, \end{cases}$$

and using that $\|z_2^i\|_\infty \lesssim \frac{1}{|\lambda|}$ for λ large enough. □

Next, we find suitable approximations for y_1^i and y_2^i for $i = 1, \cdots, N$. For this aim we define over $(0, L_i)$, the function

$$\theta^i(x) = \lambda x + \int_0^x a_i(s) ds, \forall x \in (0, L_i),$$

and the functions v_1^i and v_2^i as linear combination of $\sinh\theta^i(x)$ and $\cosh\theta^i(x)$ such that v_1^i satisfies the initial conditions in (3.3) and v_2^i satisfies those in (3.4). Note that, for $|\lambda| > M$ with $M > \max_i \|a_i(.)\|_\infty$, we have

$$v_1^i(x) = \frac{1}{\lambda}\cosh\theta^i(x), \text{ and } v_2^i(x) = \frac{1}{\lambda + a_i(0)}\sinh\theta^i(x), \forall x \in (0, L_i).$$

Note that the functions v_1^i and v_2^i depend on λ.

Lemma 3.5. *For all $i = 1, \cdots, N$ and $\lambda \in \sigma(A)$ large enough, we have*

$$\left\|v_1^i - y_1^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2} \text{ and } \left\|v_2^i - y_2^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2}.$$

Proof. For $i = 1, \cdots, N$ and $\varphi^i \in H^2(0, L_i)$, define the function $L^i(\varphi^i) = \varphi^i_{xx} - 2a_i\lambda\varphi^i - \lambda^2\varphi^i$. Then, for all $x \in (0, L_i)$, we have

$$L^i\left(v_1^i(x)\right) = \frac{a_{ix}(x)}{\lambda}\sinh\theta^i(x) + \frac{(a_i(x))^2}{\lambda}\cosh\theta^i(x),$$

and

$$L^i\left(v_2^i(x)\right) = \frac{a_{ix}(x)}{\lambda + a_i(0)}\sinh\theta^i(x) + \frac{(a_i(x))^2}{\lambda + a_i(0)}\cosh\theta^i(x).$$

Therefore, by Remark 3.1, we get that for λ large enough

$$\left\|L^i(v_1^i)\right\|_\infty \lesssim \frac{1}{|\lambda|}, \text{ and } \left\|L^i(v_2^i)\right\|_\infty \lesssim \frac{1}{|\lambda|}.$$

Since we have

$$\begin{aligned} v^i_{1xx} - 2a_i\lambda v_1^i - \lambda^2 v_1^i &= L^i(v_1^i), \\ v^i_{1x}(0) &= 0, \\ v_1^i(0) &= \frac{1}{\lambda}, \end{aligned}$$

by the variation of constants formula, we get for all $x \in (0, L_i)$

$$v_1^i(x) = y_1^i(x) + \int_0^x y_2^i(x-s)L^i(v_1^i(s))ds.$$

Therefore, by Lemma 3.4, we have

$$\left\|v_1^i - y_1^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2}.$$

Similarly, for all $x \in (0, L_i)$, we have

$$v_2^i(x) = y_2^i(x) + \int_0^x y_2^i(x-s)L^i(v_2^i(s))ds,$$

which implies that

$$\left\|v_2^i - y_2^i\right\|_\infty \lesssim \frac{1}{|\lambda|^2},$$

□

Now, we can find an approximation of the characteristic equation (3.6) from which we deduce the behavior of the high frequencies. For this aim, we introduce

$$V(\lambda) = \sum_{k=1}^N v_1^k(L_k) \prod_{l\neq k, l=1}^N v_2^l(L_l)$$

and

$$F(\lambda) = \lambda^{-N}\sum_{k=1}^N \cosh\widetilde{\theta}^k(\lambda)\prod_{l\neq k}^N \sinh\widetilde{\theta}^l(\lambda), \tag{3.8}$$

where, for $z \in \mathbb{C}$, $\widetilde{\theta}^l(z) = zL_l + \int_0^{L_l} a_l(s)ds$, for all $l = 1, \cdots, N$.

Proposition 3.6. *For $\lambda \in \sigma(A)$ large enough, we have the following estimate*

$$|Y(\lambda) - F(\lambda)| \lesssim \frac{1}{|\lambda|^{N+1}}. \tag{3.9}$$

Proof. Let λ be a large eigenvalue of A. The estimates in Lemmas 3.4 and 3.5 imply that

$$\begin{aligned} &|Y(\lambda) - V(\lambda)| \\ = &\left| \sum_{k=1}^{N} \left(y_1^k(L_k) \prod_{l\neq k, l=1}^{N} y_2^l(L_l) - v_1^k(L_k) \prod_{l\neq k, l=1}^{N} v_2^l(L_l) \right) \right| \\ = &\left| \sum_{k=1}^{N} \left(y_1^k(L_k) - v_1^k(L_k) \right) \prod_{l\neq k, l=1}^{N} y_2^l(L_l) + \sum_{k=1}^{N} v_1^k(L_k) \left(\prod_{l\neq k, l=1}^{N} y_2^l(L_l) - \prod_{l\neq k, l=1}^{N} v_2^l(L_l) \right) \right| \\ \lesssim &\frac{1}{|\lambda|^{N+1}}. \end{aligned} \tag{3.10}$$

On the other hand, we readily check that

$$\left| V(\lambda) - \frac{1}{\lambda^N} \sum_{k=1}^{N} \cosh \theta^k(L_k) \prod_{l\neq k}^{N} \sinh \theta^l(L_l) \right| \lesssim \frac{1}{|\lambda|^{N+1}}. \tag{3.11}$$

Hence, by (3.10) and (3.11), we get (3.9) since $\theta^k(L_k) = \widetilde{\theta}^k(\lambda)$. □

Estimation (3.9) suggests to apply Rouché's Theorem. Therefore, we are first interested in the roots of F that will be expressed in terms of the roots of the polynomial G given in (1.1).

Lemma 3.7. *$v \in \mathbb{C}$ is a root of F if and only if $z = e^{\frac{2L_1}{q} v}$ is a root of the polynomial G defined in (1.1). Consequently, if $v = x + \imath y$ is a root of F and $r_j e^{i\varphi_j}$ is a root of G for $1 \leq j \leq d$, then $x = \dfrac{q}{2L_1} \log r_j$ and $y = \dfrac{q}{2L_1} \varphi_j + k\pi \dfrac{q}{L_1}$ for some $k \in \mathbb{Z}$.*

Proof. The proof of Lemma 3.7 is based on writing F in an exponential form and noting that

$$\begin{aligned} &2^N v^N e^{\frac{v L_1 \sum_{i=1}^{N} p_i}{q}} F(v) \\ = &\sum_{i=1}^{N} \left(e^{\int_0^{L_i} a_i(x)dx} z^{p_i} + e^{-\int_0^{L_i} a_i(x)dx} \right) \prod_{k\neq i, k=1}^{N} \left(e^{\int_0^{L_k} a_k(x)dx} z^{p_k} - e^{-\int_0^{L_k} a_k(x)dx} \right). \end{aligned}$$

□

Remark 3.8. In the applications, the degree of the polynomial G is high, hence we use the algorithm given by the transformation of Schur (see [14]) that gives a criterion that guarantees that the roots of a given polynomial can be outside the closed unitary disk. Therefore, in applications, we use $G\left(\dfrac{1}{z}\right)$ instead of $G(z)$.

Before giving the proof of Theorem 3.2, we show that Y has the same number of roots as F in a well chosen domain. Knowing that $\alpha < \Re\lambda < \beta$ where λ is an eigenvalue of A, we consider the rectangle $R_{j,k}$ with vertices $\alpha + \imath \dfrac{q}{2L_1}\varphi_j + \left(k - \dfrac{1}{2}\right)\pi \dfrac{q}{L_1}$, $\alpha + \imath \dfrac{q}{2L_1}\varphi_j + \imath\left(k + \dfrac{1}{2}\right)\pi \dfrac{q}{L_1}$, $\beta + \imath \dfrac{q}{2L_1}\varphi_j + \imath\left(k - \dfrac{1}{2}\right)\pi \dfrac{q}{L_1}$, and $\beta + \imath \dfrac{q}{2L_1}\varphi_j + \imath\left(k + \dfrac{1}{2}\right)\pi \dfrac{q}{L_1}$ where we recall that $r_j e^{i\varphi_j}$, $j = 1, \cdots, d$, are the roots of G.

Proposition 3.9. *There exists $k_0 > 0$ such that for all $|k| \geq k_0$ and $z \in \partial R_{j,k}$,*

$$|Y(z) - F(z)| < |F(z)|. \tag{3.12}$$

Proof. Let $z \in \partial R_{j,k}$ and $|k| \geq k_0$ for some $k_0 > 0$ large enough. Similar to (3.9), we can show that there exists $C > 0$ such that

$$|Y(z) - F(z)| \leq \frac{C}{|z|^{N+1}}.$$

Therefore, in order to complete the proof, it is enough to show that for $z \in \partial R_{j,k}$

$$\frac{C}{|z|} < |F_0(z)|,$$

where

$$F_0(z) = \sum_{k=1}^{N} \cosh \widetilde{\theta}^k(z) \prod_{l \neq k}^{N} \sinh \widetilde{\theta}^l(z). \tag{3.13}$$

We remark that $|F_0|$ is $i\pi \dfrac{q}{L_1}$ periodic, hence, $\min\limits_{z \in \partial R_{j,k}} |F_0(z)| = m_j$ is independent of k. Therefore, for $k_0 \geq 1$, there exists $\widetilde{C} > 0$ such that for $|k| \geq k_0$ and $z \in \partial R_{j,k}$, we have

$$\frac{C}{|z|} \leq \frac{\widetilde{C}}{|k|}.$$

Chosing k_0 large enough, we deduce that

$$\frac{C}{|z|} \leq \frac{\widetilde{C}}{|k|} < m_j$$

and the proof follows. □

Proof. **of Theorem 3.2** We shall prove that the large eigenvalues of A are asymptotically close to the roots of F.

First Lemma 3.7 yields that all the roots of F are given by

$$z_{j,k} = \frac{q}{2L_1} \log r_j + \imath \frac{q}{2L_1} \varphi_j + \imath k \pi \frac{q}{L_1}.$$

for all $1 \leq j \leq d$, $k \in \mathbb{Z}$.

Let $0 < \rho < \min\limits_{j} \left| \dfrac{q}{2L_1} \varphi_j + \pi \dfrac{q}{L_1} \right|$ so that $B(z_{j,k}, \rho)$ contains only one root of F. From Proposition 3.6, in order to prove that $|Y(z) - F(z)| < |F(z)|$ for $z \in \partial B(z_{j,k}, \rho)$, it is enough to show that $\dfrac{C}{|z|} < |F_0(z)|$ where F_0 was defined by (3.13).

Let $h_{j,k}(\rho) = \min\limits_{z \in \partial B(z_{j,k}, \rho)} |F_0(z)|$. Since $|F_0|$ is $i\pi \dfrac{q}{L_1}$ periodic, then $h_{j,k}(\rho)$ is independent of k; i.e., $h_{j,k}(\rho) = h_{j,0}(\rho) = h_j(\rho)$. We denote by $h(\rho) = \min\limits_{1 \leq j \leq d} h_j(\rho)$. It is clear that $h(\rho) > 0$ and $h(\rho) \to 0$ as $\rho \to 0$. Therefore, there exists $k_0 > 0$ such that for $|k| > k_0$, $\dfrac{C}{|z|} \leq \dfrac{\widetilde{C}}{|k|} < h(\rho)$. Consequently, we define ρ_k by

$$\rho_k = \min_{\rho} \left\{ \frac{\widetilde{C}}{|k|} < h(\rho) \right\}. \tag{3.14}$$

We notice that $\rho_k \to 0$ as $|k| \to +\infty$. Therefore, for every $|k| > k_0$ and $z \in \partial B(z_{j,k}, \rho_k)$, we have $\frac{C}{|z|} \leq \frac{\widetilde{C}}{|k|} < h(\rho_k) \leq |F_0(z)|$.

By Rouché's Theorem, we conclude that Y and F have the same roots inside $B(z_{j,k}, \rho_k)$. As Proposition 3.9 and again the application of Rouché's theorem imply that Y and F has the same number of roots in $R_{j,k}$, for all $j = 1, \cdots, d$ and $|k| \geq k_0$, we deduce that all eigenvalues of A in $R_{j,k}$ are inside $B(z_{j,k}, \rho_k)$. This completes the proof since $\cup_{|k|\geq k_0} R_{j,k}$ covers the possible set of large eigenvalues of A. □

Remark 3.10. Using Taylor expansion in ρ, we find that $h(\rho) = O(\rho^{n_0})$ for some $n_0 \in \mathbb{N}^*$. Therefore, according to the definition of ρ_k in (3.14), we conclude that $\rho_k = \frac{\widetilde{\widetilde{C}}}{|k|^{\frac{1}{n_0}}}$ for some $\widetilde{\widetilde{C}} > 0$. Consequently, there exists some $k_0 \in \mathbb{N}^*$ large enough such that

$$\lambda_{j,k} = \frac{q}{2L_1} \log r_j + \imath \frac{q}{2L_1} \varphi_j + \imath k\pi \frac{q}{L_1} + O\left(\frac{1}{|k|^{\frac{1}{n_0}}}\right), \forall |k| > k_0.$$

4 Riesz basis with parentheses of $\mathcal{H}$ and sine-type functions

In this section, we first prove that the root vectors of A form a Riesz basis with parentheses of $\mathcal{H}$. According to [33], we recall the definition of a Riesz basis of subspaces and a Riesz basis with parenthesis.

Definition 4.1.
- A family of subspaces $\{W_k\}_{k\in\mathbb{N}}$ is called a Riesz basis of subspaces of $\mathcal{H}$ if
  (i) for every $f \in \mathcal{H}$, and every $k \in \mathbb{N}$, there is a unique $f_k \in W_k$ such that $f = \sum_{k\in\mathbb{N}} f_k$, and
  (ii) there are positive constants C_1 and C_2 such that
  $$C_1 \sum_{k\in\mathbb{N}} \|f_k\|^2 \leq \left\| \sum_{k\in\mathbb{N}} f_k \right\|^2 \leq C_2 \sum_{k\in\mathbb{N}} \|f_k\|^2.$$
- A sequence $\{y_i\}_{i\in\mathbb{N}}$ is called a Riesz basis with parenthesis of $\mathcal{H}$ if there a family $\{W_k\}_{k\in\mathbb{N}}$ of finite-dimensional spaces spanned by some y_i with $W_k \cap W_j = \{0\}$ for $k \neq j$ that forms a Riesz basis of subspaces of $\mathcal{H}$. The spaces W_k are called the parentheses.

Now, we state a Theorem which can be proved exactly as Theorem 2 in [31] and which gives sufficient conditions for which the generalized eigenfunctions of a bounded perturbation of a selfadjoint operator form a Riesz basis with parenthesis.

Theorem 4.2. *Let T be a selfadjoint operator over a Hilbert space H with discrete spectrum $\{\mu_k\}_{k\in\mathbb{Z}}$ which satisfies the generalized gap condition, i.e, there exists $k_0 > 0$ and $c > 0$ such that*

$$\mu_{k+k_0} - \mu_k > c, \forall k \in \mathbb{Z}.$$

Let B be a bounded operator from H into itself. Then the root vectors of the perturbation $T + B$ form a Riesz basis with parenthesis of H. In this case, only terms corresponding to merging eigenvalues should be put in parenthesis, i.e, there exist $r > 0$ and $N \in \mathbb{N}^*$ such that if we set

$$\Lambda_p = \Lambda_p(r),$$

then

$$\#\Lambda_p \leq N, \forall p \in \mathbb{Z},$$

$$\sigma(T+B) = \cup_{p\in\mathbb{Z}}\Lambda_p,$$

and we can take as parenthesis W_p, $p \in \mathbb{Z}$, the space spanned by the root vectors of $T+B$ corresponding to the eigenvalues in Λ_p where for any $f \in H$, $f_p = \mathbb{P}_p f$ is the the Riesz projection of A, i.e.,

$$f_p = \mathbb{P}_p f = \frac{1}{2i\pi}\int_{\gamma_p} (\lambda - T - B)^{-1} f \, d\lambda,$$

where γ_p is a contour surrounding Λ_p.

A direct consequence of this result concerns our operator A:

Proposition 4.3. *The family of root vectors of A forms a Riesz basis with parentheses of $\mathcal{H}$, which means that the statements of the previous theorem are valid for $\imath A$.*

Proof. It suffices to apply Theorem 4.2 with the choice

$$T = i\begin{pmatrix} 0 & A_1^0 \\ A_1^2 & A_0^0 \end{pmatrix} \quad \text{and} \quad B = i\begin{pmatrix} 0 & 0 \\ 0 & A_{-2a}^0 \end{pmatrix}.$$

that clearly satisfies the assumptions of the previous Theorem. □

Another consequence of the previous results is that the packet Λ_p can be splitted up into subpackets, namely there exists $N_p \in \mathbb{N}^*$ with $N_p \leq N$ such that

$$\Lambda_p = \cup_{j=1}^{N_p}\{\lambda_{p,j}\},$$

where each $\lambda_{p,j} \in \sigma(A)$ are different and of multiplicity $m_{p,j}$ (uniformly bounded in p) and therefore

$$\mathbb{P}_p f = \sum_{j=1}^{N_p} \mathbb{P}_{p,j} f,$$

with

$$\mathbb{P}_{p,j} f = \frac{1}{2i\pi}\int_{\gamma_{p,j}} (\lambda - T - B)^{-1} f \, d\lambda,$$

$\gamma_{p,j}$ being a contour surrounding $\lambda_{p,j}$ and small enough so that only the eigenvalue $\lambda_{p,j}$ of A is inside $\gamma_{p,j}$.

In the next section, we also need to show that Y defined in (3.6) is a sine-type function in the following sense:

Definition 4.4. Let f be an entire complex valued function. f is said to be of sine-type if

(a) There exists $l > 0$ such that for all $z \in \mathbb{C}$, $|f(z)| \lesssim e^{l|z|}$.

(b) The zeros of f lie in a strip $\{z \in \mathbb{C};\ |\Re z| \leq c\}$ for some $c > 0$.

(c) There exist constants $c_1, c_2 > 0$ and $x_0 \in \mathbb{R}$ such that for, all $y \in \mathbb{R}$, $c_1 \leq |f(x_0 + iy)| \leq c_2$.

The class of sine-type functions is used to deal with problems of the Riesz basis property of the complex exponentials in $L^2(0,T)$ space, with $T > 0$. When f is a sine-type function, then we can write the explicit expression of f as $f(z) = \lim\limits_{R\to+\infty} \prod\limits_{|\widetilde{\lambda}_k|\leq R}\left(1 - \frac{z}{\widetilde{\lambda}_k}\right)$, where $\{\widetilde{\lambda}_k\}_{k\in\mathbb{Z}}$ is the set of zeros of f (see [3]). If $\widetilde{\lambda}_k = 0$, then we replace the term $\left(1 - \frac{z}{\widetilde{\lambda}_k}\right)$ by z.

In our problem, we remark that the function F defined in the approximated characteristic equation (3.8) is a sine-type function. In order to deduce the same property for Y defined in (3.6), we recall a Corollary of Section 2 of [20]:

Lemma 4.5. *Given* $S(z) = \lim_{R\to+\infty} \prod_{|\widetilde{\lambda}_k|\le R} \left(1 - \frac{z}{\widetilde{\lambda}_k}\right)$ *a sine-type function, where* $\{\widetilde{\lambda}_k\}_{k\in\mathbb{Z}}$ *is the set of zeros of* $S(z)$*. Then* $S_0(z) = \lim_{R\to+\infty} \prod_{|\widetilde{\lambda}_k|\le R} \left(1 - \frac{z}{\widetilde{\lambda}_k + \psi_k}\right)$ *is also a sine-type function if* $\{\psi_k\}_{k\in\mathbb{Z}} \in \ell^p$*, for some* $p > 1$.

Lemma 4.6. *Y defined in (3.6) is sine-type, or equivalently the eigenvalues of A are the zeros of a sine-type function.*

Proof. According to Corollary 3.2 and Remark 3.10, the large eigenvalues A are close to the ones of F with a remainder $\{\psi_k\}_{k\in\mathbb{Z}}$ such that

$$\psi_k = O\left(\frac{1}{|k|^{\frac{1}{n_0}}}\right),$$

for $|k| > k_0$ that then belongs to ℓ^{n_0+1}. □

5 Exponential stability of (S_1) and proof of Theorem 1.1

Taking advantage of the fact that the root vectors of A form a Riesz basis with parenthesis of $\mathcal{H}$, our aim is now to prove that problem (S_1) is exponentially stable up to an finite dimensional space.

For our proof we recall the following lemma that can be found in Lemma 3.1 of [17].

Lemma 5.1. *Let H be a separable Hilbert space. Suppose that $\{e_n(t)\}_{n\in J}$ forms a Riesz basis for the closed subspace spanned by itself in $L^2(0,T)$, $T > 0$. Then for any $\varphi(t) = \sum_{n\in J} e_n(t)\phi_n \in L^2(0,T;H)$, there exist two positive constants $C_1(T)$, $C_2(T)$ such that*

$$C_1(T)\sum_{n\in J}\|\phi_n\|_H^2 \le \|\varphi\|_{L^2(0,T;H)}^2 \le C_2(T)\sum_{n\in J}\|\phi_n\|_H^2.$$

To apply the above lemma, we need to search for a Riesz basis in $L^2(0,T)$. Since the eigenvalues are not necessary simple, the family $\{e^{\lambda_k t}\}_{k\in\mathbb{Z}}$ does not form a Riesz basis in $L^2(0,T)$ for any $T > 0$. However, as $\sigma(A)$ is a discrete union of separated and finite sets, hence we can use the family of generalized divided differences (see [4, 17]).

Definition 5.2. Let $M \in \mathbb{N}^*$ be a fixed and let v_k, $k = 1, \cdots, M$, be arbitrary complex numbers, not necessarily distinct. Then the generalized divided differences (denoted by GDD) of order $m = 0, \cdots, M-1$ are defined by recurrence as follows: the GDD of order zero is defined as $[v_1](t) = e^{v_1 t}$, the GDD of order $m-1, 1 \le m \le M$ is defined as

$$[v_1, v_2, \cdots, v_m](t) =: \begin{cases} \dfrac{[v_1, v_2 ..., v_{m-1}](t) - [v_2, v_3, \cdots, v_m](t)}{v_1 - v_m}, & v_1 \ne v_m \\ \dfrac{\partial}{\partial v}[v, v_2, \cdots, v_{m-1}](t)\,|_{v=v_1}\,, & v_1 = v_m. \end{cases}$$

An equivalent expression is given by

$$[v_1, v_2, \cdots, v_m](t) = t^{m-1}\int_0^1\int_0^{\tau_1}...\int_0^{\tau_m-2} e^{t(v_1+\tau_1(v_2-v_1)+...+\tau_{n-1}(v_m-v_{m-1}))}d\tau_{m-1}...d\tau_2 d\tau_1.$$

Hence, if $\Re v_m \leq \Re v_{m-1} \leq ... \leq \Re v_1$, then for all $t \geq 0$

$$|[v_1, v_2..., v_m](t)| \leq t^{m-1}e^{\Re v_1 t}. \tag{5.1}$$

Now as some v_j can be repeated, we write $\{v_1, v_2, \cdots, v_M\} = \{w_1, w_2, \cdots, w_n\}$ such that $w_i \neq w_j$ for all $1 \leq i, j \leq n$ such that $i \neq j$. Supposing that each w_j is repeated n_j times, i.e, $\sum_{j=1}^{n} n_j = M$, then we can recall Proposition 3.1 of [17] which shows that for any $1 \leq k \leq n_l$, $t^{k-1}e^{w_l t}$, $l = 1, \cdots, n$ is a linear combination of $[v_1](t)$, $[v_1, v_2](t), \cdots, [v_1, v_2, \cdots, v_M](t)$.

Proposition 5.3. *Any $\varphi(t) = \sum_{j=1}^{n} e^{w_j t}\sum_{i=1}^{n_j} a_{ij}t^{i-1}$ with $a_{ij} \in \mathcal{H}$ can be rewritten as*

$$\varphi(t) = \sum_{i=1}^{M} G_i[v_1.v_2, \cdots, v_i](t),$$

with some $G_i \in \mathcal{H}$, in particular $G_1 = \sum_{j=1}^{n} a_{1j}$.

If we go back to our problem, for every $p \in \mathbb{Z}$, we construct the family of GDD of the form

$$E_p(t) = \left\{[\lambda_{p,1}](t),\ [\lambda_{p,1}, \lambda_{p,2}], \cdots, [\lambda_{p,1}, \lambda_{p,2}, \cdots, \lambda_{p,M_p}](t)\right\},$$

associated with the set $\lambda_{p,1}, \cdots, \lambda_{p,N_p}$ but the eigenvalues being repeated according to their multiplicity (and consequently $M_p = \sum_{j=1}^{N_p} m_{p,j}$).

Proposition 5.4. *There exists $T > 0$ such that the family of GDD $\{E_p(t)\}_{p\in\mathbb{Z}}$ forms a Riesz basis for the closed subspace spanned by itself in $L^2(0, T)$.*

Proof. According to Lemma 4.6, the eigenvalues of A are roots of a sine-type function. Hence, the proof becomes a direct consequence of Theorem 3 of [5] where $T > 0$ is chosen large enough (note also that a sine-type function automatically satisfies the Helson-Szego condition due to its equivalent form (condition (A_2) page 2 in [4]) and the condition (c) in our definition 4.4). □

Proof. **of Theorem 1.1.** Given an initial datum $U(0) \in \mathcal{H}$, by Proposition 4.3, it can be written as

$$U(0) = \sum_{p\in\mathbb{Z}}\sum_{j=1}^{N_p}\mathbb{P}_{\lambda_{p,j}}\left(U(0)\right),$$

where we recall that $\mathbb{P}_{\lambda_{p,j}}$ denotes the Riesz projection of A corresponding to the eigenvalue $\lambda_{p,j}$, then, for any $t > 0$, we have

$$\begin{aligned} e^{tA}U(0) &= \sum_{p\in\mathbb{Z}}\sum_{j=1}^{N_p} e^{\lambda_{p,j}t}\sum_{i=1}^{m_{p,j}}\frac{(A-\lambda_{p,j})^{i-1}}{(i-1)!}t^{i-1}\mathbb{P}_{\lambda_{p,j}}\left(U(0)\right) \\ &= \sum_{p\in\mathbb{Z}}\sum_{j=1}^{N_p} e^{\lambda_{p,j}t}\sum_{i=1}^{m_{p,j}} a_{ij,p}t^{i-1}, \end{aligned} \tag{5.2}$$

where $a_{ij,p} = \dfrac{(A-\lambda_{p,j})^{i-1}}{(i-1)!}\mathbb{P}_{\lambda_{p,j}}(U(0))$. By Proposition 5.3, we get

$$e^{tA}U(0) = \sum_{p\in\mathbb{Z}}\sum_{i=1}^{M_p} G_{p,i}[\lambda_{p,1},\lambda_{p,2},\cdots,\lambda_{p,i}](t). \tag{5.3}$$

Lemma 5.1 and Proposition 5.4 yield for some $T > 0$

$$\sum_{p\in\mathbb{Z}}\sum_{i=1}^{M_p}\|G_{p,i}\|_{\mathcal{H}}^2 \lesssim \int_0^T \left\|e^{tA}U(0)\right\|_{\mathcal{H}}^2 dt.$$

By the semigroup property, we know that there are $C,\omega > 0$ such that for all $t \geq 0$

$$\|e^{tA}\|_{\mathcal{L}(\mathcal{H})} \leq Ce^{\omega t}.$$

Therefore, the previous estimate becomes

$$\sum_{p\in\mathbb{Z}}\sum_{i=1}^{N_p}\|G_{p,i}\|_{\mathcal{H}}^2 \lesssim \frac{C^2}{2\omega}(e^{2\omega T}-1)\left\|U(0)\right\|_{\mathcal{H}}^2. \tag{5.4}$$

Finally, since the root vectors of A form a Riesz basis with parenthesis of $\mathcal{H}$, then by (5.1), (5.3), and (5.4) we get for $t \geq 1$

$$\begin{aligned}\left\|e^{tA}U(0)\right\|_{\mathcal{H}}^2 &\lesssim \sum_{p\in\mathbb{Z}}\left\|\sum_{i=1}^{M_p}G_{p,i}[\lambda_{1,p},\lambda_{2,p},\cdots,\lambda_{i+\widetilde{m}_{j-1}^p,p}](t)\right\|_{\mathcal{H}}^2 \\ &\lesssim \sum_{p\in\mathbb{Z}}t^{2(M_p-1)}e^{2\mu_p t}\sum_{i=1}^{M_p}\|G_{p,i}\|_{\mathcal{H}}^2,\end{aligned} \tag{5.5}$$

where $\mu_p = \max_{1\leq j\leq N_p}\Re\lambda_{p,j}$.

Now, by Theorem 3.2, we know that if the roots of the polynomial G are in the open unit disk, then there exists $\mu < 0$ and $p_0 \in \mathbb{N}$ such that

$$\mu_p \leq \mu < 0, \forall |p| > p_0.$$

Hence by (5.5), we deduce the exponential stability of problem (S_1) up to the finite dimensional space spanned by the roots vectors of A corresponding to the eigenvalues $\lambda_{p,j}$ such that $|p| \leq p_0$. The proof of Theorem 1.1 is complete. □

6 Exponential stability of (S_ϵ) for small values of ϵ and proof of Theorem 1.4

In this section, we consider constant damping coefficients and equal lengths $L_i = 1$, for all $i = 1,\cdots,N$. Without loss of generality we can assume that the a_i are non decreasing, i.e., $a_1 \leq a_2 \leq \cdots \leq a_N$. In the sequel, we replace the damping coefficients a_i by ϵa_i, where the parameter ϵ is positive. Our goal is to find sufficient conditions for which (S_ϵ) is exponentially stable in the whole energy space for every ϵ small enough.

Based on the results of the previous section, it seems enough to find sufficient conditions on the damping coefficients so that the low eigenvalues have negative real parts for every ϵ small enough. However, we remark that Rouché's Theorem used in the proof of Theorem 3.2 yields a constant k_0 dependent of ϵ (mainly of order $\frac{1}{\epsilon}$). Consequently, it seems difficult to separate the large eigenvalues from the low eigenvalues uniformly in ϵ for all ϵ small enough.

As previously mentioned, the exponential stability of (S_ϵ) has been studied in [22] under some abstract hypothesis. Consequently, our aim is to interpret the hypothesis from [22] to find explicit conditions on the damping coefficients. Our strategy is based on the asymptotic behavior of the spectrum of the generator $A = A(\epsilon)$ as a function of ϵ. In the sequel, we use some notations from [19], we refer the reader to this book for the exact definitions. First, we notice that the generator $A = A(\epsilon)$ is holomorphic of type (A) in the parameter ϵ in the sense of (2.1) of chapter VII.2 in [19]. Indeed, we simply have

$$A(\epsilon) = A(0) + \epsilon B,$$

where $A(0)$ is a skewadjoint operator and B is a bounded selfadjoint operator defined by

$$A(0) = \begin{pmatrix} 0 & A_1^0 \\ A_1^2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & A_{-2a}^0 \end{pmatrix}.$$

Since $A(0)$ is a skew adjoint operator with a compact resolvent, there is an orthonormal system of eigenvectors of $A(0)$ which is complete in $\mathcal{H}$. The eigenvalues of $A(0)$ are $\lambda_{1,k}(0) = \imath(k\pi + \frac{\pi}{2})$ with multiplicity one, for all $k \in \mathbb{Z}$, and $\lambda_{2,k}(0) = \imath k\pi$ with geometric and algebraic multiplicity $N-1$, for all $k \in \mathbb{Z}^*$. For shortness we write $\{\lambda_k(0)\}_{k\in\mathbb{Z}} = \{\imath k\pi\}_{k\in\mathbb{Z}^*} \cup \left\{\imath(k\pi + \frac{\pi}{2})\right\}_{k\in\mathbb{Z}}$ and we set m_k the multiplity of $\lambda_k(0)$ (hence $m_k = 1$ or $m_k = N-1$).

Now according to section VII.2.4 in [19], there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, $A(\epsilon)$ has exactly m_k eigenvalues (algebraic multiplicity counted) in $B(\lambda_k(0), \rho)$, with $\rho > 0$ fixed small enough. This set of eigenvalues is called the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$ generated by the splitting from the common eigenvalue $\lambda_k(0)$ of the unperturbed operator $A(0)$ (see page 74 in [19]). Consequently as ϵ increases, a splitting of the eigenvalues may occur and the eigenvalues of $A(\epsilon)$ can go to the left or to the right of the imaginary axis (or both). Hence, our aim is to find some sufficient conditions for which each $\lambda_k(0)$-group is strictly to the left of the imaginary axis.

For further use, for $\epsilon \in (0, \epsilon_0)$ and all $k \in \mathbb{Z}$, the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$ will be denoted by $\{\lambda_{k,j}(\epsilon)\}_{j=1}^{m_k}$.

In a first step, consider $\Gamma_{k,\rho}$ a positively-oriented circle around $\lambda_k(0)$ with radius $\rho < \frac{\pi}{4}$ such that $\lambda_k(0)$ is isolated. For $\zeta \in \Gamma_{k,\rho}$, we denote by $R(\zeta) = (A(0) - \zeta)^{-1}$. The following lemma gives a uniform estimate of $\|R(\zeta)\|$ for all $\zeta \in \Gamma_{k,\rho}$.

Lemma 6.1. *For all $\zeta \in \Gamma_{k,\rho}$, we have*

$$\|R(\zeta)\| = \frac{1}{\rho},\ \forall \zeta \in \Gamma_{k,\rho}. \tag{6.1}$$

Proof. For convenience and for a moment, we rename $\{i\beta_k\}_{k\in\mathbb{Z}^*}$ the set of eigenvalues of $A(0)$ and arrange it in increasing order (i.e ...$\beta_{k-1} \le \beta_k \le \beta_{k+1}$...). We denote by $\{\phi_k\}_{k\in\mathbb{Z}^*}$ the associated system of eigenvectors which forms an orthonormal basis of $\mathcal{H}$. Let $f = \sum_{k\in\mathbb{Z}^*} f_k\phi_k \in \mathcal{H}$, then by the spectral theorem, for all $\zeta \in \Gamma_{k,\rho}$, we can write

$$R(\zeta)f = \sum_{k\in\mathbb{Z}^*} \frac{f_k}{i\beta_k - \zeta}\phi_k.$$

Since $|i\beta_k - \zeta| \le \rho$, for all $k \in \mathbb{Z}^*$, we deduce that

$$\|R(\zeta)f\|^2 = \sum_{k\in\mathbb{Z}^*} \frac{|f_k|^2}{|i\beta_k - \zeta|^2} \le \frac{1}{\rho^2}\sum_{k\in\mathbb{Z}^*}|f_k|^2 = \frac{1}{\rho^2}\|f\|^2.$$

This proves (6.1) by taking f corresponding to one eigenvector associated with the eigenvalue $\lambda_k(0)$. □

Now we characterize the asymptotic behaviour of the real parts of the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$.

Lemma 6.2. *There exists $\epsilon_1 > 0$ and $c > 0$ such that for all $\epsilon \in (0,\epsilon_1)$, all $k \in \mathbb{Z}$ and all $j = 1, \cdots, m_k$,*

$$\Re\lambda_{k,j}(\epsilon) \le \epsilon \max_{1\le j\le m_k} \mu_{k,j} + c\epsilon^2,$$

when $\{\mu_{k,j}\}_{j=1}^{m_k}$ denotes the set of eigenvalues of $P_k(0)BP_k(0)$ and $P_k(0)$ denotes the eigenprojection corresponding to $\lambda_k(0)$, i.e., $P_k(0) = -\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_{k,\rho}} R(\xi)d\xi$.

Proof. **Step 1.** Let $\lambda_k(0)$ be an eigenvalue of $A(0)$. Define the space $M_k(\epsilon) = P_k(\epsilon)\mathcal{H}$, where $P_k(\epsilon)$ is the eigenprojection (see (1.16) page 67 of [19]) defined by

$$P_k(\epsilon) = -\frac{1}{2\pi i}\int_{\Gamma_{k,\rho}} (A(\epsilon)-\xi)^{-1}\, d\xi.$$

Notice that $(A(\epsilon)-\xi)^{-1}$ is well defined for $\xi \in \Gamma_{k,\rho}$ when $\epsilon < \dfrac{1}{\|B\|\|R(\xi)\|} = \dfrac{\rho}{\|B\|}$. Indeed, according to (1.13) and (1.14) page 67 of [19], we have by the second Neumann series for the resolvent

$$(A(\epsilon)-\xi)^{-1} = R(\xi)\left(1+\epsilon BR(\xi)\right)^{-1} = R(\xi)\sum_{n=0}^{\infty}(-\epsilon BR(\xi))^n = R(\xi) + \sum_{n=1}^{\infty}\epsilon^n R_k^{(n)}(\xi), \tag{6.2}$$

where

$$R_k^{(n)}(\xi) = R(\xi)(-BR(\xi))^n. \tag{6.3}$$

Hence the series in the right-hand side of (6.2) converges if $\epsilon < \frac{\rho}{\|B\|}$ (thanks to Lemma 6.1, we notice that the upper bound of ϵ is independent of $k \in \mathbb{Z}$) and we get

$$P_k(\epsilon) = P_k(0) + \sum_{n=1}^{\infty}\epsilon^n P_k^{(n)},$$

where $P_k^{(n)} = -\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_{k,\rho}} R_k^{(n)}(\xi)d\xi$ and $P_k^{(0)} = P_k(0)$.

As already said before if ϵ is sufficiently small, the eigenvalues of $A(\epsilon)$ lying in $\Gamma_{k,\rho}$ form exactly the $\lambda_k(0)$-group eigenvalues. Therefore, since $\lambda_k(0)$ is semisimple (since its a eigenvalue of a skewadjoint operator), then according to the identities (5.13) and (5.14) of [19, p. 112], the $\lambda_k(0)$-group eigenvalues of $A(\epsilon)$ are of the form

$$\lambda_{k,j}(\epsilon) = \lambda_k(0) + \epsilon\mu_{k,j}^{(1)}(\epsilon), \quad j = 1, \cdots, m_k, \tag{6.4}$$

where $\{\mu_{k,j}^{(1)}(\epsilon)\}_{j=1}^{m_k}$ are the eigenvalues of the operator

$$\widetilde{A}_k^{(1)}(\epsilon) = \epsilon^{-1}(A(\epsilon) - \lambda_k(0))P_k(\epsilon) = -\frac{\epsilon^{-1}}{2\pi i}\int_{\Gamma_{k,\rho}} (\xi - \lambda_k(0))\,(A(\epsilon) - \xi)^{-1}\,d\xi, \tag{6.5}$$

in the subspace $M_k(\epsilon) = P_k(\epsilon)\mathcal{H}$. The second equality in (6.5) follows from the fact that

$$(A(\epsilon) - \lambda_k(0))\,(A(\epsilon) - \xi)^{-1} = 1 + (\xi - \lambda_k(0))\,(A(\epsilon) - \xi)^{-1}\,.$$

Step 2. We estimate the difference between $\widetilde{A}_k^{(1)}(\epsilon)$ and $P_k(0)BP_k(0)$. According to the identity (2.16) page 77 of [19], we have

$$(A(\epsilon) - \lambda_k(0))P_k(\epsilon) = (A(0) - \lambda_k(0))P_k(0) + \sum_{n=1}^{\infty}\epsilon^n \widetilde{A}_k^{(n)}, \tag{6.6}$$

where

$$\widetilde{A}_k^{(n)} = (-1)^{n+1}\frac{1}{2\pi i}\int_{\Gamma_{k,\rho}} R(\xi)(BR(\xi))^n(\xi - \lambda_k(0))d\xi, \tag{6.7}$$

in particular (see (2.19) page 77 of [19])

$$\widetilde{A}_k^{(1)} = P_k(0)BP_k(0). \tag{6.8}$$

Since $\lambda_k(0)$ is semisimple, then $A(0)P_k(0) = \lambda_k(0)P_k(0)$. Thus (6.6) implies that

$$\widetilde{A}_k^{(1)}(\epsilon) = \sum_{n=0}^{\infty}\epsilon^n \widetilde{A}_k^{(n+1)}.$$

On the other hand, from (6.7) and Lemma 6.1, we have for all $n \geq 1$

$$\|\widetilde{A}_k^{(n)}\| \leq \frac{\|B\|^n}{\rho^n}.$$

Therefore, for ϵ small enough, there exists $c > 0$ independent of k such that

$$\|\widetilde{A}_k^{(1)}(\epsilon) - \widetilde{A}_k^{(1)}\| \leq \sum_{n=1}^{\infty}\epsilon^n\frac{\|B\|^n}{\rho^n} = \frac{\epsilon\frac{\|B\|}{\rho}}{1 - \epsilon\frac{\|B\|}{\rho}} \leq c\epsilon. \tag{6.9}$$

Step 3. We compare the eigenvalues $\{\mu_{k,j}^{(1)}(\epsilon)\}_{j=1}^{m_k}$ of $\widetilde{A}_k^{(1)}(\epsilon)$ with the eigenvalues of $\widetilde{A}_k^{(1)} = P_k(0)BP_k(0)$. Consider $\mu_{k,j}^{(1)}(\epsilon)$ and $\phi_{k,j}^{(1)}(\epsilon)$ an associated normalized eigenvector, then

$$\widetilde{A}_k^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon) = \mu_{k,j}^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon).$$

From (6.9), we have

$$\|\widetilde{A}_k^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon) - \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon)\| \leq c\epsilon.$$

Thus, by Cauchy-Schwarz's inequality, we have

$$| < \mu_{k,j}^{(1)}(\epsilon)\phi_{k,j}^{(1)}(\epsilon) - \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon), \phi_{k,j}^{(1)}(\epsilon) > | \leq c\epsilon,$$

or equivalently

$$|\mu_{k,j}^{(1)}(\epsilon) - < \widetilde{A}_k^{(1)}\phi_{k,j}^{(1)}(\epsilon), \phi_{k,j}^{(1)}(\epsilon) > | \leq c\epsilon.$$

Therefore,

$$\Re(\mu_{k,j}^{(1)}(\epsilon)) \leq < \widetilde{A}_k^{(1)} \phi_{k,j}^{(1)}(\epsilon), \phi_{k,j}^{(1)}(\epsilon) > +c\epsilon,$$

or

$$\Re(\mu_{k,j}^{(1)}(\epsilon)) \leq \max_{1 \leq j \leq m_k} \mu_{k,j} + c\epsilon.$$

We conclude by using this estimate and (6.4). □

According to Lemma 6.2, to prove that the spectrum of $A(\epsilon)$ is situated to the left of the imaginary axis for $\epsilon > 0$ small enough, we have to prove that, for every $k \in \mathbb{Z}$, the eigenvalues of $\widetilde{A}_k^{(1)} = P_k(0)BP_k(0)$ are strictly to the left of the imaginary axis independently of $k \in \mathbb{Z}$ and $\epsilon > 0$. In fact, the hypothesis imposed in [22] to get the exponential stability of (S_ϵ) can be interpreted as a condition on the negativity of the eigenvalues of $P_k(0)BP_k(0)$. Therefore, our aim in the next two lemmas is to find the eigenvalues of $P_k(0)BP_k(0)$ and to investigate the conditions for which their real parts are negative independently of $k \in \mathbb{Z}$ and $\epsilon > 0$.

Lemma 6.3. *If $\mu_{k,0}$ denotes the eigenvalue of $P_k(0)BP_k(0)$, where $P_k(0)$ is the eigenprojection corresponding to $\lambda_k(0) = \imath(k\pi + \frac{\pi}{2})$, with $k \in \mathbb{Z}$, then $\mu_{k,0} = -\frac{1}{N}\sum_{i=1}^{N} a_i$.*

Proof. We recall that $\lambda_k(0) = \imath(k\pi + \frac{\pi}{2})$ is simple for all $k \in \mathbb{Z}$. Some elementary calculations show that the associated normalized eigenvector is of the form

$$\phi_0 = \frac{1}{\sqrt{N}}(u, v)^\top,$$

where, for all $x \in (0,1)$ and $i = 1, \cdots, N$, $u_i(x) = \dfrac{\sin(\lambda_k(0)(1-x))}{\lambda_k(0)}$ and $v_i(x) = \sin(\lambda_k(0)(1-x))$. For any $\psi \in \mathcal{H}$, we find that

$$P_k(0)BP_k(0)\psi = -\frac{1}{N}\left(\sum_{i=1}^{N} a_i\right)(\psi, \phi_0)\phi_0,$$

hence ϕ_0 is the eigenvector of $P_k(0)BP_k(0)$ of eigenvalue $-\dfrac{1}{N}\displaystyle\sum_{i=1}^{N} a_i$. □

Lemma 6.4. *If $\{\mu_{k,j}\}_{j=1}^{N-1}$ denotes the set of eigenvalues of $P_k(0)BP_k(0)$, where $P_k(0)$ is the eigenprojection corresponding to $\lambda_k(0) = \imath k\pi$, with $k \in \mathbb{Z}^*$, then $\{\mu_{j,k}\}_{j=1}^{N-1}$ is the set of zeros of the polynomial Q defined by*

$$Q(z) = (z + a_1)(z + a_N)\sum_{i=2}^{N-1}\prod_{\substack{l \neq i \\ l=2}}^{N-1}(z + a_l) + \prod_{l=2}^{N-1}(z + a_l)(2z + a_1 + a_N) \tag{6.10}$$

Proof. First, we notice that, for all $k \in \mathbb{Z}^*$, $\lambda_k(0) = \imath k\pi$ is of multiplicity $N-1$ and that the associated eigenvectors are of the form $(u,v)^\top$ where, for $i = 1, \cdots, N$ and $x \in (0,1)$, $u_i(x) = \frac{\alpha_i}{\imath k\pi}\sin(k\pi(1-x))$ and $v_i(x) = \alpha_i \sin(k\pi(1-x))$ with $\alpha = (\alpha_i)_{i=1}^N \in \mathbb{R}^N$ such that $\sum_{i=1}^{N}\alpha_i = 0$. As a basis of the subspace $P_k(0)\mathcal{H}$, we can choose the system of eigenvectors $\{\phi^{(i)}\}_{i=1,\cdots,N-1}$ corresponding to the choice

$$\alpha^{(1)} = (1, -1, 0 \cdots, 0), \alpha^{(2)} = (1, 0, -1, 0 \cdots, 0), \cdots, \alpha^{(N-1)} = (1, 0, \cdots, 0, -1).$$

Therefore, for all $i = 1, \cdots, N-1$, $P_k(0)BP_k(0)\phi^{(i)} = \sum_{k=1}^{N-1} \alpha_{ik}\phi^{(k)}$ where $\alpha_{ik} \in \mathbb{C}$. Moreover, for all $i, j = 1, \cdots, N-1$,

$$\left\langle P_k(0)BP_k(0)\phi^{(i)}, \phi^{(j)} \right\rangle = \left\langle B\phi^{(i)}, \phi^{(j)} \right\rangle = \sum_{k=1}^{N-1} \alpha_{ik} \left\langle \phi^{(k)}, \phi^{(j)} \right\rangle.$$

Hence, $P_k(0)BP_k(0) = \Gamma G^{-1}$, where $\Gamma = (\langle B\phi^{(i)}, \phi^{(j)} \rangle)_{i,j}$ and G is the Gramian matrix defined by $G = (\langle \phi^{(i)}, \phi^{(j)} \rangle)_{i,j}$. But some elementary calculations yield

$$\Gamma = \begin{pmatrix} -a_1 - a_2 & -a_1 & -a_1 & \cdots & -a_1 \\ -a_1 & -a_1 - a_3 & -a_1 & \cdots & -a_1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & -a_1 \\ -a_1 & -a_1 & \cdots & -a_1 & -a_1 - a_N \end{pmatrix}$$

and

$$G = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}.$$

Since

$$G^{-1} = \frac{1}{N} \begin{pmatrix} N-1 & -1 & -1 & \cdots & -1 \\ -1 & N-1 & -1 & \cdots & -1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & -1 \\ -1 & -1 & \cdots & -1 & N-1 \end{pmatrix} = I - \frac{1}{N} \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix},$$

we get

$$\begin{aligned} & P_k(0)BP_k(0) \\ = \; & \frac{1}{N} \begin{pmatrix} -a_1 - (N-1)a_2 & -a_1 + a_2 & -a_1 + a_2 & \cdots & -a_1 + a_2 \\ -a_1 + a_3 & -a_1 - (N-1)a_3 & -a_1 + a_3 & \cdots & -a_1 + a_3 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ -a_1 + a_N & -a_1 + a_N & \cdots & -a_1 + a_N & -a_1 - (N-1)a_N \end{pmatrix}. \end{aligned}$$

Therefore, $\{\mu_{j,k}\}_{j=1}^{N-1}$ are the roots of the characteristic polynomial $\det(zI - P_k(0)BP_k(0))$

or equivalently

$$Q(z) = \det \begin{pmatrix} z+a_2 & 0 & \cdots & \cdots & -z-a_N \\ 0 & z+a_3 & 0 & \cdots & -z-a_N \\ \vdots & \cdots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & z+a_{N-1} & -z-a_N \\ z+a_1 & z+a_1 & \cdots & z+a_1 & 2z+a_1+a_N \end{pmatrix}.$$

Developing with respect to the last line, we find (6.10). □

Before going on let us notice that the above lemmas show that the eigenvalues $\mu_{k,j}$ of $P_k(0)BP_k(0)$ are independent of k. In the first case we directly find a condition on the damping coefficients to have $\mu_{k,0} < 0$, for the second case we need that the roots of Q are negative. For this aim, we first localize the roots of Q. Before doing so let us introduce the following notation: as the a_i are not necessarily different, we denote by $M \leq N$ the number of different a_i's and denote by $\{b_j\}_{j=1}^M$ the set of the different coefficients in increasing order, which means that

$$\{b_j\}_{j=1}^M = \{a_i\}_{i=1}^N,$$

and

$$b_1 < b_2 < \cdots < b_M.$$

Further for all $j = 1, \cdots M$, denote by k_j the number of repeated values of b_j in the initial set of coefficients a_i, namely

$$k_j = \#\{i \in \{1, \cdots, N\} : b_j = a_i\}.$$

Lemma 6.5. *If Q is the polynomial defined by (6.10), then its has $N-1$ real roots $\mu_i, i = 1, \cdots, N-1$, in $[-a_N, -a_1]$ such that*

$$-b_{j+1} < \mu_j < -b_j, \forall j = 1, \cdots, M-1,$$

the other roots are $-b_j$ of multiplicity $k_j - 1$, for all $j = 1, \cdots, M$ such that $k_j \geq 2$.

Proof. We first notice that

$$Q(-a_i) = \prod_{\substack{l=1 \\ l \neq i}}^N (a_l - a_i).$$

Hence we see that $-a_i$ is a root of Q if and only if there exists at least one $\ell \neq i$ such that $a_i = a_\ell$. Unfortunately this argument is not sufficient in the case of a triple equality $a_i = a_\ell = a_k$ with i, k, ℓ different from each other since it does not give the multiplicity of $-a_i$.

But for a complex number μ such that $\mu \notin \{-a_i\}_{i=1}^N$, we notice that

$$Q(\mu) = \prod_{l=1}^N (\mu + a_l) \left(\sum_{i=1}^N \frac{1}{\mu + a_i} \right). \tag{6.11}$$

Therefore $\mu \notin \{-a_i\}_{i=1}^N$ is a root of Q if and only if

$$\widetilde{Q}(\mu) = \sum_{i=1}^N \frac{1}{\mu + a_i} = 0.$$

As $\widetilde{Q}$ has vertical asymptotes $\mu = -b_j$, for all $j = 1, \ldots, M$ and is a decreasing function on $(-b_{j+1}, -b_j)$, for all $j = 1, \cdots, M-1$ (see Figure 2 for the graph of $\widetilde{Q}$ when $N = M = 4$, $a_1 = -2$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$), we deduce that Q has one and only one real root between two consecutive vertical asymptotes.

Now for $j = 1, \cdots, M$ such that $k_j \geq 2$, we take $\mu \neq -b_j$ but close to it and use the expression (6.11) to find that

$$Q(\mu) = \prod_{\ell: a_\ell \neq b_j} (\mu + a_l)\,(\mu + b_j)^{k_j} \left(\frac{k_j}{\mu + b_j} + \sum_{i: a_i \neq b_j} \frac{1}{\mu + a_i} \right) \tag{6.12}$$

$$= (\mu + b_j)^{k_j - 1} \prod_{\ell: a_\ell \neq b_j} (\mu + a_l) \left(k_j + (\mu + b_j) \sum_{i: a_i \neq b_j} \frac{1}{\mu + a_i} \right). \tag{6.13}$$

Since $\prod_{\ell: a_\ell \neq b_j} (\mu + a_l) \left(k_j + (\mu + b_j) \sum_{i: a_i \neq b_j} \frac{1}{\mu + a_i} \right)$ is holomorphic in a neighborhood of $-b_j$, we deduce that $-b_j$ is a root of Q of multiplicity $k_j - 1$. □

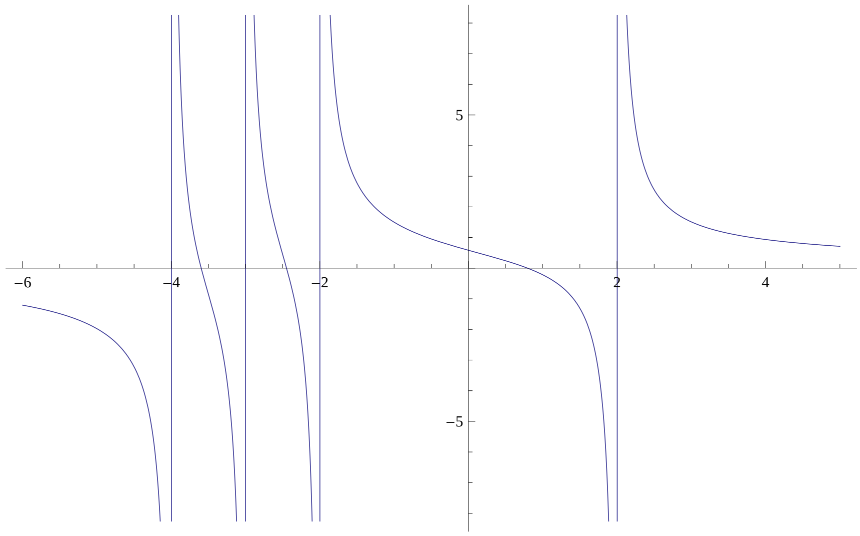

Figure 2. $N = 4, a_1 = -2, a_2 = 2, a_3 = 3, a_4 = 4$

Corollary 6.6. *The polynomial Q defined by (6.10) has negative roots if and only if one of the following two conditions is satisfied:*

(i) $a_1 \geq 0$ *and* $a_i > 0$*, for all* $i = 2, \cdots, N$*,*

(ii) $a_1 < 0$*,* $a_i > 0$*, for all* $i = 2, \cdots, N$ *and*

$$\sum_{i=1}^{N} \frac{1}{a_i} < 0.$$

Proof. According to the previous lemma, if $-b_2 \geq 0$, then Q has a positive root, hence b_2 has to be positive. Now if $b_1 = a_1$ is positive, all roots are trivially negative. On the other hand, if $b_1 \le 0$ with $k_1 > 1$, then Q has a non negative root $-b_1$. Hence k_1 has to be equal to 1. This covers the first item. For the second item, we have $b_1 < 0$ with $k_1 = 1$ and therefore again according to the previous lemma, Q has a root μ (or equivalently $\widetilde{Q}$) between $-a_2 < 0$ and $-a_1 > 0$ that potentially could be positive, but since $\widetilde{Q}$ is decreasing on $(-a_2, -a_1)$ the condition

$$\widetilde{Q}(0) = \sum_{i=1}^{N} \frac{1}{a_i} < 0$$

is a necessary and sufficient condition to get $\mu < 0$. □

Summing up the results of Lemmas 6.2, 6.3, and Corollary 6.6, we give the proof of Theorem 1.4.

Proof. **of Theorem 1.4:** According to Lemma 6.2, if $\max_{k\in\mathbb{Z}} \max_{j=1,\cdots,m_k} \mu_{k,j} = -C < 0$, then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, and all $k \in \mathbb{Z}$, $\Re\lambda_k(\epsilon) \le -\frac{C}{2}\epsilon$. Using Lemmas 6.3 and Corollary 6.6, this is satisfied if either one of the two conditions is satisfied:

(i) $a_1 \ge 0$ and $a_i > 0$, for all $i = 2, \cdots, N$,

(ii) $a_1 < 0$, $a_i > 0$, for all $i = 2, \cdots, N$ and

$$\sum_{i=1}^{N} a_i > 0 \text{ as well as } \sum_{i=1}^{N} \frac{1}{a_i} < 0.$$

Since the root vectors of $A(\epsilon)$ form a Riesz basis with parenthesis (see Proposition 4.3), we deduce the exponential stability of the solution of (S_ϵ) for all $\epsilon \in (0, \epsilon_0)$ under one of the conditions (i) or (ii) stated above. □

Remark 6.7. (i) Owing to Lemma 6.3 and Corollary 6.6, if $\sum_{i=1}^{N} a_i < 0$ or if Q has a positive root, then (S_ϵ) is unstable for all $\epsilon > 0$ small enough. For example, in the case $N = 3$, $a_1 = a_2 = 1$, and $a_3 = -\frac{1}{2}$, Figure 6 shows that there are eigenvalues to the right of the imaginary axis when $\epsilon = 0.1$.

(ii) If $\sum_{i=1}^{N} a_i = 0$ or if Q has a root equal to zero, then the exponential stability of (S_ϵ) for $\epsilon > 0$ small enough is an open question.

Remark 6.8. The previous analysis can be adapted to the case when $a_i \in L^\infty(0, 1)$ and $L_i = 1$ for all $i = 1, \cdots, N$. As before we can prove that the solution of (S_ϵ) is exponentially stable for all $\epsilon \in (0, \epsilon_0)$ for $\epsilon_0 > 0$ small enough if there exists $c_0 > 0$ and $c_1 > 0$ such that for all $k \in \mathbb{Z}$, one of the following two conditions holds:

(a) There exists at most one $j_0 \in \{1, \cdots, N\}$ such that $\int_0^1 a_{j_0}(x) \sin^2(k\pi(1-x))dx = 0$, $\int_0^1 a_i(x) \sin^2(k\pi(1-x))dx > c_0$ for all $i \ne j_0$ and $\sum_{i=1}^{N} \int_0^1 a_i(x) \sin^2((k\pi + \frac{\pi}{2})(1-x))dx > c_0$.

(b) There exists only one $i_0 \in \{1, \cdots, N\}$ such that $\displaystyle\int_0^1 a_{i_0}(x)\sin^2(k\pi(1-x))dx < 0$,
$\displaystyle\int_0^1 a_i(x)\sin^2(k\pi(1-x))dx > c_0$ for all $i \neq i_0$, $\displaystyle\sum_{i=1}^N \int_0^1 a_i(x)\sin^2((k\pi+\frac{\pi}{2})(1-x))dx > c_0$,
and $\displaystyle\sum_{i=1}^N \frac{1}{\displaystyle\int_0^1 a_i(x)\sin^2(k\pi(1-x))dx} < -c_1$.

Indeed, the results of Lemma 6.2 still hold. Lemma 6.3 also holds but in this case, for all $k \in \mathbb{Z}$, $\mu_{k,0} = -\dfrac{2}{N}\displaystyle\sum_{i=1}^N \int_0^1 a_i(x)\sin^2((k\pi+\frac{\pi}{2})(1-x))dx$. Similarly, in Lemma 6.4, we can repeat the same analysis and find that, for all $k \in \mathbb{Z}^*$, $\{\mu_{j,k}\}_{j=1}^{N-1}$ is the set of zeros of

$$\widehat{Q}(z) = (z+I_1)(z+I_N)\sum_{i=2}^{N-1}\prod_{\substack{l\neq i\\ l=2}}^{N-1}(z+I_l) + \prod_{l=2}^{N-1}(z+I_l)(2z+I_1+I_N),$$

where for all $i = 1, \cdots, N$, $I_i = 2\displaystyle\int_0^1 a_i(x)\sin^2(k\pi(1-x))dx$ (which here depends on k). As Lemma 6.5 can be used for $\widehat{Q}$, we find the same results but with a_i replaced by I_i for all $i = 1, \cdots, N$. Therefore, thanks to Lemma 6.2 and under one of the conditions (a) or (b) stated above, we deduce the existence of $\widehat{C} > 0$ such that for all $k \in \mathbb{Z}$, $\Re\lambda_k(\epsilon) \leq -\epsilon\widehat{C}$ for all $\epsilon \in (0, \epsilon_0)$.

7 Examples

In order to illustrate our general results we present some concrete examples where we can give explicit conditions on the damping coefficients to get exponential decay (up to a finite-dimensional space) for both problems (S_1) and (S_ϵ). In the first case, this is reduced to the calculation of the roots of the polynomial G defined by (1.1), in the second one since the conditions from Theorem 1.4 are easy to check, we concentrate on a limit case (see Remark 6.7) and on the characterization of the limit values of ϵ for which the global stability is lost.

7.1 Examples for (S_1)

We consider (S_1) with three edges ($N = 3$) of length $L_i = 1$ and $a_i(.) \in W^{1,\infty}(0,1)$ such that $\displaystyle\int_0^1 a_1(x)dx = \int_0^1 a_2(x)dx = 1$ and $\displaystyle\int_0^1 a_3(x)dx = \alpha \leq 0$. Using Theorem 1.1, we will find the critical value of α for which (S_1) is exponentially stable up to a finite dimensional space. Indeed, for this example, the polynomial G is given by

$$G(z) = 3e^{2+\alpha}z^3 - (e^{2-\alpha}+2e^{\alpha})z^2 - (e^{-2+\alpha}+2e^{-\alpha})z + 3e^{-2-\alpha}.$$

The roots of G are given by

$$\begin{aligned}
z_1 &= e^{-2},\\
z_2 &= -\frac{e^{-2}}{6} + \frac{e^{-2\alpha}}{6} - \frac{e^{-2-2\alpha}}{6}\sqrt{e^4+e^{4\alpha}+34e^{2+2\alpha}},\\
z_3 &\ -\frac{e^{-2}}{6} + \frac{e^{-2\alpha}}{6} + \frac{e^{-2-2\alpha}}{6}\sqrt{e^4+e^{4\alpha}+34e^{2+2\alpha}}.
\end{aligned}$$

Recall that according to Theorem 1.1, (S_1) is exponentially stable up to a finite dimensional space if and only if $|z_i| < 1$ for all $i = 1, 2, 3$. Hence we need to analyze the behavior of the z_i with respect to α. Clearly $z_1 < 1$ is independent of α, while the two other ones depend on α. For z_2, we easily check that $z_2 < 0$ is an increasing function of α with $\lim_{\alpha \to -\infty} z_2 = -3e^{-3} > -1$. Hence, $-1 < z_2 < 0$ for all $\alpha \leq 0$. Next, we notice that $|z_2||z_3| = e^{-2-2\alpha}$. So, if $\alpha \leq -1$, then $|z_2||z_3| \geq 1$ which means that $|z_3| \geq 1$. Therefore, to get the exponential stability of (S_1), we must have $\alpha > -1$. In this case, z_3 is a decreasing function of α and for $\alpha_0 = \frac{1}{2}\ln\left(\frac{3+e^2}{1+3e^2}\right)$ we get $z_3 \geq 1$ if $\alpha \leq \alpha_0$ and $0 < z_3 < 1$ if $\alpha > \alpha_0$. In conclusion, (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha \in (\alpha_0, 0]$.

As a second example, we still take three edges, but consider $L_1 = L_2 = 1$ and $L_3 = 2$ with $\int_0^1 a_1(x)dx = \int_0^1 a_2(x)dx = 1$ and $\int_0^2 a_3(x)dx = \alpha$. With this choice, the polynomial G is given by

$$G(z) = (ez - e^{-1})p(z).$$

where

$$p(z) = 3e^{1+\alpha}z^3 + e^{-1+\alpha}z^2 - e^{1-\alpha}z - 3e^{-1-\alpha}.$$

As the roots of the first factor is $e^{-2} < 1$, we only have to consider the roots of the second factor p. Let $z_i = z_i(\alpha)$ for $i = 1, 2, 3$ be the roots of p and define $\varphi(\alpha) = \max_{i \in \{1,2,3\}} |z_i(\alpha)|$. With the help of a formal computation software (Mathematica), we can find the roots $z_i(\alpha)$ for $i = 1, 2, 3$ as well as $\varphi(\alpha)$.

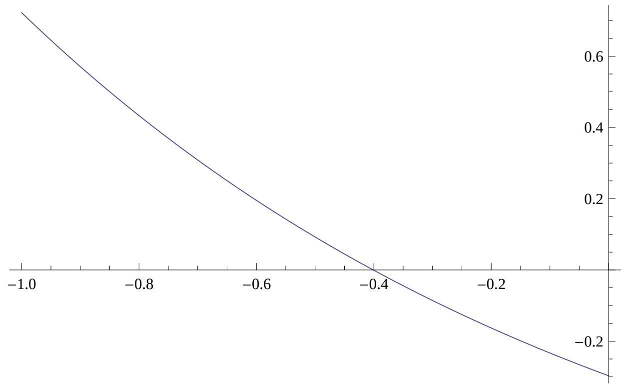

Figure 3. Graph of $\varphi(\alpha) - 1$ when $\int_0^1 a_1(x)dx = \int_0^1 a_2(x)dx = 1$, $\int_0^2 a_3(x)dx = \alpha$.

The explicit form of φ allows to check that when $\alpha > \alpha_0 = \frac{1}{2}\ln\left(\frac{3+e^2}{1+3e^2}\right)$ then $\varphi(\alpha) < 1$ (see Figure 3). Hence (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha \in (\alpha_0, 0]$.

The same study can be done when changing L_3, namely by taking $L_3 = \frac{1}{2}$ or $L_3 = 3$ and we surprisingly obtain the same critical value α_0 of α so that (S_1) is exponentially stable up to a finite dimensional space. Moreover, if we choose $L_1 = 1$ and $L_2 = 2$ such that $\int_0^1 a_1(x)dx = \int_0^2 a_2(x)dx = 1$, then for $L_3 = 1$ or $L_3 = 2$, we still obtain the same condition, $\alpha > \alpha_0 = \frac{1}{2}\ln\left(\frac{3+e^2}{1+3e^2}\right)$ to get the exponential stability of (S_1) up to a finite dimensional space. Furthermore, if we change the mean values, by considering $L_1 = L_2 = 1$, but

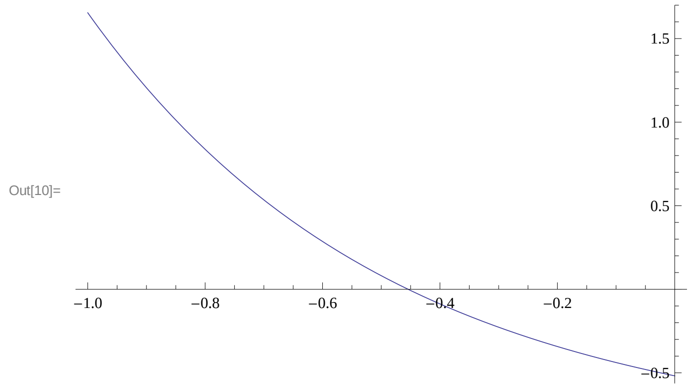

Figure 4. Graph of $\varphi(\alpha) - 1$ when $\int_0^1 a_1(x)dx = 1,\ \int_0^1 a_2(x)dx = 2,\ L_3 = 2.$

$\int_0^1 a_1(x)dx = 1$ and $\int_0^1 a_2(x)dx = 2$, then whether $L_3 = 1$ or $L_3 = 2$, we still obtain the same critical value α_1 with $0.45 < \alpha_1 < 0.46$ such that (S_1) is exponentially stable up to a finite dimensional space if and only if $\alpha > \alpha_1$ (see Figure 4).

In conclusion, we find that the critical value of α depends on $\int_0^{L_1} a_1(x)dx$ and $\int_0^{L_2} a_2(x)dx$ and not on the choice of the lengths. This opens the question whether the abstract condition given in Theorem 1.1 can be expressed explicitly in terms of $\int_0^{L_i} a_i(x)dx$ for all $i \in \{1, \cdots, N\}$, see Conjecture 1.2.

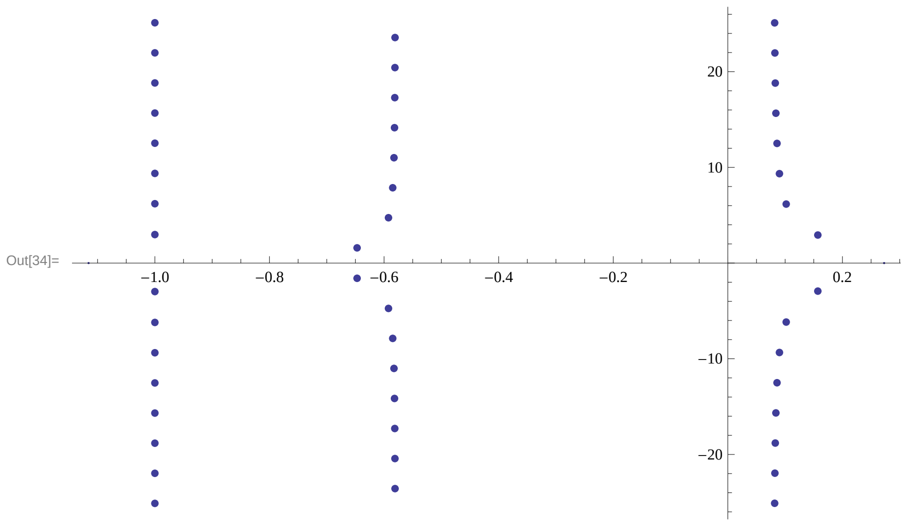

Figure 5. $a_1 = a_2 = 1,\ a_3 = -\frac{1}{2},\ \epsilon = 1.$

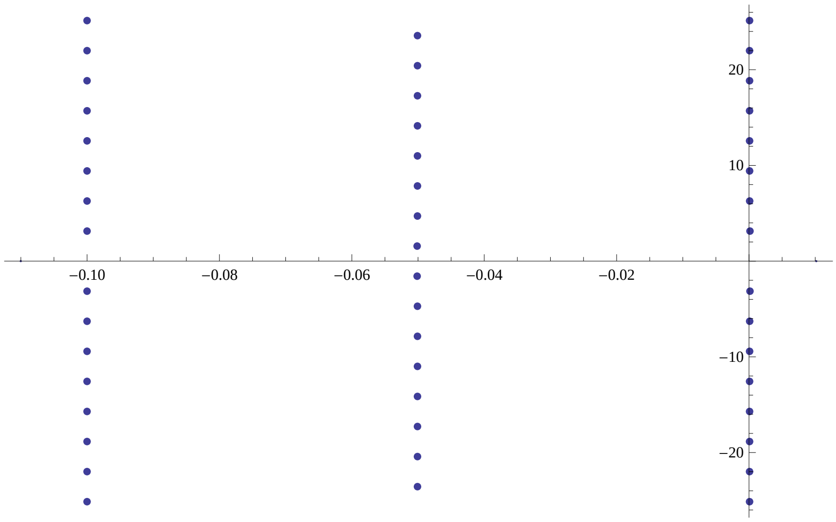

Figure 6. $a_1 = a_2 = 1,\ a_3 = -\dfrac{1}{2},\ \epsilon = 0.1.$

7.2 Examples for problem (S_ϵ)

We start with a limit case in Theorem 1.4, namely we take $N = 3$, $a_1 = a_2 = 1$, $a_3 = -\dfrac{1}{2}$, and $L_1 = L_2 = L_3 = 1$. Hence neither the first condition holds nor the second one since $\displaystyle\sum_{i=1}^{N} a_i > 0$ but $\displaystyle\sum_{i=1}^{N} \frac{1}{a_i} = 0$.

But Lemma 6.2 yields that for all $\epsilon > 0$ small enough, the eigenvalues are of the form

$$\begin{aligned}
\lambda_{1,k}(\epsilon) &= -\epsilon + \imath k\pi + o(\epsilon),\\
\lambda_{2,k}(\epsilon) &= \imath k\pi + o(\epsilon),\\
\lambda_{3,k}(\epsilon) &= -\frac{\epsilon}{2} + \imath\frac{(2k+1)\pi}{2} + o(\epsilon),
\end{aligned}$$

Hence the problem of stability would come from $\lambda_{2,k}(\epsilon)$ but a more precise asymptotic analysis yields $\Re\lambda_{2,k}(\epsilon) = \dfrac{\epsilon^3}{12} + o(\epsilon^3)$, hence the problem is not exponentially stable for ϵ small. Figure 6 shows the existence of a positive asymptote when $\epsilon = 0.1$, since the asymptotes are $x_1 = -0.1,\ x_2 \approx -0.0500833$, and $x_3 \approx 0.000083333$.

Note that for $\epsilon = 1$, then by Theorem 1.1 there is a positive asymptote, since the asymptotes are $x_1 = -1,\ x_2 \approx -0.580322$, and $x_3 \approx 0.0803219$ (see Figure 5).

In general, if we consider $a_1 = a_2 = a$ and $a_3 \neq a$, then according to Theorem 1.4, the problem becomes exponentially stable for all ϵ small enough if one of the following two conditions holds:

(i) $a > 0$ and $a_3 \geq 0$

(ii) $a > 0$ and $a_3 < 0$ such that $2a + a_3 > 0$ and $a + 2a_3 > 0$.

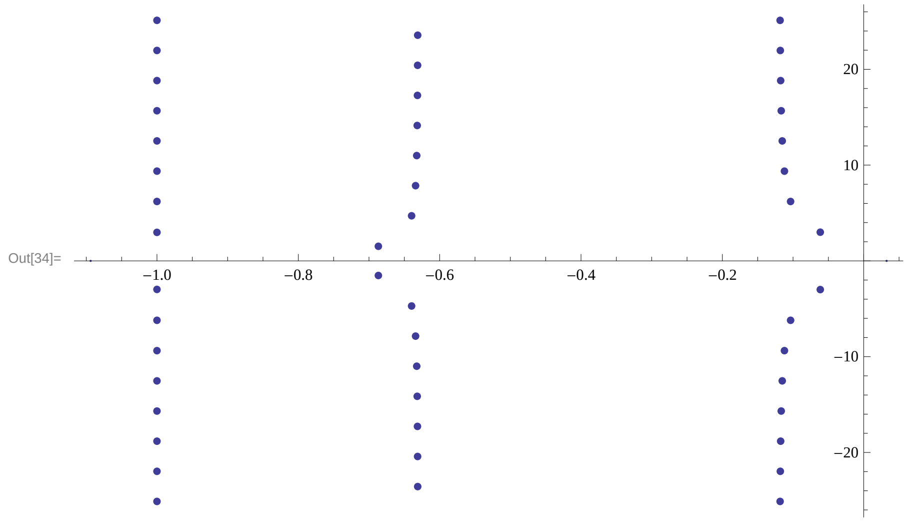

Figure 7. $a_1 = a_2 = 1,\ a_3 = -\dfrac{1}{4},\ \epsilon = 1.$

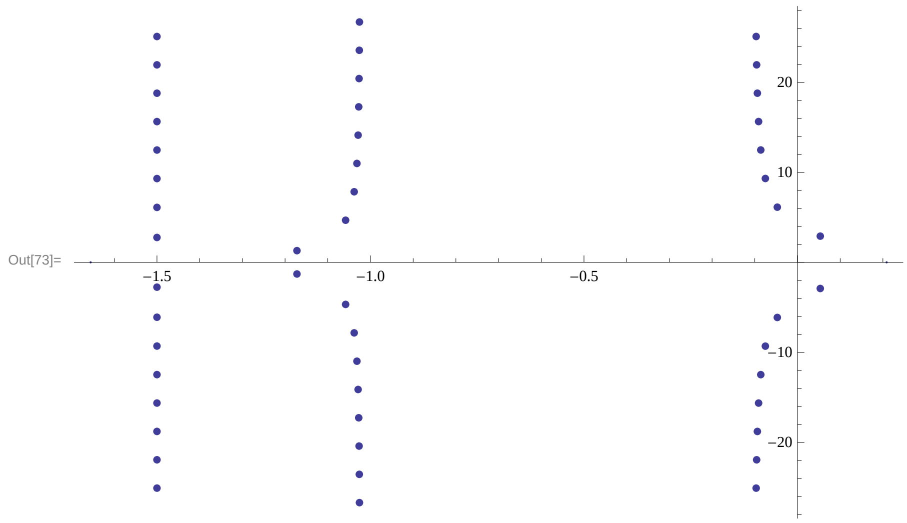

Figure 8. $a_1 = a_2 = 1,\ a_3 = -\dfrac{1}{4},\ \epsilon = 1.5.$

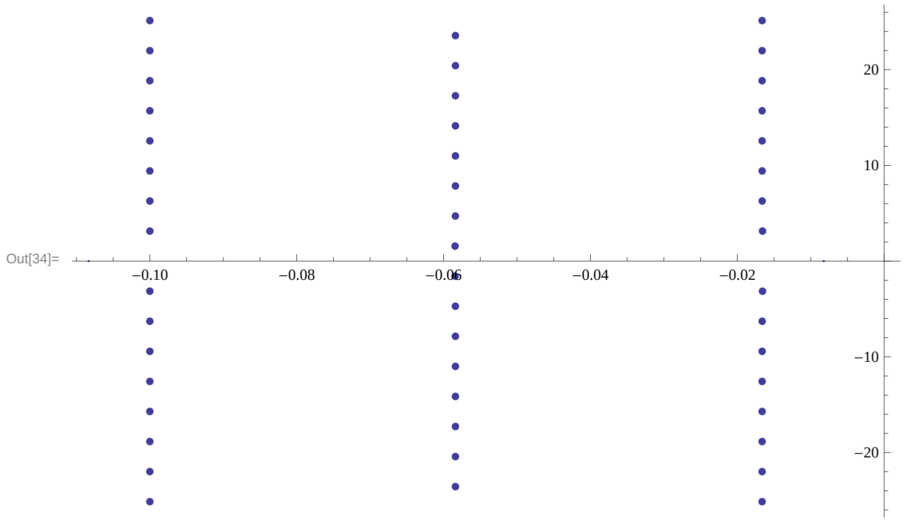

Figure 9. $a_1 = a_2 = 1,\ a_3 = -\frac{1}{4},\ \epsilon = 1/10.$

These results are coherent with the numerical results shown in Figure 9 with $a_1 = a_2 = 1,\ a_3 = -\frac{1}{4}$ and $\epsilon = \frac{1}{10}$ where the asymptotes ($x_1 = -0.1,\ x_2 \approx -0.0166184$, and $x_3 \approx -0.0583816$) are to the left of the imaginary axis. If we increase ϵ and take $\epsilon = 1$, then Figure 7 still shows the exponential stability in the whole energy space where the asymptotes are $x_1 = -1,\ x_2 \approx -0.630695$, and $x_3 \approx -0.119305$. But for $\epsilon = 1.5$, then Figure 8 shows the exponential stability up to a finite dimensional space. Indeed, the asymptotes found in Figure 8 are $x_1 = -1.5,\ x_2 \approx -1.02451$, and $x_3 \approx -0.100488$ which show that the large eigenvalues are to the left of the imaginary axis although there are some low eigenvalues with positive real parts. In fact, in the case $a_1 = a_2 = 1,\ a_3 = -\frac{1}{4}$, all the eigenvalues are to the left of the imaginary axis for all $\epsilon < \epsilon_0$, where numerically we have found that $1.30 < \epsilon_0 < 1.31$.

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Author information

Farah Abdallah, Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, Institut des Sciences et Techniques of Valenciennes, F-59313 - Valenciennes Cedex 9 France and Université Libanaise, Ecole Doctorale des Sciences et de Technologie, Hadath, Beyrouth, Liban, France.
E-mail: Farah.Abdallah@meletu.univ-valenciennes.fr

Denis Mercier, Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, Institut des Sciences et Techniques of Valenciennes, F-59313 - Valenciennes Cedex 9, France.
E-mail: denis.mercier@univ-valenciennes.fr

Serge Nicaise, Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, Institut des Sciences et Techniques of Valenciennes, F-59313 - Valenciennes Cedex 9, France.
E-mail: Serge.Nicaise@univ-valenciennes.fr

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