A General Boundary Value Problem For Impulsive Fractional Differential Equations

Hilmi Ergoren and Cemil Tunc

Abstract. In this study, we consider a boundary value problem for a class of impulsive fractional differential equations having general boundary conditions. We establish some sufficient conditions for the existence of solutions for the problem by using some standard fixed point theorems and give an illustrative example.

1 Introduction

This work deals with the existence and uniqueness of the solutions to the boundary value problem (BVP for short), for the following impulsive fractional differential equation,

$$\begin{cases} {}^C D^\alpha y(t) = f(t, y(t)), \ \ t \in J := [0, T], t \neq \theta_j, \ 1 < \alpha \leq 2, \\ \Delta y(\theta_j) = I_j(y(\theta_j^-)), \ \ \Delta y'(\theta_j) = I_j^*(y(\theta_j^-)), \ j = 1, 2, ..., p, \\ \quad y(0) = \lambda_1 y(T) + \lambda_2 y(\xi) + \lambda_3 \int_0^T w_1(s, y(s))ds + k_1, \\ \quad y'(0) = \beta_1 y'(T) + \beta_2 y'(\xi) + \beta_3 \int_0^T w_2(s, y(s))ds + k_2, \end{cases} \tag{1.1}$$

where ${}^C D^\alpha$ is the Caputo fractional derivative, $f : J \times R \to R$, $I_j, I_j^* : R \to R$, $\{\theta_j\}_{j=1}^p$ is a strictly increasing $B-$sequence(to be defined later) of impulse points θ_j such that $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_p < \theta_{p+1} = T$, $\Delta y(\theta_j) = y(\theta_j^+) - y(\theta_j^-)$ with $y(\theta_j^+) = \lim_{h \to 0^+} y(\theta_j + h)$, $y(\theta_j^-) = \lim_{h \to 0^-} y(\theta_j + h)$,and $\Delta y'(\theta_j)$ has a similar meaning for $y'(t)$, and $\lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_2, \beta_3, k_1, k_2$ are real constants with $\lambda_1 + \lambda_2 \neq 1$, $\beta_1 + \beta_2 \neq 1$.

The subject of fractional differential equations has been recently paid increasing attention and it is gaining much importance. That is why, the fractional derivatives serve an excellent tool for the description of hereditary properties of different materials and processes. Actually, fractional differential equations arise in many engineering and scientific disciplines such as, physics, chemistry, biology, electrochemistry, electromagnetic, control theory, economics, signal and image processing, aerodynamics, porous media, etc.(see [1, 2, 3, 4, 5, 6, 7, 8] and references therein).

On the other hand, theory of impulsive differential equations for integer order has become important and found its extensive applications in mathematical modeling of phenomena and practical situations in both physical and social sciences in recent years. One can see a remarkable development in impulsive theory. For instance, for the general theory and applications of impulsive differential equations we refer the readers to [9, 10, 11, 12].

Boundary value problems take place in the studies of fractional differential equations differently many times (see [13, 14, 15, 16, 17, 18, 19, 20] and the relevant references therein). More precisely, one can encounter some boundary value problems for impulsive fractional differential equations including periodic, anti periodic, hybrid, closed, nonlocal, integral boundary etc. conditions. We can refer the interested researchers to [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36].

As to this study, motivated by the mentioned recent works above, we form a class of general boundary value problem, which includes some afore-mentioned boundary conditions, for impulsive fractional differential equations as in (1.1) and investigate the existence and uniqueness of solutions to the BVP (1.1). We hope that this study will become a good generalization of some initial-boundary value problems for impulsive differential equations and make contribution to this emerging field.

The rest of the paper is organized as follows. In Section 2, we present some notations and preliminary results about fractional calculus and differential equations to be used in the following

sections. In Section 3, we discuss some existence and uniqueness results for solutions of BVP (1.1). Namely, the first one is based on O' Regan fixed point theorem , the second one is based on Banach's fixed point theorem. At the end, we give an illustrative example for our results.

2 Preliminaries

Definition 2.1. ([1, 2]) The fractional (arbitrary) order integral of the function $h \in L^1 (J, R_+)$ of order $\alpha \in R_+$ is defined by

$$I_{0^+}^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. ([1, 2]) For a function h given on the interval J, Caputo fractional derivative of order $\alpha > 0$ is defined by

$$^C D_{0^+}^\alpha h(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} h^{(n)}(s)ds, \ n = [\alpha] + 1,$$

where the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.3. *([1, 13]) Let $\alpha > 0$, then the differential equation*

$$^C D_{0^+}^\alpha h(t) = 0$$

has solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \ \ c_i \in R, \ i = 0, 1, 2, ..., n-1, \ n = [\alpha] + 1.$$

Lemma 2.4. *([1]) Let $\alpha > 0$, $h(t) \in C^n [a, b]$, then*

$$I_{a^+}^\alpha {}^C D_{a^+}^\alpha h(t) = h(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + ... + c_{n-1}(t-a)^{n-1},$$

for some $c_i \in R$, $i = 0, 1, 2, ..., n-1$, $n = [\alpha] + 1$.

Now, we introduce O' Regan fixed point theorem to be applied to prove our main results later.

Lemma 2.5. *([37]) Denote by U an open set in a closed, convex set Y of a Banach space E. Assume that $0 \in U$. Also assume that $F(\overline{U})$ is bounded and $F : \overline{U} \to Y$ is given by $F = F_1 + F_2$, in which $F_1 : \overline{U} \to E$ is continuous and completely continuous and $F_2 : \overline{U} \to E$ is a nonlinear contraction(i.e., there exists a nonnegative nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \leq \phi(\|x-y\|)$ for all $x, y \in \overline{U}$), then either*
    (C_1) F has a fixed point $u \in \overline{U}$, or
    (C_2) there exists a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$, where $\overline{U}$, ∂U represent the closure and boundary of U, respectively.

Definition 2.6. ([38])If U is a subset of a Banach space X and $T : U \to X$, then T is said to be completely continuous if T is continuous and for any bounded set $B \subseteq U$, the closure of TU is compact.

Denote by $\theta = \{\theta_j\}$ a collection of some impulse points mentioned above and a strictly increasing sequence of real numbers such that the set $\mathcal{A}$ of indexes j is an interval in Z.

Definition 2.7. ([39]) θ is a B-sequence, if one of the following alternatives is valid:
    (a) θ is empty,
    (b) θ is non-empty and finite set,
    (c) θ is an infinite set such that $|\theta_j| \to \infty$ as $|j| \to \infty$.

Let us set $J_0 = [\theta_0, \theta_1]$, $J_1 = (\theta_1, \theta_2], ..., J_{j-1} = (\theta_{j-1}, \theta_j]$, $J_j = (\theta_j, \theta_{i+1}]$, $J' := [0, T] \backslash \{\theta_1, \theta_2, ..., \theta_p\}$ and define the set of functions:
    $PC(J, R) = \{y : J \to R : y \in C((\theta_j, \theta_{j+1}], R), \ j = 0, 1, 2, ..., p$ and there exist $y(\theta_j^+)$ and $y(\theta_j^-)$, $j = 1, 2, ..., p$ with $y(\theta_j^-) = y(\theta_j)\}$ and
    $PC^1(J, R) = \{y \in PC(J, R), \ y' \in C((\theta_j, \theta_{j+1}], R), \ j = 0, 1, 2, ..., p$ and there exist $y'(\theta_j^+)$ and $y'(\theta_j^-)$, $j = 1, 2, ..., p$ with $y'(\theta_j^-) = y'(\theta_j)\}$ which is a Banach space with the norm $\|y\| = \sup_{t \in J} \left\{ \|y\|_{PC}, \|y'\|_{PC} \right\}$, where $\|y\|_{PC} := \sup \{|y(t)| : t \in J\}$.

3 Main Results

To begin with, in order to deal with BVP(1.1), we shall consider the following linear problem associated with BVP(1.1) and give its solution.

Lemma 3.1. *Let $\xi \in (\theta_l, \theta_{l+1})$; l is non-negative integer in $\mathcal{A}$ i.e. $0 \le l \le p$, $1 < \alpha \le 2$, $j = 1, 2, ..., p$ and $\sigma, q_1, q_2 : J \to R$ be continuous. A function $y(t) \in PC(J, R)$ is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + b_0 + b_1 t, \ if \ t \in J_0, \\ \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + I_i(y(\theta_i^-)) \right] \\ + \sum_{i=1}^j (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_i^*(y(\theta_i^-)) \right] + b_0 + b_1 t, \ if \ t \in J_j \end{cases} \tag{3.1}$$

if and only if $y(t)$ is a solution of the fractional BVP

$$\begin{cases} {}^C D^\alpha y(t) = \sigma(t), \ t \in J' \\ \Delta y(\theta_j) = I_j(y(\theta_j^-)), \ \Delta y'(\theta_j) = I_j^*(y(\theta_j^-)), \\ y(0) = \lambda_1 y(T) + \lambda_2 y(\xi) + \lambda_3 \int_0^T q_1(s)ds + k_1, \\ y'(0) = \beta_1 y'(T) + \beta_2 y'(\xi) + \beta_3 \int_0^T w_2(s, y(s))ds + k_2, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} b_0 = & \ \frac{-1}{\lambda_1 + \lambda_2 - 1} \left[\lambda_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + I_i(y(\theta_i^-)) \right] \right. \right. \\ & + \sum_{i=1}^p (T - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_i^*(y(\theta_i^-)) \right] \bigg) \\ & + \lambda_2 \left(\int_{\theta_l}^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + I_i(y(\theta_i^-)) \right] \right. \\ & + \sum_{i=1}^l (\xi - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_i^*(y(\theta_i^-)) \right] \bigg) \\ & + \lambda_3 \int_0^T w_1(s, y(s))ds + k_1 \bigg] - \frac{\lambda_1 T + \lambda_2 \xi}{(\lambda_1 + \lambda_2 - 1)} b_1 \end{aligned}$$

and

$$\begin{aligned} b_1 = & \ \frac{-1}{\beta_1 + \beta_2 - 1} \left[\beta_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\ & + \beta_2 \left(\int_{\theta_l}^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_i^*(y(\theta_i^-)) \right] \right) \\ & + \beta_3 \int_0^T w_2(s, y(s))ds + k_2 \bigg]. \end{aligned}$$

Proof. Let y be the solution of (3.2). If $t \in J_0 = [\theta_0, \theta_1]$, then Lemma 2.4 implies that

$$y(t) = I^\alpha \sigma(t) - c_0 - c_1 t = \int_{\theta_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - c_0 - c_1 t,$$

$$y'(t) = \int_{\theta_0}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds - c_1,$$

for some $c_0, c_1 \in R$.

If $t \in J_1 = (\theta_1, \theta_2]$, then Lemma 2.4 implies that

$$y(t) = \int_{\theta_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - d_0 - d_1(t - \theta_1),$$

$$y'(t) = \int_{\theta_1}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds - d_1$$

for some $d_0, d_1 \in R$. Thus we have

$$y(\theta_1^-) = \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - c_0 - c_1\theta_1, \ y(\theta_1^+) = -d_0,$$

$$y'(\theta_1^-) = \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds - c_1, \ y'(\theta_1^+) = -d_1.$$

In view of $\Delta y(\theta_1) = y(\theta_1^+) - y(\theta_1^-) = I_1(y(\theta_1^-))$ and $\Delta y'(\theta_1) = y'(\theta_1^+) - y'(\theta_1^-) = I_1^*(y(\theta_1^-))$, we have

$$-d_0 = \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - c_0 - c_1\theta_1 + I_1(y(\theta_1^-)),$$

$$-d_1 = \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds - c_1 + I_1^*(y(\theta_1^-)),$$

hence, for $t \in J_1$,

$$y(t) = \int_{\theta_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds$$

$$+ (t-\theta_1)\int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_1(y(\theta_1^-))$$

$$+ (t-\theta_1)I_1^*(y(\theta_1^-)) - c_0 - c_1t,$$

$$y'(t) = \int_{\theta_1}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \int_0^{t_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_1^*(y(\theta_1^-)) - c_1.$$

If $t \in J_2 = (\theta_2, \theta_3]$, then Lemma 2.4 implies that

$$y(t) = \int_{\theta_2}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - e_0 - e_1(t-\theta_2),$$

$$y'(t) = \int_{\theta_2}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds - e_1,$$

for some $e_0, e_1 \in R$. Thus we have

$$y(\theta_2^-) = \int_{\theta_1}^{\theta_2} \frac{(\theta_2-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds$$

$$+ (\theta_2-\theta_1)\int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_1(y(\theta_1^-))$$

$$+ (\theta_2-\theta_1)I_1^*(y(\theta_1^-)) - c_0 - c_1\theta_2,$$

$$y(\theta_2^+) = -e_0,$$

and

$$y'(\theta_2^-) = \int_{\theta_1}^{\theta_2} \frac{(\theta_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_1^*(y(\theta_1^-)) - c_1,$$

$$y'(\theta_2^+) = -e_1.$$

In view of $\Delta y(\theta_2) = y(\theta_2^+) - y(\theta_2^-) = I_2(y(\theta_2^-))$ and $\Delta y'(\theta_2) = y'(\theta_2^+) - y'(\theta_2^-) = I_2^*(y(\theta_2^-))$, we have

$$-e_0 = y(\theta_2^-) + I_2(y(\theta_2^-)),$$

$$-e_1 = y'(\theta_2^-) + I_2^*(y(\theta_2^-)),$$

hence, for $t \in J_2$,

$$
\begin{aligned}
y(t) &= \int_{\theta_2}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \int_{0}^{\theta_1} \frac{(\theta_1-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \int_{\theta_1}^{\theta_2} \frac{(\theta_2-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds \\
&\quad + (\theta_2 - \theta_1) \left[\int_{0}^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_1^*(y(\theta_1^-)) \right] \\
&\quad + (t - \theta_2) \left[\int_{0}^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + \int_{\theta_1}^{\theta_2} \frac{(\theta_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_1^*(y(\theta_1^-)) + I_2^*(y(\theta_2^-)) \right] \\
&\quad + I_1(y(\theta_1^-)) + I_2(y(\theta_2^-)) - c_0 - c_1 t,
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= \int_{\theta_2}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^{2} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \\
&\quad + \sum_{i=1}^{2} (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] - c_0 - c_1 t.
\end{aligned}
$$

By repeating the same process, if $t \in J_j$, then again from Lemma 2.4, we get

$$
y(t) = \begin{cases}
\int_{\theta_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^{j} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \\
+ \sum_{i=1}^{j} (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] - c_0 - c_1 t
\end{cases}
$$

and

$$
y'(t) = \begin{cases}
\int_{\theta_j}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds \\
+ \sum_{i=1}^{j} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] - c_1.
\end{cases}
$$

Applying the conditions $y(0) = \lambda_1 y(T) + \lambda_2 y(\xi) + \lambda_3 \int_0^T w_1(s, y(s)) ds + k_1$, $y'(0) = \beta_1 y'(T) + \beta_2 y'(\xi) + \beta_3 \int_0^T w_2(s, y(s)) ds + k_2$ we have

$$
\begin{aligned}
c_1 &= \frac{1}{\beta_1 + \beta_2 - 1} \left[\beta_1 \left(\int_{\theta_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + \sum_{i=1}^{p} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\
&\quad + \beta_2 \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + \sum_{i=1}^{l} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \\
&\quad \left. + \beta_3 \int_0^T w_2(s, y(s)) ds + k_2 \right]
\end{aligned}
$$

and

$$
\begin{aligned}
c_0 &= \frac{1}{\lambda_1 + \lambda_2 - 1} \left[\lambda_1 \left(\int_{\theta_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^{p} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \right. \right. \\
&\quad \left. + \sum_{i=1}^{p} (T - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \\
&\quad + \lambda_2 \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^{l} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \right. \\
&\quad \left. + \sum_{i=1}^{l} (\xi - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \\
&\quad \left. + \lambda_3 \int_0^T w_1(s, y(s)) ds + k_1 \right] - \frac{\lambda_1 T + \lambda_2 \xi}{(\lambda_1 + \lambda_2 - 1)} c_1.
\end{aligned}
$$

Then, replacing $-c_0$ and $-c_1$ with b_0 and b_1 respectively, we obtain (3.1).

Conversely, assume that y satisfies the impulsive fractional integral equation (3.1). Then, in view of the Caputo differentiation (2.2) and the property $(^C D^\alpha I^\alpha \sigma)(t) = \sigma(t)$, $\alpha > 0$ ([1], Lemma 2.21), it is obtained that $^C D^\alpha y(t) = \sigma(t)$ in (3.2). Moreover, it can be easily verified that Eq. (3.1) holds the boundary conditions in (3.2). Therefore, the solution y given by (3.1) satisfies (3.2). The proof is complete.   □

Definition 3.2. A function $y \in PC^1(J, R)$ with its $\alpha-$derivative existing on J' is said to be a solution of (1.1) if y satisfies the equation $^C D^\alpha y(t) = f(t, y(t))$ on J' and the conditions

$$\Delta y(\theta_j) = I_j(y(\theta_j^-)), \ \Delta y'(\theta_j) = I_j^*(y(\theta_j^-)),$$
$$y(0) = \lambda_1 y(T) + \lambda_2 y(\xi) + \lambda_3 \int_0^T w_1(s, y(s))ds + k_1,$$
$$y'(0) = \beta_1 y'(T) + \beta_2 y'(\xi) + \beta_3 \int_0^T w_2(s, y(s))ds + k_2$$

are satisfied for y.

Let $\xi \in (\theta_l, \theta_{l+1})$; l is a non-negative integer in $\mathcal{A}$ i.e., $0 \le l \le p$. In view of (3.1) and Definition 3.2, let us define the operator $T : PC^1(J, R) \to PC^1(J, R)$ by

$$(Ty)(t) = \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds + I_i(y(\theta_i^-)) \right]$$

$$+ \sum_{i=1}^j (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s))ds + I_i^*(y(\theta_i^-)) \right]$$

$$\frac{-1}{\lambda_1 + \lambda_2 - 1} \left\{ \lambda_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds + I_i(y(\theta_i^-)) \right] \right.$$

$$+ \sum_{i=1}^p (T - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s))ds + I_i^*(y(\theta_i^-)) \right] \right)$$

$$+ \lambda_2 \left(\int_{\theta_l}^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds + I_i(y(\theta_i^-)) \right] \right.$$

$$+ \sum_{i=1}^l (\xi - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s))ds + I_i^*(y(\theta_i^-)) \right] \right)$$

$$+ \lambda_3 \int_0^T w_1(s, y(s))ds + k_1 \right\} + \frac{\lambda_1 (T-t) + \lambda_2 (\xi - t) + t}{(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)} \times$$

$$\left\{ \beta_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s))ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s))ds + I_i^*(y(\theta_i^-)) \right] \right) \right.$$

$$+ \beta_2 \left(\int_{\theta_l}^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s))ds + \sum_{i=1}^l \left[\int_{t_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s))ds + I_i^*(y(\theta_i^-)) \right] \right)$$

$$+ \beta_3 \int_0^T w_2(s, y(s))ds + k_2 \right\}, \ if \ t \in J_j, \qquad (3.3)$$

where $T = T_1 + T_2$ with

$$(T_1y)(t) = \begin{cases} \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds + I_i(y(\theta_i^-)) \right] \\ \quad + \sum_{i=1}^j (t-\theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s))ds + I_i^*(y(\theta_i^-)) \right] \\ \quad - \frac{1}{\lambda_1+\lambda_2-1} \left\{ \lambda_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds + I_i(y(\theta_i^-)) \right] \right. \right. \\ \qquad + \sum_{i=1}^p (T-\theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s))ds + I_i^*(y(\theta_i^-)) \right] \bigg) \\ \qquad + \lambda_2 \left(\int_{\theta_l}^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds + I_i(y(\theta_i^-)) \right] \right. \\ \qquad + \sum_{i=1}^l (\xi-\theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s))ds + I_i^*(y(\theta_i^-)) \right] \bigg) \right\} \\ \quad + \frac{\lambda_1(T-t)+\lambda_2(\xi-t)+t}{(\lambda_1+\lambda_2-1)(\beta_1+\beta_2-1)} \times \\ \quad \left\{ \beta_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s))ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s))ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\ \quad + \beta_2 \left(\int_{\theta_l}^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s))ds + \sum_{i=1}^l \left[\int_{t_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s))ds + I_i^*(y(\theta_i^-)) \right] \right) \right\} \end{cases}$$

and

$$(T_2y)(t) = \begin{cases} \frac{-1}{\lambda_1+\lambda_2-1} \left(\lambda_3 \int_0^T w_1(s,y(s))ds + k_1 \right) \\ + \frac{\lambda_1(T-t)+\lambda_2(\xi-t)+t}{(\lambda_1+\lambda_2-1)(\beta_1+\beta_2-1)} \left(\beta_3 \int_0^T w_2(s,y(s))ds + k_2 \right). \end{cases}$$

In order to establish our following main results, we first define $B_r = \{y \in PC^1(J,R) : \|y\| < r\}$ for any $r > 0$.

3.1 Existence of solutions

Lemma 3.3. *Assume that*

A1) $f : J \times R \to R$ *is continuous and there exists a constant* $M_1 > 0$ *such that* $|f(t,u)| \le M_1$, $\forall t \in J$ *and* $\forall u \in R$,

A2) $I_j, I_j^* : R \to R$ *are continuous and there exist constants* $M_2 > 0$ *and* $M_3 > 0$ *such that* $|I_j(u)| \le M_2$, $\left|I_j^*(u)\right| \le M_3$ *for all* $u \in R$ *and* $j = 1, 2, ..., p$.

Then, $T_1 : \overline{B}_r \to PC^1(J,R)$ *is completely continuous.*

Proof. First, note that the continuity of f, w_1, w_2, I_j and I_j^*, $j = 1, 2, ..., p$, ensure the continuity of T_1.

Now, in view of (A1) and (A2), for each $y \in \overline{B}_r$, we have

$$|(T_1y)(t)| \le \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,y(s))|\, ds + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,y(s))|\, ds + \left|I_i(y(\theta_i^-))\right| \right]$$

$$+ \sum_{i=1}^j (t-\theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))|\, ds + \left|I_i^*(y(\theta_i^-))\right| \right]$$

$$+ \frac{1}{|\lambda_1+\lambda_2-1|} \left\{ |\lambda_1| \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,y(s))|\, ds \right. \right.$$

$$+ \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,y(s))|\, ds + \left|I_i(y(\theta_i^-))\right| \right]$$

$$+ \sum_{i=1}^p (T-\theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))|\, ds + \left|I_i^*(y(\theta_i^-))\right| \right] \bigg)$$

$$+ |\lambda_2| \left(\int_{\theta_l}^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,y(s))|\, ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,y(s))|\, ds + \left|I_i(y(\theta_i^-))\right| \right] \right.$$

$$+ \sum_{i=1}^l (\xi-\theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))|\, ds + \left|I_i^*(y(\theta_i^-))\right| \right] \bigg) \right\}$$

$$+ \left| \frac{\lambda_1 (T - t) + \lambda_2 (\xi - t) + t}{(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)} \right| \times \left\{ |\beta_1| \left(\int_{\theta_p}^{T} \frac{(T - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} |f(s, y(s))|\, ds \right. \right.$$

$$\left. + \sum_{i=1}^{p} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} |f(s, y(s))|\, ds + |I_i^*(y(\theta_i^-))| \right] \right)$$

$$+ |\beta_2| \left(\int_{\theta_l}^{\xi} \frac{(\xi - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} |f(s, y(s))|\, ds + \sum_{i=1}^{l} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} |f(s, y(s))|\, ds + |I_i^*(y(\theta_i^-))| \right] \right) \right\},$$

$$|(T_1 y)(t)| \leq M_1 \int_{\theta_j}^{t} \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} ds + \sum_{i=1}^{k} \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha - 1}}{\Gamma(\alpha)} ds + M_2 \right]$$

$$+ \sum_{i=1}^{k} (t - \theta_i) \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} ds + M_3 \right]$$

$$+ \frac{1}{|\lambda_1 + \lambda_2 - 1|} \left\{ |\lambda_1| \left(M_1 \int_{\theta_p}^{T} \frac{(T - s)^{\alpha - 1}}{\Gamma(\alpha)} ds + \sum_{i=1}^{p} \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha - 1}}{\Gamma(\alpha)} ds + M_2 \right] \right. \right.$$

$$\left. + \sum_{i=1}^{p} (T - \theta_i) \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} ds + M_3 \right] \right)$$

$$+ |\lambda_2| \left(M_1 \int_{\theta_l}^{\xi} \frac{(\xi - s)^{\alpha - 1}}{\Gamma(\alpha)} ds + \sum_{i=1}^{l} \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha - 1}}{\Gamma(\alpha)} ds + M_2 \right] \right.$$

$$\left. \left. + \sum_{i=1}^{l} (\xi - \theta_i) \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} ds + M_3 \right] \right) \right\}$$

$$+ \left| \frac{\lambda_1 (T - t) + \lambda_2 (\xi - t) + t}{(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)} \right| \times$$

$$\left\{ |\beta_1| \left(M_1 \int_{\theta_p}^{T} \frac{(T - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} ds + \sum_{i=1}^{p} \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} ds + M_3 \right] \right) \right.$$

$$\left. + |\beta_2| \left(M_1 \int_{\theta_l}^{\xi} \frac{(\xi - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} ds + \sum_{i=1}^{l} \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} ds + M_3 \right] \right) \right\},$$

$$
\begin{aligned}
|(T_1 y)(t)| \quad \leq \quad & \left(\frac{M_1 T^{\alpha}(1 + p + \alpha p)}{\Gamma(\alpha + 1)} + M_2 p + M_3 p T \right) \left(\frac{2|\lambda_1| + 2|\lambda_2| + 1}{|\lambda_1 + \lambda_2 - 1|} \right) \\
& + \left(\frac{M_1 T^{\alpha - 1}(1 + p)}{\Gamma(\alpha)} + M_3 p \right) \left(\frac{(2|\lambda_1| + 2|\lambda_2| + 1)(|\beta_1| + |\beta_2|)}{|\lambda_1 + \lambda_2 - 1||\beta_1 + \beta_2 - 1|} \right) \\
\vdots \quad = \quad & K.
\end{aligned}
$$

Thus,

$$\|T_1 y\| \leq K \tag{3.4}$$

which implies that T_1 maps bounded sets into bounded sets.

Moreover, let $y \in \overline{B}_r$, for each $t \in J_j$, $0 \le j \le p$, we have

$$
\begin{aligned}
|(T_1 y)'(t)| \ \le\ & \int_{\theta_j}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))| \, ds \\
& + \sum_{i=1}^{j} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))| \, ds + \left| I_i^*(y(\theta_i^-)) \right| \right] \\
& \frac{1}{|\beta_1 + \beta_2 - 1|} \times \left\{ |\beta_1| \left(\int_{\theta_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))| \, ds \right. \right. \\
& + \sum_{i=1}^{p} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))| \, ds + \left| I_i^*(y(\theta_i^-)) \right| \right] \bigg) \\
& + |\beta_2| \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))| \, ds \right. \\
& + \sum_{i=1}^{l} \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))| \, ds + \left| I_i^*(y(\theta_i^-)) \right| \right] \bigg) \bigg\} \\
\le\ & \left(\frac{M_1 T^{\alpha-1}(1+p)}{\Gamma(\alpha)} + M_3 p \right) \left(1 + \frac{(|\beta_1| + |\beta_2|)}{|\beta_1 + \beta_2 - 1|} \right) \\
:=\ & \widetilde{K}.
\end{aligned}
$$

Now, letting $t_1, t_2 \in J_j$, $t_1 < t_2$, $0 \le j \le p$, we have

$$
|(T_1 y)(t_2) - (T_1 y)(t_1)| \le \int_{t_1}^{t_2} |(T_1 y)'(s)| \, ds \le \widetilde{K}(t_2 - t_1)
$$

which implies that T_1 maps bounded sets into equicontinuous sets. Namely, $T_1(\overline{B}_r)$ is equicontinuous on all subintervals $J_j (j = 0, 1, 2, ..., p)$. So, by Arzela-Ascoli Theorem and Definition 2.6, we conclude that the operator T_1 is completely continuous. $\qquad\square$

Theorem 3.4. *In addition to (A1) - (A2), assume that the following conditions are satisfied:*

A3) $w_1, w_2 : J \times R \to R$ are continuous and there exist constants $M_4 > 0$ and $M_5 > 0$ such that $|w_1(t,u)| \le M_4$, $|w_2(t,u)| \le M_5$ $\forall t \in J$ and $\forall u \in R$,

A4) There exist constants $N_1 > 0$ and $N_2 > 0$ such that $|w_1(t,u) - w_1(t,v)| \le N_1$, $|w_2(t,u) - w_2(t,v)| \le N_2$ for each $u, v \in R$,

A5) There exists a constant $L > 0$ such that

$$
\frac{N_1 T |\lambda_3|}{|\lambda_1 + \lambda_2 - 1|} + \frac{N_2 T |\beta_3| (2|\lambda_1| + 2|\lambda_2| + 1)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} =: L < 1.
$$

Then, BVP (1.1) has at least one solution on J.

Proof. In order to show the existence of the solutions BVP (1.1), we need to transform BVP (1.1) to a fixed point problem by using the operator T such that $T = T_1 + T_2$. Now, we shall use O'Regan fixed point theorem in Lemma 2.5 to prove that T has a fixed point which is then a solution of BVP (1.1). First, let us consider the set $B_r = \{y \in PC^1(J,R) : \|y\| < r\}$ defined above.

In view of O'Regan fixed point theorem, let us take $U = B_r$, $Y = \overline{B}_r$ and $E = PC^1(J,R)$. Then, it is obvious that $Y = \overline{B}_r$ is closed and convex where $U \subset Y$. From Lemma 3.3, it is proven that $T_1 : \overline{U} \to E$ i.e., $T_1 : \overline{B}_r \to PC^1(J,R)$, is completely continuous.

In order to show the boundedness of the operator $T : \overline{U} \to Y$ that is, $T : \overline{B}_r \to \overline{B}_r$, it is enough that show that the boundedness of the operator T_2 since it is proven that $T_1 : \overline{B}_r \to$

$PC^1(J,R)$ is bounded in Lemma 3.3. Then, for $y \in \overline{B}_r$, we have

$$
\begin{aligned}
|(T_2 y)(t)| &\leq \frac{1}{|\lambda_1 + \lambda_2 - 1|} \left(|\lambda_3| \int_0^T |w_1(s, y(s))| \, ds + |k_1| \right) \\
&\quad + \frac{|\lambda_1 (T - t) + \lambda_2 (\xi - t) + t|}{|(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)|} \left(|\beta_3| \int_0^T |w_2(s, y(s))| \, ds + |k_2| \right),
\end{aligned}
$$

$$
|(T_2 y)(t)| \leq \frac{M_4 T |\lambda_3| + |k_1|}{|\lambda_1 + \lambda_2 - 1|} + \frac{(M_5 T |\beta_3| + |k_2|)(2 |\lambda_1| + 2 |\lambda_2| + 1)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} := H,
$$

$$
\|T_2 y\| \leq H. \tag{3.5}
$$

Hence, $T_2 : \overline{B}_r \to PC^1(J,R)$ is bounded.

Now, taking into account of (3.4) and (3.5), provided that

$$
r \geq K + H, \tag{3.6}
$$

we obtain that the operator $T : \overline{U} \to Y$ that is, $T : \overline{B}_r \to \overline{B}_r$ is bounded.

Now, let us show that $T_2 : \overline{U} \to E$ is nonlinear contraction. For x and $y \in \overline{B}_r$, we have

$$
\begin{aligned}
|(T_2 x)(t) - (T_2 y)(t)| &\leq \frac{|\lambda_3|}{|\lambda_1 + \lambda_2 - 1|} \int_0^T |w_1(s, x(s)) - w_1(s, y(s))| \, ds \\
&\quad + |\beta_3| \frac{|\lambda_1 (T - t) + \lambda_2 (\xi - t) + t|}{|(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)|} \int_0^T |w_2(s, x(s)) - w_2(s, y(s))| \, ds \\
&\leq \frac{N_1 T |\lambda_3|}{|\lambda_1 + \lambda_2 - 1|} + \frac{N_2 T |\beta_3| (2 |\lambda_1| + 2 |\lambda_2| + 1)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} \|x - y\|,
\end{aligned}
$$

$$
\|T_2 x - T_2 y\| \leq L \|x - y\|.
$$

Let us choose $\phi(z) = Lz$. Since $L < 1$ from *(A5)*, we obtain for $\forall x, y \in \overline{B}_r$,

$$
\|T_2 x - T_2 y\| \leq \phi(\|x - y\|),
$$

that is, the operator T_2 is nonlinear contraction.

At the end, assume that the hypothesis (C_2) in Lemma 2.5 is valid, then there exist $u \in \partial B_r$ and $\lambda \in (0,1)$ such that $u = \lambda T(u)$. Thus, we get $\|u\| = r$ and

$$
\begin{aligned}
|u| &= |\lambda| |Tu|, \\
\|u\| &= r < \|Tu\| \leq K + H, \\
r &< K + H,
\end{aligned}
$$

which contradict with (3.6).

Therefore, T has a fixed point $u \in \overline{B}_r$ which is the solution of BVP (1.1). □

Now, we will give some sufficient conditions for the uniqueness of the solutions of BVP (1.1).

3.2 Uniqueness of solutions

Theorem 3.5. *In addition to (A1) - (A5), assume that the following conditions are satisfied:*

(A6) There exists a constant $L_1 > 0$ such that $|f(t, u) - f(t, v)| \leq L_1 |u - v|$, $\forall t \in J$, and $u, v \in R$,

(A7) There exist constants $L_2 > 0$, $L_3 > 0$ such that $|I_j(u) - I_j(v)| \leq L_2 |u - v|$, $\left| I_j^*(u) - I_j^*(v) \right| \leq L_3 |u - v|$ for each $u, v \in R$ and $j = 1, 2, ..., p$.

Moreover,

$$
\begin{aligned}
&\left(\frac{L_1 T^\alpha (1 + p + \alpha p)}{\Gamma(\alpha + 1)} + L_2 p + L_3 p T \right) \left(\frac{2 |\lambda_1| + 2 |\lambda_2| + 1}{|\lambda_1 + \lambda_2 - 1|} \right) \\
&+ \left(\frac{L_1 T^{\alpha - 1} (1 + p)}{\Gamma(\alpha)} + L_3 p \right) \left(\frac{(2 |\lambda_1| + 2 |\lambda_2| + 1)(|\beta_1| + |\beta_2|)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} \right) \\
&+ \frac{N_1 T |\lambda_3|}{|\lambda_1 + \lambda_2 - 1|} + \frac{N_2 T |\beta_3| (2 |\lambda_1| + 2 |\lambda_2| + 1)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} \\
&: \; = \Omega_{L_1, L_2, L_3, L_4, L_5, \lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_2, \beta_3} < 1. \tag{3.7}
\end{aligned}
$$

Then, BVP (1.1) has a unique solution on J.

Proof. In view of 3.6, choosing $r = K + H$, we can easily show that $T\overline{B}_r \subset \overline{B}_r$, where $\overline{B}_r = \{y \in PC(J, R) : \|y\| \leq r\}$.

Now, for $x, y \in PC^1(J, R)$ and for each $t \in J_j$, we obtain

$$|(Tx)(t) - (Ty)(t)| \leq \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| \, ds$$

$$+ \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| \, ds + \left| I_i(x(\theta_i^-)) - I_i(y(\theta_i^-)) \right| \right]$$

$$+ \sum_{i=1}^j (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| \, ds + \left| I_i^*(x(\theta_i^-)) - I_i^*(y(\theta_i^-)) \right| \right]$$

$$+ \frac{1}{|\lambda_1 + \lambda_2 - 1|} \left\{ |\lambda_1| \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| \, ds \right. \right.$$

$$+ \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| \, ds + \left| I_i(x(\theta_i^-)) - I_i(y(\theta_i^-)) \right| \right]$$

$$+ \sum_{i=1}^p (T - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| \, ds + \left| I_i^*(x(\theta_i^-)) - I_i^*(y(\theta_i^-)) \right| \right] \right)$$

$$+ |\lambda_2| \left(\int_{\theta_l}^\xi \frac{(\xi - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| \, ds \right.$$

$$+ \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| \, ds + \left| I_i(x(\theta_i^-)) - I_i(y(\theta_i^-)) \right| \right]$$

$$+ \sum_{i=1}^l (\xi - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| \, ds + \left| I_i^*(x(\theta_i^-)) - I_i^*(y(\theta_i^-)) \right| \right] \right)$$

$$+ |\lambda_3| \int_0^T |w_1(s, x(s)) - w_1(s, y(s))| \, ds \right\} + \left| \frac{\lambda_1 (T - t) + \lambda_2 (\xi - t) + t}{(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)} \right| \times$$

$$\left\{ |\beta_1| \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| \, ds \right. \right.$$

$$+ \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| \, ds + \left| I_i^*(x(\theta_i^-)) - I_i^*(y(\theta_i^-)) \right| \right] \right)$$

$$+ |\beta_2| \left(\int_{\theta_l}^\xi \frac{(\xi - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| \, ds \right.$$

$$+ \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| \, ds + \left| I_i^*(x(\theta_i^-)) - I_i^*(y(\theta_i^-)) \right| \right] \right)$$

$$+ |\beta_3| \int_0^T |w_2(s, x(s)) - w_2(s, y(s))| \, ds \right\}, t \in J_j.$$

Then, in view of assumptions *(A5)-(A7)*, we have

$$\|Tx - Ty\| \leq \left\{ \left(\frac{L_1 T^\alpha (1 + p + \alpha p)}{\Gamma(\alpha + 1)} + L_2 p + L_3 p T \right) \left(\frac{2 |\lambda_1| + 2 |\lambda_2| + 1}{|\lambda_1 + \lambda_2 - 1|} \right) \right.$$

$$+ \left(\frac{L_1 T^{\alpha-1}(1 + p)}{\Gamma(\alpha)} + L_3 p \right) \left(\frac{(2 |\lambda_1| + 2 |\lambda_2| + 1)(|\beta_1| + |\beta_2|)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} \right)$$

$$+ \frac{N_1 T |\lambda_3|}{|\lambda_1 + \lambda_2 - 1|} + \frac{N_2 T |\beta_3| (2 |\lambda_1| + 2 |\lambda_2| + 1)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} \right\} \|x - y\|$$

$$\leq \Omega_{L_1, L_2, L_3, L_4, L_5, \lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_2, \beta_3} \|x - y\|.$$

Therefore, by (3.7) and thanks to Banach's fixed point theorem, the operator T is contraction mapping. Consequently, BVP (1.1) has a unique solution. $\square$

Remark 3.6. Indeed, the boundary value problem we dealt with in (1.1) consists of various kinds of initial and boundary conditions. For instance, when $\lambda_1 = \lambda_2 = \lambda_3 = \beta_1 = \beta_2 = \beta_3 = 0$ holds in (1.1), one can obtain an initial value problem (see [22]); when $\lambda_1 = \beta_1 = 1$ and $\lambda_2 = \lambda_3 = \beta_2 = \beta_3 = k_1 = k_2 = 0$ hold in (1.1), one can obtain a periodic boundary value problem; when $\lambda_1 = \beta_1 = -1$ and $\lambda_2 = \lambda_3 = \beta_2 = \beta_3 = k_1 = k_2 = 0$ hold in (1.1), one can obtain an anti-periodic boundary value problem (see [40, 41]); when $\lambda_2 \neq 0$, $\beta_2 \neq 0$ holds in (1.1), one can obtain a non-local boundary value problem (see [42]); when $\lambda_3 \neq 0$ and $\beta_3 \neq 0$ hold in (1.1), one can obtain an integral boundary value problem (see [24]); when $\lambda_1 \neq 1$ and $\beta_1 \neq 1$ hold in (1.1), one can obtain a non-separated boundary value problem (see [43]).

Example 3.7. Consider the following impulsive fractional boundary value problem

$$
\begin{aligned}
{}^{C}D^{\frac{3}{2}}y(t) &= \frac{(\sin 5t)\,|y(t)|}{(t+5)^3(1+|y(t)|)}, \ t \in [0,1], \ t \neq \frac{1}{2}, \\
\Delta y(\tfrac{1}{2}) &= \frac{\left|y(\frac{1}{2}^-)\right|}{150 + \left|y(\frac{1}{2}^-)\right|}, \ \Delta y^{'}(\frac{1}{2}) = \frac{\left|y(\frac{1}{2}^-)\right|}{200 + \left|y(\frac{1}{2}^-)\right|}, \\
y(0) &= 2y(1) + 3y(\xi) - \int_0^1 e^{-\frac{1}{50}y(s)}ds + \frac{1}{6}, \\
y'(0) &= y'(1) - 2y'(\xi) + \frac{1}{5}\int_0^T e^{-\frac{1}{25}y(s)}ds - \frac{1}{9}, \quad (3.8)
\end{aligned}
$$

where $0 < \xi < 1, \xi \neq \frac{1}{2}$.

Here, according to BVP (1.1), $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = -1$, $\beta_1 = 1$, $\beta_2 = -2$, $\beta_3 = -\frac{1}{5}$, $\alpha = \frac{3}{2}$, $k_1 = \frac{1}{6}$, $k_2 = \frac{-1}{9}$, $T = 1$, $p = 1$. Obviously, $L_1 = \frac{1}{125}$, $L_2 = \frac{1}{150}$, $L_3 = \frac{1}{200}$, $N_1 = \frac{1}{50}$, $N_2 = \frac{1}{25}$ and by (3.7), it can be find that

$$
\Omega_{L_1,L_2,L_3,N_1,N_2,\lambda_1,\lambda_2,\lambda_3,\beta_1,\beta_2,\beta_3} = \frac{88}{375\sqrt{\pi}} + \frac{1649}{24\,000} = 0.201\,1 < 1.
$$

Therefore, due to fact that all the assumptions of Theorem 3.5 hold, the BVP (3.8) has a unique solution. Besides, one can easily check the result of Theorem 3.4 for the BVP (3.8).

References

[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[2] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[3] N. Heymans and I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann Liouville fractional derivatives, Rheol. Acta.,45 (5) (2006) 765–772.

[4] L. Gaul, P. Klein, S. Kempfle, Damping description involving fractional operators, Mech. Syst. Signal Process. 5 (1991) 81–88.

[5] W.G. Glockle, T.F. Nonnenmacher, A fractional calculus approach of self-similar protein dynamics, Biophys. J. 68 (1995) 46–53.

[6] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[7] F. Mainardi, Fractional calculus, Some basic problems in continuum and statistical mechanics, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien, 1997, pp. 291–348.

[8] F. Metzler, W. Schick, H.G. Kilian, T.F. Nonnenmache, Relaxation in filled polymers: a fractional calculus approach, J. Chem. Phys. 103 (1995) 7180–7186.

[9] M. Benchohra, J. Henderson, and S. Ntouyas, Impulsive Differential Equations and Inclusions, Hindawi Publishing Corporation, 2006.

[10] A.M. Samoilenko and N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.

[11] Y.V. Rogovchenko, Impulsive evolution systems: main results and new trends, Dyn. Contin. Discrete Impuls. Syst. 3 (1997) 57–88.

[12] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[13] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, Electron. J. Differential Equations 36 (2006) 1–12.

[14] M. Benchohra, S. Hamani, and S. K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear Anal., 71, no. 7-8, (2009) 2391–2396.

[15] W. Zhong and W. Lin, Nonlocal and multiple-point boundary value problem for fractional differential equations, Comput. Math. Appl., 59 (3) (2010) 1345–1351.

[16] B. Ahmad and S. Sivasundaram, On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order, Appl. Math. Comput., 217(2) (2010), 480–487.

[17] B. Ahmad and J.J. Nieto, Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray–Schauder degree theory, Topol. Methods Nonlinear Anal., 35 (2010) 295–304.

[18] R. P. Agarwal, M. Benchohra and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta. Appl. Math., 109 (3) (2010), 973–1033.

[19] B. Ahmad, J.J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions. Bound. Value Probl. 2009, Article ID 708576 (2009).

[20] H. Ergören, On The Lidstone Boundary Value Problems for Fractional Differential Equations, International Journal of Mathematics and Computations, 22(1): 66-74, 2014.

[21] M. Benchohra and B.A. Slimani, Existence and uniqueness of solutions to impulsive fractional differential equations, Electron. J. Differential Equations 2009(2009), no. 10, pp. 1–11.

[22] Agarwal, R. P., Benchohra, M., Slimani, B. A., 2008. Existence results for differential equations with fractional order and impulses, Mem. Differential Equations Math. Phys., 44: 1 – 21.

[23] L. Yang and H.Chen, Nonlocal Boundary Value Problem for Impulsive Differential Equations of Fractional Order, Advances in Difference Equations (2011) 16 pages Article ID 404917, doi:10.1155/2011/404917.

[24] B. Ahmad, S. Sivasundaram, Existence of solutions for impulsive integral boundary value problems of fractional order, Nonlinear Anal. Hybrid Syst. 4 (2010) 134–141.

[25] G. Wang, B. Ahmad, L. Zhang, Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions, Computers and Mathematics with Applications 62 (2011) 1389–1397.

[26] L. Zhang, G. Wang, G. Song, On mixed boundary value problem of impulsive semilinear evolution equations of fractional order. Bound. Value Probl. 2012, 17 (2012).

[27] B. Ahmad, S. Sivasundaram, Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations, Nonlinear Anal. Hybrid Syst. 3 (2009) 251–258.

[28] H.H. Wang, Existence results for fractional functional differential equations with impulses, Journal of Applied Mathematics and Computing (2010) DOI 10. 1007/s12190-010-0465-9.

[29] G.T. Wang, B. Ahmad, L.H. Zhang, Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order, Nonlinear Analysis. 74 (2011) 792–804.

[30] L.H. Zhang, G.T. Wang, Existence of solutions for nonlinear fractional differential equations with impulses and anti-periodic boundary conditions, Electronic Journal of Qualitative Theory of Differential Equations. 7 (2011) 1–11.

[31] A. Anguraj, P. Karthikeyan, Anti-periodic boundary value problem for impulsive fractional integro differetial equations, Fractional calculus and applied analysis. 13 (3) (2010) 281–293.

[32] B. Ahmad, J.J. Nieto, Existence and approximation of solutions for a class of nonlinear impulsive functional differential equations with antiperiodic boundary conditions, Nonlinear Analysis. 69 (2008) 3291–3298.

[33] M. Benchohra, S. Hamani, J.J. Nieto, B.A. Slimani, Existence of solutions to differential inclusions with fractional order and impulses, Electronic Journal of Differential Equations. 80 (2010) 1–18.

[34] H. Ergören and A. Kiliçman, Some existence results for impulsive nonlinear fractional differential equations with closed boundary conditions, Abstract and Applied Analysis, Volume 2012, Article ID 387629, 15 pages, doi:10.1155/2012/387629.

[35] H. Ergören and A. Kiliçman, Boundary Value Problems For Impulsive Fractional Integro-differential Equations With Non-Local Conditions, Boundary Value Problems 2012, 2012:145. doi:10.1186/1687-2770-2012-145.

[36] H. Ergören, M.G.Sakar, Boundary value problems for impulsive fractional differential equations with non-local conditions, in "Advances in Applied Mathematics and Approximation Theory - Contributions from AMAT 2012", Springer Proceedings in Mathematics & Statistics, Volume 41, 2013, pp 283–297.

[37] D. O'Regan, Fixed-point theory for the sum of two operators, Appl. Math. Lett. 9(1) (1996): 1– 8.

[38] J. Hale, Theory of Functional Differential Equations, Springer, New York, 1977.

[39] M. Akhmet, Principles of Discontinuous Dynamical Systems, Springer, New York, Dordrecht Heidelberg, London, 2010.

[40] B. Ahmad, J. J. Nieto, Existence of solutions for impulsive anti – periodic boundary value problems of fractional order, Taiwanese J. Math.15(3) (2011) 981 – 993.

[41] L. Zhang, G. Wang, Existence of solutions for nonlinear fractional differential equations with impulses and anti-periodic boundary conditions, Electron. J.Qual. Theory Differ. Equ. 7 (2011) 1 – 11.

[42] Y. Tian, Z. Bai, Existence results for the three-point impulsive boundary value problem involving fractional differential equations, Comput. Math. Appl. 59 (2010) 2601 – 2609.

[43] B. Ahmad, J. J. Nieto, A. Alsaedi, Existence and uniqueness of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions, Acta Math. Sci. 31B(6) (2011) 2122 – 2130.

Author information

Hilmi Ergoren, Department of Mathematics, Faculty of Sciences, Yuzuncu Yil University, 65080, Van, Turkey.
E-mail: `hergoren@yahoo.com`

Cemil Tunc, Department of Mathematics, Faculty of Sciences, Yuzuncu Yil University, 65080, Van, Turkey.
E-mail: `cemtunc@yahoo.comn`