

A General Boundary Value Problem For Impulsive Fractional Differential Equations

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Communicated by Ayman Badawi

MSC 2010 Classifications: 26A33, 34A37.

Keywords and phrases: Caputo fractional derivative, Existence and uniqueness, Fixed point theorem, Impulsive differential equations.

Abstract. In this study, we consider a boundary value problem for a class of impulsive fractional differential equations having general boundary conditions. We establish some sufficient conditions for the existence of solutions for the problem by using some standard fixed point theorems and give an illustrative example.

1 Introduction

This work deals with the existence and uniqueness of the solutions to the boundary value problem (BVP for short), for the following impulsive fractional differential equation,

$$\begin{cases} {}^C D^\alpha y(t) = f(t, y(t)), & t \in J := [0, T], t \neq \theta_j, 1 < \alpha \leq 2, \\ \Delta y(\theta_j) = I_j(y(\theta_j^-)), \quad \Delta y'(\theta_j) = I_j^*(y(\theta_j^-)), & j = 1, 2, \dots, p, \\ y(0) = \lambda_1 y(T) + \lambda_2 y(\xi) + \lambda_3 \int_0^T w_1(s, y(s)) ds + k_1, \\ y'(0) = \beta_1 y'(T) + \beta_2 y'(\xi) + \beta_3 \int_0^T w_2(s, y(s)) ds + k_2, \end{cases} \quad (1.1)$$

where ${}^C D^\alpha$ is the Caputo fractional derivative, $f : J \times R \rightarrow R$, $I_j, I_j^* : R \rightarrow R$, $\{\theta_j\}_{j=1}^p$ is a strictly increasing B -sequence (to be defined later) of impulse points θ_j such that $0 = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_p < \theta_{p+1} = T$, $\Delta y(\theta_j) = y(\theta_j^+) - y(\theta_j^-)$ with $y(\theta_j^+) = \lim_{h \rightarrow 0^+} y(\theta_j + h)$, $y(\theta_j^-) = \lim_{h \rightarrow 0^-} y(\theta_j + h)$, and $\Delta y'(\theta_j)$ has a similar meaning for $y'(t)$, and $\lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_2, \beta_3, k_1, k_2$ are real constants with $\lambda_1 + \lambda_2 \neq 1$, $\beta_1 + \beta_2 \neq 1$.

The subject of fractional differential equations has been recently paid increasing attention and it is gaining much importance. That is why, the fractional derivatives serve an excellent tool for the description of hereditary properties of different materials and processes. Actually, fractional differential equations arise in many engineering and scientific disciplines such as, physics, chemistry, biology, electrochemistry, electromagnetic, control theory, economics, signal and image processing, aerodynamics, porous media, etc. (see [1, 2, 3, 4, 5, 6, 7, 8] and references therein).

On the other hand, theory of impulsive differential equations for integer order has become important and found its extensive applications in mathematical modeling of phenomena and practical situations in both physical and social sciences in recent years. One can see a remarkable development in impulsive theory. For instance, for the general theory and applications of impulsive differential equations we refer the readers to [9, 10, 11, 12].

Boundary value problems take place in the studies of fractional differential equations differently many times (see [13, 14, 15, 16, 17, 18, 19, 20] and the relevant references therein). More precisely, one can encounter some boundary value problems for impulsive fractional differential equations including periodic, anti periodic, hybrid, closed, nonlocal, integral boundary etc. conditions. We can refer the interested researchers to [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36].

As to this study, motivated by the mentioned recent works above, we form a class of general boundary value problem, which includes some afore-mentioned boundary conditions, for impulsive fractional differential equations as in (1.1) and investigate the existence and uniqueness of solutions to the BVP (1.1). We hope that this study will become a good generalization of some initial-boundary value problems for impulsive differential equations and make contribution to this emerging field.

The rest of the paper is organized as follows. In Section 2, we present some notations and preliminary results about fractional calculus and differential equations to be used in the following

sections. In Section 3, we discuss some existence and uniqueness results for solutions of BVP (1.1). Namely, the first one is based on O' Regan fixed point theorem, the second one is based on Banach's fixed point theorem. At the end, we give an illustrative example for our results.

2 Preliminaries

Definition 2.1. ([1, 2]) The fractional (arbitrary) order integral of the function $h \in L^1(J, R_+)$ of order $\alpha \in R_+$ is defined by

$$I_{0^+}^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. ([1, 2]) For a function h given on the interval J , Caputo fractional derivative of order $\alpha > 0$ is defined by

$${}^C D_{0^+}^\alpha h(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} h^{(n)}(s) ds, \quad n = [\alpha] + 1,$$

where the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.3. ([1, 13]) Let $\alpha > 0$, then the differential equation

$${}^C D_{0^+}^\alpha h(t) = 0$$

has solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad c_i \in R, \quad i = 0, 1, 2, \dots, n-1, \quad n = [\alpha] + 1.$$

Lemma 2.4. ([1]) Let $\alpha > 0$, $h(t) \in C^n[a, b]$, then

$$I_{a^+}^{\alpha C} D_{a^+}^\alpha h(t) = h(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \dots + c_{n-1}(t-a)^{n-1},$$

for some $c_i \in R$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

Now, we introduce O' Regan fixed point theorem to be applied to prove our main results later.

Lemma 2.5. ([37]) Denote by U an open set in a closed, convex set Y of a Banach space E . Assume that $0 \in U$. Also assume that $F(\bar{U})$ is bounded and $F : \bar{U} \rightarrow Y$ is given by $F = F_1 + F_2$, in which $F_1 : \bar{U} \rightarrow E$ is continuous and completely continuous and $F_2 : \bar{U} \rightarrow E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \leq \phi(\|x - y\|)$ for all $x, y \in \bar{U}$), then either

(C₁) F has a fixed point $u \in \bar{U}$, or

(C₂) there exists a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$, where \bar{U} , ∂U represent the closure and boundary of U , respectively.

Definition 2.6. ([38]) If U is a subset of a Banach space X and $T : U \rightarrow X$, then T is said to be completely continuous if T is continuous and for any bounded set $B \subseteq U$, the closure of TU is compact.

Denote by $\theta = \{\theta_j\}$ a collection of some impulse points mentioned above and a strictly increasing sequence of real numbers such that the set \mathcal{A} of indexes j is an interval in Z .

Definition 2.7. ([39]) θ is a B -sequence, if one of the following alternatives is valid:

- (a) θ is empty,
- (b) θ is non-empty and finite set,
- (c) θ is an infinite set such that $|\theta_j| \rightarrow \infty$ as $|j| \rightarrow \infty$.

Let us set $J_0 = [\theta_0, \theta_1]$, $J_1 = (\theta_1, \theta_2]$, ..., $J_{j-1} = (\theta_{j-1}, \theta_j]$, $J_j = (\theta_j, \theta_{i+1}]$, $J' := [0, T] \setminus \{\theta_1, \theta_2, \dots, \theta_p\}$ and define the set of functions:

$PC(J, R) = \{y : J \rightarrow R : y \in C((\theta_j, \theta_{j+1}], R), j = 0, 1, 2, \dots, p \text{ and there exist } y(\theta_j^+) \text{ and } y(\theta_j^-), j = 1, 2, \dots, p \text{ with } y(\theta_j^-) = y(\theta_j)\}$ and

$PC^1(J, R) = \{y \in PC(J, R), y' \in C((\theta_j, \theta_{j+1}], R), j = 0, 1, 2, \dots, p \text{ and there exist } y'(\theta_j^+) \text{ and } y'(\theta_j^-), j = 1, 2, \dots, p \text{ with } y'(\theta_j^-) = y'(\theta_j)\}$ which is a Banach space with the norm $\|y\| = \sup_{t \in J} \left\{ \|y\|_{PC}, \left\| y' \right\|_{PC} \right\}$, where $\|y\|_{PC} := \sup \{|y(t)| : t \in J\}$.

3 Main Results

To begin with, in order to deal with BVP(1.1), we shall consider the following linear problem associated with BVP(1.1) and give its solution.

Lemma 3.1. *Let $\xi \in (\theta_l, \theta_{l+1})$; l is non-negative integer in \mathcal{A} i.e. $0 \leq l \leq p$, $1 < \alpha \leq 2$, $j = 1, 2, \dots, p$ and $\sigma, q_1, q_2 : J \rightarrow R$ be continuous. A function $y(t) \in PC(J, R)$ is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + b_0 + b_1 t, & \text{if } t \in J_0, \\ \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \\ + \sum_{i=1}^j (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] + b_0 + b_1 t, & \text{if } t \in J_j \end{cases} \quad (3.1)$$

if and only if $y(t)$ is a solution of the fractional BVP

$$\begin{cases} {}^C D^\alpha y(t) = \sigma(t), & t \in J' \\ \Delta y(\theta_j) = I_j(y(\theta_j^-)), \Delta y'(\theta_j) = I_j^*(y(\theta_j^-)), \\ y(0) = \lambda_1 y(T) + \lambda_2 y(\xi) + \lambda_3 \int_0^T q_1(s) ds + k_1, \\ y'(0) = \beta_1 y'(T) + \beta_2 y'(\xi) + \beta_3 \int_0^T w_2(s, y(s)) ds + k_2, \end{cases} \quad (3.2)$$

where

$$\begin{aligned} b_0 = & \frac{-1}{\lambda_1 + \lambda_2 - 1} \left[\lambda_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \right) \right. \\ & + \left. \sum_{i=1}^p (T - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \\ & + \lambda_2 \left(\int_{\theta_l}^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \right) \\ & + \left. \sum_{i=1}^l (\xi - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \\ & + \lambda_3 \int_0^T w_1(s, y(s)) ds + k_1 \Big] - \frac{\lambda_1 T + \lambda_2 \xi}{(\lambda_1 + \lambda_2 - 1)} b_1 \end{aligned}$$

and

$$\begin{aligned} b_1 = & \frac{-1}{\beta_1 + \beta_2 - 1} \left[\beta_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\ & + \beta_2 \left(\int_{\theta_l}^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \\ & \left. + \beta_3 \int_0^T w_2(s, y(s)) ds + k_2 \right]. \end{aligned}$$

Proof. Let y be the solution of (3.2). If $t \in J_0 = [\theta_0, \theta_1]$, then Lemma 2.4 implies that

$$\begin{aligned} y(t) &= I^\alpha \sigma(t) - c_0 - c_1 t = \int_{\theta_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - c_0 - c_1 t, \\ y'(t) &= \int_{\theta_0}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds - c_1, \end{aligned}$$

for some $c_0, c_1 \in R$.

If $t \in J_1 = (\theta_1, \theta_2]$, then Lemma 2.4 implies that

$$y(t) = \int_{\theta_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - d_0 - d_1(t - \theta_1),$$

$$y'(t) = \int_{\theta_1}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds - d_1$$

for some $d_0, d_1 \in R$. Thus we have

$$\begin{aligned} y(\theta_1^-) &= \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - c_0 - c_1 \theta_1, \quad y(\theta_1^+) = -d_0, \\ y'(\theta_1^-) &= \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds - c_1, \quad y'(\theta_1^+) = -d_1. \end{aligned}$$

In view of $\Delta y(\theta_1) = y(\theta_1^+) - y(\theta_1^-) = I_1(y(\theta_1^-))$ and $\Delta y'(\theta_1) = y'(\theta_1^+) - y'(\theta_1^-) = I_1^*(y(\theta_1^-))$, we have

$$\begin{aligned} -d_0 &= \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - c_0 - c_1 \theta_1 + I_1(y(\theta_1^-)), \\ -d_1 &= \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds - c_1 + I_1^*(y(\theta_1^-)), \end{aligned}$$

hence, for $t \in J_1$,

$$\begin{aligned} y(t) &= \int_{\theta_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds \\ &\quad + (t-\theta_1) \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_1(y(\theta_1^-)) \\ &\quad + (t-\theta_1) I_1^*(y(\theta_1^-)) - c_0 - c_1 t, \\ y'(t) &= \int_{\theta_1}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_1^*(y(\theta_1^-)) - c_1. \end{aligned}$$

If $t \in J_2 = (\theta_2, \theta_3]$, then Lemma 2.4 implies that

$$\begin{aligned} y(t) &= \int_{\theta_2}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - e_0 - e_1(t-\theta_2), \\ y'(t) &= \int_{\theta_2}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds - e_1, \end{aligned}$$

for some $e_0, e_1 \in R$. Thus we have

$$\begin{aligned} y(\theta_2^-) &= \int_{\theta_1}^{\theta_2} \frac{(\theta_2-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds \\ &\quad + (\theta_2-\theta_1) \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_1(y(\theta_1^-)) \\ &\quad + (\theta_2-\theta_1) I_1^*(y(\theta_1^-)) - c_0 - c_1 \theta_2, \\ y(\theta_2^+) &= -e_0, \end{aligned}$$

and

$$\begin{aligned} y'(\theta_2^-) &= \int_{\theta_1}^{\theta_2} \frac{(\theta_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_1^*(y(\theta_1^-)) - c_1, \\ y'(\theta_2^+) &= -e_1. \end{aligned}$$

In view of $\Delta y(\theta_2) = y(\theta_2^+) - y(\theta_2^-) = I_2(y(\theta_2^-))$ and $\Delta y'(\theta_2) = y'(\theta_2^+) - y'(\theta_2^-) = I_2^*(y(\theta_2^-))$, we have

$$\begin{aligned} -e_0 &= y(\theta_2^-) + I_2(y(\theta_2^-)), \\ -e_1 &= y'(\theta_2^-) + I_2^*(y(\theta_2^-)), \end{aligned}$$

hence, for $t \in J_2$,

$$\begin{aligned}
 y(t) &= \int_{\theta_2}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \int_{\theta_1}^{\theta_2} \frac{(\theta_2-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds \\
 &\quad + (\theta_2 - \theta_1) \left[\int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_1^*(y(\theta_1^-)) \right] \\
 &\quad + (t - \theta_2) \left[\int_0^{\theta_1} \frac{(\theta_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + \int_{\theta_1}^{\theta_2} \frac{(\theta_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_1^*(y(\theta_1^-)) + I_2^*(y(\theta_2^-)) \right] \\
 &\quad + I_1(y(\theta_1^-)) + I_2(y(\theta_2^-)) - c_0 - c_1 t,
 \end{aligned}$$

$$\begin{aligned}
 y(t) &= \int_{\theta_2}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^2 \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \\
 &\quad + \sum_{i=1}^2 (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] - c_0 - c_1 t.
 \end{aligned}$$

By repeating the same process, if $t \in J_j$, then again from Lemma 2.4, we get

$$y(t) = \begin{cases} \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \\ + \sum_{i=1}^j (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] - c_0 - c_1 t \end{cases}$$

and

$$y'(t) = \begin{cases} \int_{\theta_j}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds \\ + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] - c_1. \end{cases}$$

Applying the conditions $y(0) = \lambda_1 y(T) + \lambda_2 y(\xi) + \lambda_3 \int_0^T w_1(s, y(s)) ds + k_1$, $y'(0) = \beta_1 y'(T) + \beta_2 y'(\xi) + \beta_3 \int_0^T w_2(s, y(s)) ds + k_2$ we have

$$\begin{aligned}
 c_1 &= \frac{1}{\beta_1 + \beta_2 - 1} \left[\beta_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\
 &\quad \left. + \beta_2 \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\
 &\quad \left. + \beta_3 \int_0^T w_2(s, y(s)) ds + k_2 \right]
 \end{aligned}$$

and

$$\begin{aligned}
 c_0 &= \frac{1}{\lambda_1 + \lambda_2 - 1} \left[\lambda_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \right) \right. \\
 &\quad \left. + \sum_{i=1}^p (T - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \\
 &\quad + \lambda_2 \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + I_i(y(\theta_i^-)) \right] \right) \\
 &\quad \left. + \sum_{i=1}^l (\xi - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \sigma(s) ds + I_i^*(y(\theta_i^-)) \right] \right) \\
 &\quad \left. + \lambda_3 \int_0^T w_1(s, y(s)) ds + k_1 \right] - \frac{\lambda_1 T + \lambda_2 \xi}{(\lambda_1 + \lambda_2 - 1)} c_1.
 \end{aligned}$$

Then, replacing $-c_0$ and $-c_1$ with b_0 and b_1 respectively, we obtain (3.1).

Conversely, assume that y satisfies the impulsive fractional integral equation (3.1). Then, in view of the Caputo differentiation (2.2) and the property ${}^C D^\alpha I^\alpha \sigma(t) = \sigma(t)$, $\alpha > 0$ ([1], Lemma 2.21), it is obtained that ${}^C D^\alpha y(t) = \sigma(t)$ in (3.2). Moreover, it can be easily verified that Eq. (3.1) holds the boundary conditions in (3.2). Therefore, the solution y given by (3.1) satisfies (3.2). The proof is complete. \square

Definition 3.2. A function $y \in PC^1(J, R)$ with its α -derivative existing on J' is said to be a solution of (1.1) if y satisfies the equation ${}^C D^\alpha y(t) = f(t, y(t))$ on J' and the conditions

$$\begin{aligned} \Delta y(\theta_j) &= I_j(y(\theta_j^-)), \quad \Delta y'(\theta_j) = I_j^*(y(\theta_j^-)), \\ y(0) &= \lambda_1 y(T) + \lambda_2 y(\xi) + \lambda_3 \int_0^T w_1(s, y(s)) ds + k_1, \\ y'(0) &= \beta_1 y'(T) + \beta_2 y'(\xi) + \beta_3 \int_0^T w_2(s, y(s)) ds + k_2 \end{aligned}$$

are satisfied for y .

Let $\xi \in (\theta_l, \theta_{l+1})$; l is a non-negative integer in \mathcal{A} i.e., $0 \leq l \leq p$. In view of (3.1) and Definition 3.2, let us define the operator $T : PC^1(J, R) \rightarrow PC^1(J, R)$ by

$$\begin{aligned} (Ty)(t) &= \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + I_i(y(\theta_i^-)) \right] \\ &\quad + \sum_{i=1}^j (t-\theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + I_i^*(y(\theta_i^-)) \right] \\ &\quad - \frac{1}{\lambda_1 + \lambda_2 - 1} \left\{ \lambda_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + I_i(y(\theta_i^-)) \right] \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^p (T-\theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\ &\quad \left. + \lambda_2 \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + I_i(y(\theta_i^-)) \right] \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^l (\xi-\theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\ &\quad \left. + \lambda_3 \int_0^T w_1(s, y(s)) ds + k_1 \right\} + \frac{\lambda_1 (T-t) + \lambda_2 (\xi-t) + t}{(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)} \times \\ &\quad \left\{ \beta_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\ &\quad \left. + \beta_2 \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\ &\quad \left. + \beta_3 \int_0^T w_2(s, y(s)) ds + k_2 \right\}, \quad \text{if } t \in J_j, \end{aligned} \tag{3.3}$$

where $T = T_1 + T_2$ with

$$(T_1y)(t) = \left\{ \begin{aligned} & \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + I_i(y(\theta_i^-)) \right] \\ & \quad + \sum_{i=1}^j (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + I_i^*(y(\theta_i^-)) \right] \\ & - \frac{1}{\lambda_1 + \lambda_2 - 1} \left\{ \lambda_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + I_i(y(\theta_i^-)) \right] \right. \right. \\ & \quad \left. \left. + \sum_{i=1}^p (T - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\ & \quad \left. + \lambda_2 \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds + I_i(y(\theta_i^-)) \right] \right. \right. \\ & \quad \left. \left. + \sum_{i=1}^l (\xi - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + I_i^*(y(\theta_i^-)) \right] \right) \right\} \\ & \quad + \frac{\lambda_1(T-t) + \lambda_2(\xi-t) + t}{(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)} \times \\ & \quad \left\{ \beta_1 \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + I_i^*(y(\theta_i^-)) \right] \right) \right. \\ & \quad \left. + \beta_2 \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, y(s)) ds + I_i^*(y(\theta_i^-)) \right] \right) \right\} \end{aligned} \right.$$

and

$$(T_2y)(t) = \left\{ \begin{aligned} & \frac{-1}{\lambda_1 + \lambda_2 - 1} \left(\lambda_3 \int_0^T w_1(s, y(s)) ds + k_1 \right) \\ & + \frac{\lambda_1(T-t) + \lambda_2(\xi-t) + t}{(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)} \left(\beta_3 \int_0^T w_2(s, y(s)) ds + k_2 \right). \end{aligned} \right.$$

In order to establish our following main results, we first define $B_r = \{y \in PC^1(J, R) : \|y\| < r\}$ for any $r > 0$.

3.1 Existence of solutions

Lemma 3.3. Assume that

(A1) $f : J \times R \rightarrow R$ is continuous and there exists a constant $M_1 > 0$ such that $|f(t, u)| \leq M_1$, $\forall t \in J$ and $\forall u \in R$,

(A2) $I_j, I_j^* : R \rightarrow R$ are continuous and there exist constants $M_2 > 0$ and $M_3 > 0$ such that $|I_j(u)| \leq M_2$, $|I_j^*(u)| \leq M_3$ for all $u \in R$ and $j = 1, 2, \dots, p$.

Then, $T_1 : \bar{B}_r \rightarrow PC^1(J, R)$ is completely continuous.

Proof. First, note that the continuity of f, w_1, w_2, I_j and $I_j^*, j = 1, 2, \dots, p$, ensure the continuity of T_1 .

Now, in view of (A1) and (A2), for each $y \in \bar{B}_r$, we have

$$\begin{aligned} |(T_1y)(t)| & \leq \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds + |I_i(y(\theta_i^-))| \right] \\ & \quad + \sum_{i=1}^j (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds + |I_i^*(y(\theta_i^-))| \right] \\ & \quad + \frac{1}{|\lambda_1 + \lambda_2 - 1|} \left\{ |\lambda_1| \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds \right. \right. \\ & \quad \left. \left. + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds + |I_i(y(\theta_i^-))| \right] \right) \right. \\ & \quad \left. + \sum_{i=1}^p (T - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds + |I_i^*(y(\theta_i^-))| \right] \right\} \\ & \quad + |\lambda_2| \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds + |I_i(y(\theta_i^-))| \right] \right. \\ & \quad \left. + \sum_{i=1}^l (\xi - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds + |I_i^*(y(\theta_i^-))| \right] \right) \end{aligned}$$

$$\begin{aligned}
& + \left| \frac{\lambda_1 (T-t) + \lambda_2 (\xi-t) + t}{(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)} \right| \times \left\{ |\beta_1| \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds + |I_i^*(y(\theta_i^-))| \right] \right) \right. \\
& \quad \left. + |\beta_2| \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds + |I_i^*(y(\theta_i^-))| \right] \right) \right\}, \\
| (T_1 y)(t) | & \leq M_1 \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \sum_{i=1}^k \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} ds + M_2 \right] \\
& \quad + \sum_{i=1}^k (t-\theta_i) \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + M_3 \right] \\
& + \frac{1}{|\lambda_1 + \lambda_2 - 1|} \left\{ |\lambda_1| \left(M_1 \int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \sum_{i=1}^p \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} ds + M_2 \right] \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^p (T-\theta_i) \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + M_3 \right] \right) \right. \\
& \quad \left. + |\lambda_2| \left(M_1 \int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \sum_{i=1}^l \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} ds + M_2 \right] \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^l (\xi-\theta_i) \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + M_3 \right] \right) \right\} \\
& \quad + \left| \frac{\lambda_1 (T-t) + \lambda_2 (\xi-t) + t}{(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)} \right| \times \\
& \quad \left\{ |\beta_1| \left(M_1 \int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + \sum_{i=1}^p \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + M_3 \right] \right) \right. \\
& \quad \left. + |\beta_2| \left(M_1 \int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + \sum_{i=1}^l \left[M_1 \int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + M_3 \right] \right) \right\}, \\
| (T_1 y)(t) | & \leq \left(\frac{M_1 T^\alpha (1+p+\alpha p)}{\Gamma(\alpha+1)} + M_2 p + M_3 p T \right) \left(\frac{2|\lambda_1| + 2|\lambda_2| + 1}{|\lambda_1 + \lambda_2 - 1|} \right) \\
& \quad + \left(\frac{M_1 T^{\alpha-1} (1+p)}{\Gamma(\alpha)} + M_3 p \right) \left(\frac{(2|\lambda_1| + 2|\lambda_2| + 1)(|\beta_1| + |\beta_2|)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} \right) \\
& : = K.
\end{aligned}$$

Thus,

$$\|T_1 y\| \leq K \quad (3.4)$$

which implies that T_1 maps bounded sets into bounded sets.

Moreover, let $y \in \overline{B}_r$, for each $t \in J_j, 0 \leq j \leq p$, we have

$$\begin{aligned} |(T_1 y)'(t)| &\leq \int_{\theta_j}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds \\ &\quad + \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds + |I_i^*(y(\theta_i^-))| \right] \\ &\quad \frac{1}{|\beta_1 + \beta_2 - 1|} \times \left\{ |\beta_1| \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds + |I_i^*(y(\theta_i^-))| \right] \right) \right. \\ &\quad \left. + |\beta_2| \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, y(s))| ds + |I_i^*(y(\theta_i^-))| \right] \right) \right\} \\ &\leq \left(\frac{M_1 T^{\alpha-1} (1+p)}{\Gamma(\alpha)} + M_3 p \right) \left(1 + \frac{(|\beta_1| + |\beta_2|)}{|\beta_1 + \beta_2 - 1|} \right) \\ &=: \tilde{K}. \end{aligned}$$

Now, letting $t_1, t_2 \in J_j, t_1 < t_2, 0 \leq j \leq p$, we have

$$|(T_1 y)(t_2) - (T_1 y)(t_1)| \leq \int_{t_1}^{t_2} |(T_1 y)'(s)| ds \leq \tilde{K}(t_2 - t_1)$$

which implies that T_1 maps bounded sets into equicontinuous sets. Namely, $T_1(\overline{B}_r)$ is equicontinuous on all subintervals $J_j (j = 0, 1, 2, \dots, p)$. So, by Arzela-Ascoli Theorem and Definition 2.6, we conclude that the operator T_1 is completely continuous. \square

Theorem 3.4. *In addition to (A1) - (A2), assume that the following conditions are satisfied:*

A3) $w_1, w_2 : J \times R \rightarrow R$ are continuous and there exist constants $M_4 > 0$ and $M_5 > 0$ such that $|w_1(t, u)| \leq M_4, |w_2(t, u)| \leq M_5 \forall t \in J$ and $\forall u \in R$,

A4) There exist constants $N_1 > 0$ and $N_2 > 0$ such that $|w_1(t, u) - w_1(t, v)| \leq N_1, |w_2(t, u) - w_2(t, v)| \leq N_2$ for each $u, v \in R$,

A5) There exists a constant $L > 0$ such that

$$\frac{N_1 T |\lambda_3|}{|\lambda_1 + \lambda_2 - 1|} + \frac{N_2 T |\beta_3| (2|\lambda_1| + 2|\lambda_2| + 1)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} =: L < 1.$$

Then, BVP (1.1) has at least one solution on J .

Proof. In order to show the existence of the solutions BVP (1.1), we need to transform BVP (1.1) to a fixed point problem by using the operator T such that $T = T_1 + T_2$. Now, we shall use O'Regan fixed point theorem in Lemma 2.5 to prove that T has a fixed point which is then a solution of BVP (1.1). First, let us consider the set $B_r = \{y \in PC^1(J, R) : \|y\| < r\}$ defined above.

In view of O'Regan fixed point theorem, let us take $U = B_r, Y = \overline{B}_r$ and $E = PC^1(J, R)$. Then, it is obvious that $Y = \overline{B}_r$ is closed and convex where $U \subset Y$. From Lemma 3.3, it is proven that $T_1 : \overline{U} \rightarrow E$ i.e., $T_1 : \overline{B}_r \rightarrow PC^1(J, R)$, is completely continuous.

In order to show the boundedness of the operator $T : \overline{U} \rightarrow Y$ that is, $T : \overline{B}_r \rightarrow \overline{B}_r$, it is enough that show that the boundedness of the operator T_2 since it is proven that $T_1 : \overline{B}_r \rightarrow$

$PC^1(J, R)$ is bounded in Lemma 3.3. Then, for $y \in \overline{B}_r$, we have

$$\begin{aligned} |(T_2y)(t)| &\leq \frac{1}{|\lambda_1 + \lambda_2 - 1|} \left(|\lambda_3| \int_0^T |w_1(s, y(s))| ds + |k_1| \right) \\ &\quad + \frac{|\lambda_1(T-t) + \lambda_2(\xi-t) + t|}{|(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)|} \left(|\beta_3| \int_0^T |w_2(s, y(s))| ds + |k_2| \right), \\ |(T_2y)(t)| &\leq \frac{M_4T|\lambda_3| + |k_1|}{|\lambda_1 + \lambda_2 - 1|} + \frac{(M_5T|\beta_3| + |k_2|)(2|\lambda_1| + 2|\lambda_2| + 1)}{|\lambda_1 + \lambda_2 - 1||\beta_1 + \beta_2 - 1|} := H, \\ \|T_2y\| &\leq H. \end{aligned} \quad (3.5)$$

Hence, $T_2 : \overline{B}_r \rightarrow PC^1(J, R)$ is bounded.

Now, taking into account of (3.4) and (3.5), provided that

$$r \geq K + H, \quad (3.6)$$

we obtain that the operator $T : \overline{U} \rightarrow Y$ that is, $T : \overline{B}_r \rightarrow \overline{B}_r$ is bounded.

Now, let us show that $T_2 : \overline{U} \rightarrow E$ is nonlinear contraction. For x and $y \in \overline{B}_r$, we have

$$\begin{aligned} |(T_2x)(t) - (T_2y)(t)| &\leq \frac{|\lambda_3|}{|\lambda_1 + \lambda_2 - 1|} \int_0^T |w_1(s, x(s)) - w_1(s, y(s))| ds \\ &\quad + |\beta_3| \frac{|\lambda_1(T-t) + \lambda_2(\xi-t) + t|}{|(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)|} \int_0^T |w_2(s, x(s)) - w_2(s, y(s))| ds \\ &\leq \frac{N_1T|\lambda_3|}{|\lambda_1 + \lambda_2 - 1|} + \frac{N_2T|\beta_3|(2|\lambda_1| + 2|\lambda_2| + 1)}{|\lambda_1 + \lambda_2 - 1||\beta_1 + \beta_2 - 1|} \|x - y\|, \\ \|T_2x - T_2y\| &\leq L \|x - y\|. \end{aligned}$$

Let us choose $\phi(z) = Lz$. Since $L < 1$ from (A5), we obtain for $\forall x, y \in \overline{B}_r$,

$$\|T_2x - T_2y\| \leq \phi(\|x - y\|),$$

that is, the operator T_2 is nonlinear contraction.

At the end, assume that the hypothesis (C_2) in Lemma 2.5 is valid, then there exist $u \in \partial B_r$ and $\lambda \in (0, 1)$ such that $u = \lambda T(u)$. Thus, we get $\|u\| = r$ and

$$\begin{aligned} |u| &= |\lambda| |Tu|, \\ \|u\| &= r < \|Tu\| \leq K + H, \\ r &< K + H, \end{aligned}$$

which contradict with (3.6).

Therefore, T has a fixed point $u \in \overline{B}_r$ which is the solution of BVP (1.1). \square

Now, we will give some sufficient conditions for the uniqueness of the solutions of BVP (1.1).

3.2 Uniqueness of solutions

Theorem 3.5. *In addition to (A1) - (A5), assume that the following conditions are satisfied:*

(A6) *There exists a constant $L_1 > 0$ such that $|f(t, u) - f(t, v)| \leq L_1 |u - v|$, $\forall t \in J$, and $u, v \in R$,*

(A7) *There exist constants $L_2 > 0$, $L_3 > 0$ such that $|I_j(u) - I_j(v)| \leq L_2 |u - v|$, $|I_j^*(u) - I_j^*(v)| \leq L_3 |u - v|$ for each $u, v \in R$ and $j = 1, 2, \dots, p$.*

Moreover,

$$\begin{aligned} &\left(\frac{L_1 T^\alpha (1 + p + \alpha p)}{\Gamma(\alpha + 1)} + L_2 p + L_3 p T \right) \left(\frac{2|\lambda_1| + 2|\lambda_2| + 1}{|\lambda_1 + \lambda_2 - 1|} \right) \\ &+ \left(\frac{L_1 T^{\alpha-1} (1 + p)}{\Gamma(\alpha)} + L_3 p \right) \left(\frac{(2|\lambda_1| + 2|\lambda_2| + 1)(|\beta_1| + |\beta_2|)}{|\lambda_1 + \lambda_2 - 1||\beta_1 + \beta_2 - 1|} \right) \\ &+ \frac{N_1 T |\lambda_3|}{|\lambda_1 + \lambda_2 - 1|} + \frac{N_2 T |\beta_3| (2|\lambda_1| + 2|\lambda_2| + 1)}{|\lambda_1 + \lambda_2 - 1||\beta_1 + \beta_2 - 1|} \\ &:= \Omega_{L_1, L_2, L_3, L_4, L_5, \lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_2, \beta_3} < 1. \end{aligned} \quad (3.7)$$

Then, BVP (1.1) has a unique solution on J .

Proof. In view of 3.6, choosing $r = K + H$, we can easily show that $T\overline{B}_r \subset \overline{B}_r$, where $\overline{B}_r = \{y \in PC(J, R) : \|y\| \leq r\}$.

Now, for $x, y \in PC^1(J, R)$ and for each $t \in J_j$, we obtain

$$\begin{aligned}
 |(Tx)(t) - (Ty)(t)| &\leq \int_{\theta_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| ds \\
 &+ \sum_{i=1}^j \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| ds + |I_i(x(\theta_i^-)) - I_i(y(\theta_i^-))| \right] \\
 &+ \sum_{i=1}^j (t - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| ds + |I_i^*(x(\theta_i^-)) - I_i^*(y(\theta_i^-))| \right] \\
 &+ \frac{1}{|\lambda_1 + \lambda_2 - 1|} \left\{ |\lambda_1| \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| ds \right. \right. \\
 &+ \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| ds + |I_i(x(\theta_i^-)) - I_i(y(\theta_i^-))| \right] \\
 &+ \sum_{i=1}^p (T - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| ds + |I_i^*(x(\theta_i^-)) - I_i^*(y(\theta_i^-))| \right] \left. \right\} \\
 &+ |\lambda_2| \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| ds \right. \\
 &+ \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| ds + |I_i(x(\theta_i^-)) - I_i(y(\theta_i^-))| \right] \\
 &+ \sum_{i=1}^l (\xi - \theta_i) \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| ds + |I_i^*(x(\theta_i^-)) - I_i^*(y(\theta_i^-))| \right] \left. \right\} \\
 &+ |\lambda_3| \int_0^T |w_1(s, x(s)) - w_1(s, y(s))| ds \left\} + \left| \frac{\lambda_1(T-t) + \lambda_2(\xi-t) + t}{(\lambda_1 + \lambda_2 - 1)(\beta_1 + \beta_2 - 1)} \right| \times \\
 &\quad \left\{ |\beta_1| \left(\int_{\theta_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| ds \right. \right. \\
 &+ \sum_{i=1}^p \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| ds + |I_i^*(x(\theta_i^-)) - I_i^*(y(\theta_i^-))| \right] \left. \right\} \\
 &\quad + |\beta_2| \left(\int_{\theta_l}^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| ds \right. \\
 &+ \sum_{i=1}^l \left[\int_{\theta_{i-1}}^{\theta_i} \frac{(\theta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s, x(s)) - f(s, y(s))| ds + |I_i^*(x(\theta_i^-)) - I_i^*(y(\theta_i^-))| \right] \left. \right\} \\
 &\quad + |\beta_3| \int_0^T |w_2(s, x(s)) - w_2(s, y(s))| ds \left. \right\}, t \in J_j.
 \end{aligned}$$

Then, in view of assumptions (A5)-(A7), we have

$$\begin{aligned}
 \|Tx - Ty\| &\leq \left\{ \left(\frac{L_1 T^\alpha (1 + p + \alpha p)}{\Gamma(\alpha + 1)} + L_2 p + L_3 p T \right) \left(\frac{2|\lambda_1| + 2|\lambda_2| + 1}{|\lambda_1 + \lambda_2 - 1|} \right) \right. \\
 &+ \left(\frac{L_1 T^{\alpha-1} (1 + p)}{\Gamma(\alpha)} + L_3 p \right) \left(\frac{(2|\lambda_1| + 2|\lambda_2| + 1)(|\beta_1| + |\beta_2|)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} \right) \\
 &+ \left. \frac{N_1 T |\lambda_3|}{|\lambda_1 + \lambda_2 - 1|} + \frac{N_2 T |\beta_3| (2|\lambda_1| + 2|\lambda_2| + 1)}{|\lambda_1 + \lambda_2 - 1| |\beta_1 + \beta_2 - 1|} \right\} \|x - y\| \\
 &\leq \Omega_{L_1, L_2, L_3, L_4, L_5, \lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_2, \beta_3} \|x - y\|.
 \end{aligned}$$

Therefore, by (3.7) and thanks to Banach's fixed point theorem, the operator T is contraction mapping. Consequently, BVP (1.1) has a unique solution. \square

Remark 3.6. Indeed, the boundary value problem we dealt with in (1.1) consists of various kinds of initial and boundary conditions. For instance, when $\lambda_1 = \lambda_2 = \lambda_3 = \beta_1 = \beta_2 = \beta_3 = 0$ holds in (1.1), one can obtain an initial value problem (see [22]); when $\lambda_1 = \beta_1 = 1$ and $\lambda_2 = \lambda_3 = \beta_2 = \beta_3 = k_1 = k_2 = 0$ hold in (1.1), one can obtain a periodic boundary value problem; when $\lambda_1 = \beta_1 = -1$ and $\lambda_2 = \lambda_3 = \beta_2 = \beta_3 = k_1 = k_2 = 0$ hold in (1.1), one can obtain an anti-periodic boundary value problem (see [40, 41]); when $\lambda_2 \neq 0, \beta_2 \neq 0$ holds in (1.1), one can obtain a non-local boundary value problem (see [42]); when $\lambda_3 \neq 0$ and $\beta_3 \neq 0$ hold in (1.1), one can obtain an integral boundary value problem (see [24]); when $\lambda_1 \neq 1$ and $\beta_1 \neq 1$ hold in (1.1), one can obtain a non-separated boundary value problem (see [43]).

Example 3.7. Consider the following impulsive fractional boundary value problem

$$\begin{aligned} {}^c D^{\frac{3}{2}} y(t) &= \frac{(\sin 5t) |y(t)|}{(t+5)^3 (1+|y(t)|)}, \quad t \in [0, 1], \quad t \neq \frac{1}{2}, \\ \Delta y\left(\frac{1}{2}\right) &= \frac{|y(\frac{1}{2}^-)|}{150 + |y(\frac{1}{2}^-)|}, \quad \Delta y'\left(\frac{1}{2}\right) = \frac{|y(\frac{1}{2}^-)|}{200 + |y(\frac{1}{2}^-)|}, \\ y(0) &= 2y(1) + 3y(\xi) - \int_0^1 e^{-\frac{1}{50}y(s)} ds + \frac{1}{6}, \\ y'(0) &= y'(1) - 2y'(\xi) + \frac{1}{5} \int_0^T e^{-\frac{1}{25}y(s)} ds - \frac{1}{9}, \end{aligned} \quad (3.8)$$

where $0 < \xi < 1, \xi \neq \frac{1}{2}$.

Here, according to BVP (1.1), $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = -1, \beta_1 = 1, \beta_2 = -2, \beta_3 = -\frac{1}{5}, \alpha = \frac{3}{2}, k_1 = \frac{1}{6}, k_2 = \frac{-1}{9}, T = 1, p = 1$. Obviously, $L_1 = \frac{1}{125}, L_2 = \frac{1}{150}, L_3 = \frac{1}{200}, N_1 = \frac{1}{50}, N_2 = \frac{1}{25}$ and by (3.7), it can be find that

$$\Omega_{L_1, L_2, L_3, N_1, N_2, \lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_2, \beta_3} = \frac{88}{375\sqrt{\pi}} + \frac{1649}{24000} = 0.2011 < 1.$$

Therefore, due to fact that all the assumptions of Theorem 3.5 hold, the BVP (3.8) has a unique solution. Besides, one can easily check the result of Theorem 3.4 for the BVP (3.8).

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Received: January 23, 2015.

Accepted: February 25, 2015