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DERIVED FUNCTORS USING $\mathcal{X}$-INJECTIVE CORESOLUTIONS AND DIMENSIONS

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Abstract In this paper, we take $\mathcal{X}$ is the class of all pure projective modules and introduce a right derived functor $Fixt^n_R(-,-)$ using $\mathcal{X}$-injective coresolutions. It is shown that $Fixt^1_R(M,N) \to Ext^1_R(M,N)$ is an isomorphism for all R modules N if and only if M is $\mathcal{X}^\perp$-projective. If R is a Noetherian ring, then we show that $glLcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq n-2$ if and only if $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq n$. Finally, we show that every left R-module has an $\mathcal{X}$-injective cover with the unique mapping property if and only if every pure injective R-module is $\mathcal{X}$-injective.

1 Introduction

The notions of covers and envelopes of modules were introduced by Enochs in [1]. Let $\mathscr{C}$ be a class of left R-modules. Following [1], we say that a map $f \in Hom_R(C,M)$ with $C \in \mathscr{C}$ is a $\mathscr{C}$-precover of M, if the group homomorphism $Hom_R(C',f) : Hom_R(C',C) \to Hom_R(C',M)$ is surjective for each $C' \in \mathcal{C}$. A $\mathscr{C}$-precover $f \in Hom_R(C,M)$ of M is called a $\mathscr{C}$-cover of M if f is right minimal. That is, if $fg = f$ implies that g is an automorphism for each $g \in End_R(C)$. $\mathscr{C} \subseteq R\text{-}Mod$ is a precovering class (covering class) provided that each module has a $\mathscr{C}$-precover ($\mathcal{C}$-cover). Dually, we have the definition of $\mathscr{C}$ preenvelope ($\mathscr{C}$ envelope).

Given a class $\mathscr{C}$ of left R-modules, we write

$$\begin{aligned} \mathscr{C}^\perp &= \{N \in R\text{-}Mod \mid Ext^1_R(M,N) = 0, \ \forall\, M \in \mathscr{C}\} \\ {}^\perp\mathscr{C} &= \{N \in R\text{-}Mod \mid Ext^1_R(N,M) = 0, \ \forall\, M \in \mathscr{C}\}. \end{aligned}$$

A $\mathscr{C}$-precover f of M is said to be *special* if f is an epimorphism and $\ker f \in \mathscr{C}^\perp$.

A $\mathscr{C}$-preenvelope f of M is said to be *special* if f is a monomorphism and $coker f \in {}^\perp\mathscr{C}$.

A module is said to be *pure projective* [5] if it is projective with respect to pure exact sequence.

The notions of FP-injective modules and FP-injective dimensions of modules and rings were first introduced by Stenström in [9]. The *FP-injective dimension* of an R-module M, denoted by $FP\text{-}id(M)$, is defined to be the smallest nonnegative integer n such that M has an FP-injective resolution of length n. L. Mao and N. Ding in [4] introduced by $\mathcal{X}$-injective modules. An R-module M is called $\mathcal{X}$-injective if $Ext^1_R(A,M) = 0$ for all R-modules $A \in \mathcal{X}$.

In this paper, we take $\mathcal{X}$ is the class of all pure projective modules. $\mathcal{X}^\perp$ is the class of all $\mathcal{X}$-injective modules. An R-module M is $\mathcal{X}^\perp$-projective if $Ext^1_R(M,U) = 0$ for all R-modules $U \in \mathcal{X}^\perp$. ${}^\perp(\mathcal{X}^\perp)$ is the class of all $\mathcal{X}^\perp$-projective modules. Clearly, $({}^\perp(\mathcal{X}^\perp), \mathcal{X}^\perp)$ is a cotorsion theory.

Throughout this paper, R denotes a non-trivial associative ring with identity and $\mathcal{M}$ denotes a category of left R-modules. Clearly, $\mathcal{M}$ is an abelian category with enough injectives. The class $\mathcal{X}^\perp$ of $\mathcal{X}$-injective modules is a full subcategory which is closed under isomorphisms. Similarly, a subcategory of a subcategory $\mathcal{X}^\perp$ of $\mathcal{M}$ always means a full subcategory of $\mathcal{X}^\perp$ which is closed under isomorphisms. By $\mathcal{I}(\mathcal{M})$ we denote the classes of all injective objects of a category $\mathcal{M}$.

If $\mathcal{X}$ is the class of all pure projective modules, then every left R-module has an $\mathcal{X}^\perp$-coresolution over an arbitrary ring by [8, Theorem 2.2]. Then, the functor $Hom(-,-)$ is right balanced on $R\text{-}Mod \times R\text{-}Mod$ by $\mathcal{X}^\perp \times \mathcal{X}^\perp$. In [7], if $\mathcal{X}$ is the class of all pure projective modules, then every R-module has an $\mathcal{X}$-injective precover over a Noetherian ring R. It follows that every R-module has an $\mathcal{X}^{\perp}$-resolution. Hence, if R is a Noetherian ring, then $Hom(-,-)$ is left balanced.

Let $Fixt^n(-,-)$ denote the nth right derived functor of $Hom(-,-)$ with respect to the pair $\mathcal{X}^{\perp} \times \mathcal{X}^{\perp}$. Then, for two left R-modules M and N, $Fixt^n(M,N)$ can be computed using a $\mathcal{X}^{\perp}$-resolution of M or a $\mathcal{X}^{\perp}$-coresolution of N. Also we denote $Fixt_n(-,-)$ is the nth left derived functor of $Hom(-,-)$ with respect to the pair $\mathcal{X}^{\perp} \times \mathcal{X}^{\perp}$. Then, for two left R-modules M and N, $Fixt_n(M,N)$ can be computed using a $\mathcal{X}^{\perp}$-coresolution of M or a $\mathcal{X}^{\perp}$-resolution of N.

The left $\mathcal{X}^{\perp}$-dimension of a left R-module M, denoted by $Lcores.dim_{\mathcal{X}^{\perp}}(M)$, is defined as $\inf\{n\colon$ there is a $\mathcal{X}^{\perp}$-resolution of the form $0 \to G_n \to \cdots \to G_0 \to M \to 0$ of $M\}$. If there is no such n, set left $Lcores.dim_{\mathcal{X}^{\perp}}(M) = \infty$. The *global left $\mathcal{X}^{\perp}$-dimension of $\mathcal{M}$*, denoted by $glLcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M})$, is defined to be $sup\{Lcores.dim_{\mathcal{X}^{\perp}}(M)\colon M \in \mathcal{M}\}$ and is infinite otherwise. The right versions can be defined similarly. We denote by $Rcores.dim_{\mathcal{X}^{\perp}}(M)$ the right $\mathcal{X}^{\perp}$-dimension of a left R-module M and we denote by $glRcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M})$ the *global right $\mathcal{X}^{\perp}$-dimension of $\mathcal{M}$*.

This paper is organized as follows: In Section 2, we take $\mathcal{X}$ is the class of all pure projective modules and we introduce a right derived functor $Fixt^n_R(-,-)$ using $\mathcal{X}^{\perp}$-injective coresolutions. It is shown that $Fixt^1_R(M,N) \to Ext^1_R(M,N)$ is an isomorphism for all R modules N if and only if M is $\mathcal{X}^{\perp}$-projective.

In Section 3, we investigate the $\mathcal{X}$-injective dimension of modules and rings and the left derived functors $Fixt_n(-,-)$. Let R be a Noetherian ring. We prove that $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq 1$ if and only if the canonical map $\mu\colon Fixt_0({}_RR,N) \to Hom({}_RR,N)$ is a monomorphism for any left R-module N. Then, it is shown that $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq n (n \geq 2)$ if and only if $Fixt_{n+k}(M,N) = 0$ for all left R-modules N and all $k \geq -1$. Moreover, $glLcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq n-2 (n \geq 2)$ if and only if $Fixt_{n+k}(M,N) = 0$ for all pure injective R-modules N and all $k \geq -1$. Finally, every left R-module has an $\mathcal{X}^{\perp}$-injective cover with the unique mapping property if and only if $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq 2$.

2 The Right Derived Functors using $\mathcal{X}$-injective coresolutions:

In this section, we investigate right derived functors using $\mathcal{X}^{\perp}$-coresolution. By [8, Theorem 2.2], every R-module has an $\mathcal{X}^{\perp}$-coresolution. Let $0 \to N \to G^0 \to G^1 \to \cdots$ be an $\mathcal{X}^{\perp}$-coresolution of N. This complex is unique up to homotopy. This leads us to new derived functors $Fixt^n_R(-,-)$, which are well defined. Applying the functor $Hom_R(M,-)$, we obtain the deleted complex

$$\mathbb{G}^{\bullet}\colon 0 \to Hom_R(M,G^0) \to Hom_R(M,G^1) \to \cdots.$$

Then, we can define $Fixt^n_R(M,N) = H^n(Hom(M,\mathbb{G}^{\bullet}))$.

Theorem 2.1. *For $M,N \in Obj(\mathcal{M})$, then the $Fixt^n_R(M,N)$ are well defined.*

Proof. For any R-module N, there is an $\mathcal{X}^{\perp}$-coresolution of M

$$0 \to N \xrightarrow{\epsilon} G^0 \xrightarrow{\alpha_1} G^1 \xrightarrow{\alpha_2} \cdots$$

and an R-module $\overline{N}$, there is an $\mathcal{X}^{\perp}$-coresolution of $\overline{N}$

$$0 \to \overline{N} \xrightarrow{\epsilon'} \overline{G^0} \xrightarrow{\alpha'_1} \overline{G^1} \xrightarrow{\alpha'_2} \cdots$$

Let $\gamma \in Hom_R(N,\overline{N})$. We only need to show that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\epsilon"] \arrow[d, "\gamma"] & G^0 \arrow[r, "\alpha_1"] \arrow[d, dashed, "\gamma_0"] & G^1 \arrow[r, "\alpha_2"] \arrow[d, dashed, "\gamma_1"] & \cdots \arrow[r] & G^n \arrow[r, "\alpha_{n+1}"] \arrow[d, dashed, "\gamma_n"] & G^{n+1} \arrow[r] \arrow[d, dashed, "\gamma_{n+1}"] & \cdots \\
0 \arrow[r] & \overline{N} \arrow[r, "\epsilon'"] & \overline{G^0} \arrow[r, "\alpha'_1"] & \overline{G^1} \arrow[r, "\alpha'_2"] & \cdots \arrow[r] & \overline{G^n} \arrow[r, "\alpha'_{n+1}"] & \overline{G^{n+1}} \arrow[r] & \cdots
\end{tikzcd}$$

and the associated map of $\mathcal{X}^{\perp}$-coresolutions is unique up to homotopy.

Every R-module has a special $\mathcal{X}^{\perp}$-preenvelope. Then, N has an $\mathcal{X}^{\perp}$-preenvelope G^0 and $\overline{N}$ has an $\mathcal{X}^{\perp}$-preenvelope $\overline{G^0}$such that $\gamma_0 \in Hom_R(G^0,\overline{G^0}), \epsilon' \in Hom_R(\overline{N},\overline{G^0})$ and $\gamma_0\epsilon = \epsilon'\gamma$. Then the following diagram is commutative

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\epsilon"] \arrow[d, "\gamma"'] & G^0 \arrow[r] \arrow[d, dashed, "\gamma_0"'] & L^1 \arrow[r] \arrow[d, dashed, "l_1"'] & 0 \\
0 \arrow[r] & \overline{N} \arrow[r, "\epsilon'"] & \overline{G^0} \arrow[r] & \overline{L^1} \arrow[r] & 0,
\end{tikzcd}
$$

where L^1 and $\overline{L^1}$ are cokernel of ϵ and ϵ'. Let G^1 and $\overline{G^1}$ are $\mathcal{X}^{\perp}$-preenvelop of L^1 and $\overline{L^1}$, respectively. Then, the diagram

$$
\begin{tikzcd}
0 \arrow[r] & L^1 \arrow[r] \arrow[d, "l_1"'] & G^1 \arrow[d, dashed, "\gamma_1"'] \\
0 \arrow[r] & \overline{L^1} \arrow[r] & \overline{G^1}
\end{tikzcd}
$$

is commutative. Continuing this process, there is $\gamma_{n-1} \in Hom_R(G^{n-1}, \overline{G^{n-1}})$ such that the following diagram

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\epsilon"] \arrow[d, "\gamma"'] & G^0 \arrow[r, "\alpha_1"] \arrow[d, dashed, "\gamma_0"'] & G^1 \arrow[r] \arrow[d, dashed, "\gamma_1"'] & \cdots \arrow[r] & G^{n-1} \arrow[r] \arrow[d, dashed, "\gamma_{n-1}"'] & L_n \arrow[r] \arrow[d, dashed, "l_n"'] & 0 \\
0 \arrow[r] & \overline{N} \arrow[r, "\epsilon'"] & \overline{G^0} \arrow[r, "\alpha_1'"] & \overline{G^1} \arrow[r] & \cdots \arrow[r] & \overline{G^{n-1}} \arrow[r] & \overline{L^n} \arrow[r] & 0,
\end{tikzcd}
$$

is commutative, where G^i and $\overline{G^i}$ are $\mathcal{X}$-injective for each $i \in \{0, 1, \cdots, n-1\}$, L^n and $\overline{L^n}$ are cokernals. Let G^n and $\overline{G^n}$ are $\mathcal{X}^{\perp}$-preenvelop of L^n and $\overline{L^n}$, respectively. Then, there exists $\gamma_n \in Hom_R(G^n, \overline{G^n})$ such that the diagram

$$
\begin{tikzcd}
0 \arrow[r] & L^n \arrow[r] \arrow[d, "l_n"'] & G^n \arrow[d, dashed, "\gamma_n"'] \\
0 \arrow[r] & \overline{L^n} \arrow[r] & \overline{G^n}
\end{tikzcd}
$$

is commutative. It follows that we can complete the diagram.

We are now to prove the uniqueness up to homotopy, that is, to prove that from the following diagram

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\alpha_0"] \arrow[d, equal] & G^0 \arrow[r, "\alpha_1"] \arrow[dl, "s_{-1}"'] \arrow[d, shift right, "\gamma_0"'] \arrow[d, shift left, "\gamma_0'"] & \cdots \arrow[r, "\alpha_n"] & G^n \arrow[r, "\alpha_{n+1}"] \arrow[dl, "s_{n-1}"'] \arrow[d, shift right, "\gamma_n"'] \arrow[d, shift left, "\gamma_n'"] & G^{n+1} \arrow[r] \arrow[dl, "s_n"'] \arrow[d, shift right, "\gamma_{n+1}"'] \arrow[d, shift left, "\gamma_{n+1}'"] & \cdots \\
0 \arrow[r] & N \arrow[r, "\alpha_0'"] & \overline{G^0} \arrow[r, "\alpha_1'"] & \cdots \arrow[r] & \overline{G^n} \arrow[r, "\alpha_{n+1}'"] & \overline{G^{n+1}} \arrow[r] & \cdots
\end{tikzcd}
$$

there exist $s_0, \cdots, s_n, \cdots$, with $s_n \colon G^{n+1} \to \overline{G^n}$ such that $\gamma_n - \gamma_n' = \alpha_n' \circ s_{n-1} + s_n \circ \alpha_{n+1}$, where $s_{-1} = 0$. We know that $\gamma_0 \alpha_0 = \alpha_0' = \gamma_0' \alpha_0$. It follows that $(\gamma_0 - \gamma_0')\alpha_0 = 0$. Then, we have the diagram

$$
\begin{tikzcd}
N \arrow[r] \arrow[d, equal] & G^0 \arrow[r] \arrow[d, "\gamma_0 - \gamma_0'"'] & G^0/N \arrow[r] \arrow[d] & G^1 \arrow[dl, dashed] \\
N \arrow[r] & \overline{G^0} \arrow[r] & \overline{G_0/N} &
\end{tikzcd}
$$

which can be completed since G^1 is an $\mathcal{X}^{\perp}$-preenvelope, that is, there exists $s_0 \in Hom_R(G^1, \overline{G^0})$ such that $\gamma_0 - \gamma_0' = s_0 \circ \alpha_1$.

We will show that the following diagram is commutative, that is, a map on G^0 which is zero.

$$\begin{tikzcd}
N \arrow[r] \arrow[d, equal] & G^0 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_0 - \gamma_0'"'] & G^1 \arrow[r, "\alpha_2"] \arrow[d, "\gamma_1 - \gamma_1'"'] \arrow[dl, "s_0"'] & G^2 \arrow[dl, dashed, "s_1"'] \\
N \arrow[r] & \overline{G^0} \arrow[r, "\alpha_1'"] & \overline{G^1} \arrow[r, "\alpha_2'"] & \overline{G^2}.
\end{tikzcd}$$

Let s_1 be the map which completes the following diagram

$$\begin{tikzcd}
G^1/im\alpha_1 \arrow[r] \arrow[d, "\gamma_1 - \gamma_1' - \alpha_1' s_0"'] & G^2 \arrow[dl, dashed, "s_1"'] \\
\overline{G^1} &
\end{tikzcd}$$

Hence,

$$\begin{aligned}
(\gamma_1 - \gamma_1' - \alpha_1' s_0)\alpha_1 &= (\gamma_1 - \gamma_1')\alpha_1 - (\alpha_1' s_0)\alpha_1 \\
&= (\gamma_1 - \gamma_1')\alpha_1 - \alpha_1'(\gamma_0 - \gamma_0') \\
&= 0,
\end{aligned}$$

as desired. Continuing this process, we can find $s_2, \cdots, s_{n-1}$. Define s_n as the completion of the following diagram

$$\begin{tikzcd}
G^1/im\alpha_1 \arrow[r] \arrow[d, "\gamma_1 - \gamma_1' - \alpha_1' s_0"'] & G^2 \arrow[dl, dashed, "s_1"'] \\
\overline{G^1} &
\end{tikzcd}$$

It follows that we can get a following commutative diagram

$$\begin{tikzcd}
 & G^{n-1} \arrow[r, "\alpha_n"] \arrow[d, "\gamma_{n-1} - \gamma_{n-1}'"'] \arrow[dl, "s_{n-2}"'] & G_n \arrow[r, "\alpha_{n+1}"] \arrow[d, "\gamma_n - \gamma_n'"'] \arrow[dl, "s_{n-1}"'] & G^{n+1} \arrow[dl, dashed, "s_n"'] \\
\overline{G^{n-2}} \arrow[r, "\alpha_{n-1}'"'] & \overline{G^{n-1}} \arrow[r] & \overline{G^n}. &
\end{tikzcd}$$

Since the previous diagram has exact rows, then

$$\begin{aligned}
(\gamma_n - \gamma_n' - \alpha_n' s_{n-1})\alpha_n &= (\gamma_n - \gamma_n')\alpha_n - \alpha_n'(s_{n-1}\alpha_n) \\
&= (\gamma_n - \gamma_n')\alpha_n - \alpha_n'(\gamma_{n-1} - \gamma_{n-1}' - \alpha_{n-1}' s_{n-1}) \\
&= (\gamma_n - \gamma_n')\alpha_n - \alpha_n'(\gamma_{n-1} - \gamma_{n-1}') + (\alpha_{n-1}' s_{n-1}) \\
&= 0.
\end{aligned}$$

Hence, the $\mathcal{X}^{\perp}$-coresolution is unique up to homotopy. □

From the homology groups of this $\mathcal{X}^{\perp}$-coresolution gives a well defined derived functor which we will call $Fixt_R^n(M, N)$. Let $0 \to M \to G^0 \to G^1 \to \cdots$ be an $\mathcal{X}^{\perp}$-coresolution of M and $0 \to M \to I^0 \to I^1 \to \cdots$ an $\mathcal{I}(\mathcal{M})$-coresolution of M. Since I_0 is injective, there exists $g_0 \in Hom_R(G^0, I^0)$ such that $\alpha' = g \circ \alpha$. By the injectivity of I^1, there exists $g_1 \in Hom_R(G^1, I^1)$ such that $g_1\alpha_0 = \alpha_0' g_0$. Continuing this process and using analog proof of Theorem 2.1, we can complete the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal, "\gamma"'] & G^0 \arrow[r, "\alpha_1"] \arrow[d, dashed, "g_0"'] & G^1 \arrow[r, "\alpha_2"] \arrow[d, dashed, "g_1"'] & \cdots \\
0 \arrow[r] & M \arrow[r, "\alpha'"] & I^0 \arrow[r, "\alpha'_1"] & I^1 \arrow[r, "\alpha'_2"] & \cdots
\end{tikzcd}$$

to a commutative diagram uniquely, up to homotopy. Now applying $Hom_R(M, -)$ to the diagram

$$\begin{tikzcd}
0 \arrow[r] & G^0 \arrow[r] \arrow[d, "g_0"'] & G^1 \arrow[r, "\alpha_2"] \arrow[d, "g_1"'] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

gives natural maps $Fixt^n_R(M, G) \to Ext^n_R(M, G)$ for all $n \geq 0$.

Proposition 2.2. *For $M, G \in Obj(\mathcal{M})$, $Fixt^0_R(M, G) \cong Hom_R(M, G)$.*

Proof. Let $0 \to M \to G^0 \to G^1 \to \cdots$ be a $\mathcal{X}^\perp$-coresolution of M. Then, the homotopy groups of the complex

$$0 \to Hom_R(M, G^0) \to Hom_R(M, G^1) \to \cdots$$

gives us the groups $Fixt^n_R(M, G)$. Hence, $Fixt^0_R(M, G)$ is the kernel of $Hom_R(M, G^0) \to Hom_R(M, G^1)$. But the functor $Hom_R(M, -)$ is left exact. So if the sequence $0 \to G \to G^0 \to G^1$ is exact, then the sequence $0 \to Hom_R(M, G^0) \to Hom_R(M, G^1) \to \cdots$ is exact. Now $Hom_R(M, G)$ is isomorphic to the kernel of $Hom_R(M, G^0) \to Hom_R(M, G^1)$, that is, $Hom_R(M, G) \cong Fixt^0_R(M, G)$, as desired. □

Proposition 2.3. *For $M, G \in Obj(\mathcal{M})$, $Fixt^1_R(M, N) \to Ext^1_R(M, N)$ is injective.*

Proof. Let $0 \to N \to G^0 \to G^1 \to \cdots$ be an $\mathcal{X}^\perp$-coresolution of N and $0 \to N \to E^0 \to E^1 \to \cdots$ be an $\mathcal{I}(\mathcal{M})$-coresolution of N. Consider the exact sequences $0 \to N \to G^0 \to G^0/N \to 0$ and $0 \to N \to E^0 \to E^0/N \to 0$. From the following diagram with exact rows

$$\begin{tikzcd}
 & & & M \arrow[dl, dashed] \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & G^0 \arrow[r] \arrow[d] & G^0/N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & E^0 \arrow[r] & E^0/N \arrow[r] & 0,
\end{tikzcd}$$

we get the following diagram

$$\begin{tikzcd}
 & M \arrow[ddl, dashed] \arrow[d] \\
 & L \arrow[d] \\
E^0 \arrow[r] & E^0/N
\end{tikzcd}$$

is commutative. Then, the diagram

$$\begin{tikzcd} & M \arrow[dl, dashed] \arrow[d] \\ G^0 \arrow[r] & G^0/N \end{tikzcd}$$

can also be completed, as desired. $\square$

Theorem 2.4. *For $M \in Obj(\mathcal{M})$, M is $\mathcal{X}^{\perp}$-projective if and only if $Fixt^1_R(M,N) \to Ext^1_R(M,N)$ is an isomorphism for all R modules N.*

Proof. Suppose M is $\mathcal{X}^{\perp}$-projective. By Theorem [8, Theorem 2.2], N has an $\mathcal{X}$-injective coresolution, $0 \to N \xrightarrow{f} G^0 \to G^1 \to \cdots$. Consider the short exact sequence $0 \to N \xrightarrow{f} G^0 \to G^0/imf \to 0$. Then, we get the following commutative diagram with exact rows:

$$\begin{tikzcd} Hom_R(M,G^0) \arrow[r] \arrow[d, "\cong"] & Hom_R(M,G^0/imf) \arrow[r] \arrow[d, "\cong"] & Fixt^1_R(M,N) \arrow[r] \arrow[d] & Fixt^1_R(M,G^0)=0 \arrow[d] \\ Hom_R(M,G^0) \arrow[r] & Hom_R(M,G^0/imf) \arrow[r] & Ext^1_R(M,N) \arrow[r] & Ext^1_R(M,G^0)=0. \end{tikzcd}$$

Hence, $Fixt^1_R(M,N) \cong Ext^1_R(M,N)$ for any R-module N. Conversely, we may assume that $Fixt^1_R(M,N) \to Ext^1_R(M,G)$ is an isomorphism for all R-modules N. Then, $Ext^1_R(M,G)=0$ for all $\mathcal{X}$-injective R-modules G since $Fixt^1_R(M,G)=0$ for all $\mathcal{X}$-injective R-modules G and R-modules M. Thus, M is $\mathcal{X}^{\perp}$-projective. $\square$

3 $\mathcal{X}$-injective dimensions and the left derived functors of Hom

In this section, we deals with the $\mathcal{X}$-injective dimensions of modules and the left derived functors $Fixt_n(-,-)$.

For $M \in Obj\mathcal{M}$, there exists an $\mathcal{X}$-injective coresolution of M such that $0 \to M \xrightarrow{f} G^0 \xrightarrow{g_0} G^1 \to \cdots$. Then, we obtain the deleted complex

$$\mathbb{G}_{\bullet}\colon\ \cdots \to Hom(G', N) \xrightarrow{g_0^{\star}} Hom(G^0,N) \to 0$$

when we apply the functor $Hom(-,N)$. It follows that $Fixt_n(M,N)$ is exactly nth homology of the complex, that is, $Fixt_n(M,N) = H_n(Hom(\mathbb{G}_{\bullet},N))$. Then, there is a canonical map

$$\mu\colon Fixt_0(M,N) = Hom(G^0,N)/im(g_0^{\star}) \to Hom(M,N)$$

defined by $\mu(h + im(g_0^{\star})) = hg_0$ for $h \in Hom(G^0,N)$.

Proposition 3.1. *Let R be a Noetherian ring. For $M \in Obj\mathcal{M}$, M is $\mathcal{X}$-injective if and only if the canonical map $\mu\colon Fixt_0(M,N) \to Hom(M,N)$ is an epimorphism for any left R-module N.*

Proof. The direct implication is clear if $G^0 = M$. Conversely, if $N = M$ then there exists $h \in Hom(G^0,M)$ such that $\mu(h+im(g_0^{\star})) = hg_0 = 1_M$. Thus, M is isomorphic to a direct summand of G^0. Since the direct summand of $\mathcal{X}$-injective module is $\mathcal{X}$-injective, M is $\mathcal{X}$-injective. $\square$

Corollary 3.2. *Let R be a Noetherian ring. Then, the following conditions are equivalent:*

(i) ${}_RR$ *is $\mathcal{X}$-injective;*

(ii) The canonical map $\mu\colon Fixt_0({}_RR,N) \to Hom({}_RR,N)$ is an epimorphism for any left R-module N;

(iii) The canonical map $\mu\colon Fixt_0({}_RR,{}_RR) \to Hom({}_RR,{}_RR)$ is an epimorphism;

(iv) Every R-module has a special $\mathcal{X}$-injective cover.

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ follows from Proposition 3.1.

$(1) \Rightarrow (4)$. For any left R-module M, M has an $\mathcal{X}$-injective precover. Consider the exact sequence $F \to M \to 0$ with F a free module. By hypothesis, F is $\mathcal{X}$-injective.

$(4) \Rightarrow (1)$. Let $\phi\colon M \to {}_RR$ be a special $\mathcal{X}$-injective cover. Then, ${}_RR$ is isomorphic to a direct summand of M. Thus, ${}_RR$ is $\mathcal{X}$-injective. $\square$

Proposition 3.3. *Let R be a Noetherian ring. Then, the following are equivalent:*

(i) $Rcores.dim_{\mathcal{X}^\perp}(M) \leq 1$;

(ii) The canonical map $\mu\colon Fixt_0(M,N) \to Hom(M,N)$ *is a monomorphism for any left R-module N.*

Proof. (1) $\Rightarrow$ (2). By hypothesis, M has a $\mathcal{X}^\perp$-coresolution $0 \to G^0 \to G^1 \to 0$. It follows that we get an exact sequence $0 \to Hom(G^1,N) \to Hom(G^0,N) \to Hom(M,N)$ for any R-module N. Hence, μ is a monomorphism.

(2) $\Rightarrow$ (1). Consider the exact sequence $0 \to M \to G^0 \to L^1 \to 0$, where $M \to G^0$ is an $\mathcal{X}$-injective preenvelope. We only need to show that L^1 is $\mathcal{X}$-injective. By [2, Theorem 8.2.3], we have the commutative diagram with exact rows:

$$\begin{tikzcd}
& Fixt_0(L^1,L^1) \arrow[r] \arrow[d, "\mu_1"'] & Fixt_0(G^0,L^1) \arrow[r] \arrow[d, "\mu_2"'] & Fixt_0(M,L^1) \arrow[r] \arrow[d, "\mu_3"'] & 0 \\
0 \arrow[r] & Hom(L^1,L^1) \arrow[r] & Hom(G^0,L^1) \arrow[r] & Hom(M,L^1).
\end{tikzcd}$$

Note that α_2 is an epimorphism by Proposition 3.1 and μ_3 is a monomorphism by hypothesis. Hence, μ_1 is an epimorphism by the Snake Lemma [6, Theorem 6.5]. By Proposition 3.1, L^1 is $\mathcal{X}$-injective. $\square$

Lemma 3.4. *Let R be a Noetherian ring. Then,* $Rcores.dim_{\mathcal{X}^\perp}(M) = cores.dim_{\mathcal{X}^\perp}(M)$ *for any left R-module M.*

Proof. Clearly, $cores.dim_{\mathcal{X}^\perp}(M) \leq Rcores.dim_{\mathcal{X}^\perp}(M)$. Conversely, we assume that $cores.dim_{\mathcal{X}^\perp}(M) = n < \infty$. Consider the partial $\mathcal{X}^\perp$-coresolution of M, $0 \to G^0 \to G^1 \to \cdots \to G^{n-1}$. Then, we get an exact sequence $0 \to G^0 \to G^1 \to \cdots \to G^{n-1} \to L \to 0$. Hence, L is $\mathcal{X}$-injective, and so $Rcores.dim_{\mathcal{X}^\perp}(M) \leq n$, as desired. $\square$

The finitistic $\mathcal{X}^\perp$-coresolution of dimension, denoted by $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M})$, is defined as $\sup\{cores.dim_{\mathcal{X}^\perp}(M)\colon M \in Obj(\mathcal{M})\}$.

Proposition 3.5. *Let R be a Noetherian ring. Then, the following are equivalent:*

(i) $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq 1$;

(ii) The canonical map $\mu\colon Fixt_0({}_RR,N) \to Hom({}_RR,N)$ *is a monomorphism for any left R-module N.*

Proof. (1) $\Leftrightarrow$ (2). It follows by Proposition 3.3 and Lemma 3.4. $\square$

Proposition 3.6. *Let R be a Noetherian ring and an integer $n \geq 2$. Then, the following conditions are equivalent:*

(i) $Rcores.dim_{\mathcal{X}^\perp}(M) \leq n$;

(ii) $Fixt_{n+k}(M,N) = 0$ *for all R-modules N and all $k \geq -1$;*

(iii) $Fixt_{n-1}(M,N) = 0$ *for all R-modules N.*

Proof. (1) $\Rightarrow$ (2). Note that $Fixt_{n+k}(M,N) = 0$ for all $k \geq 1$. We only need to show that $Fixt_{n+k}(M,N) = 0$ for all $k \in \{-1,0\}$. Consider $\mathcal{X}^\perp$-coresolution of M, $0 \to G^0 \to G^1 \to \cdots \to G^n \to 0$. Which induces an exact sequence $0 \to Hom(G^n,N) \to Hom(G^{n-1},N) \to Hom(G^{n-2},N)$ for any R-module N. Hence, $Fixt_n(M,N) = Fixt_{n-1}(M,N) = 0$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Consider the $\mathcal{X}^\perp$-coresolution of M, $0 \to G^0 \to G^1 \to \cdots \to G^n \to 0$. with $L^n = coker(G^{n-2} \to G^{n-1})$. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & G^0 \arrow[r] & \cdots G^{n-2} \arrow[r, "\alpha"] & G^{n-1} \arrow[rr, "\beta"] \arrow[dr, "\gamma"] & & G^n \arrow[r] & \cdots \\
& & & & & L^n \arrow[ur, "\eta"] \arrow[dr] & & \\
& & & & 0 \arrow[ur] & & 0. &
\end{tikzcd}$$

Since $Fixt_{n-1}(M, L^n) = 0$, the sequence

$$Hom(G^n, L^n) \xrightarrow{\beta^\star} Hom(G^{n-1}, L^n) \xrightarrow{\alpha^\star} Hom(G^{n-2}, L^n)$$

is exact. Now $\alpha^\star(\gamma) = \gamma\alpha = 0$. It follows that $\gamma \in ker(\alpha^\star) = im(\beta^\star)$. Then, there exists $\nu \in Hom(G^n, L^n)$ such that $\gamma = \beta^\star(\nu) = \nu\beta = \nu\eta\gamma$, and hence $\nu\eta = 1$ since γ is epic. Therefore L^n is $\mathcal{X}$-injective, as desired. □

Corollary 3.7. *Let R be a Noetherian ring and an integer $n \geq 2$. Then, the following are equivalent:*

(i) $cores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq n$;

(ii) $Fixt_{n+k}(M, N) = 0$ *for all R-modules M, N and all $k \geq -1$;*

(iii) $Fixt_{n-1}(M, N) = 0$ *for all R-modules M and N.*

Proposition 3.8. *Let R be a Noetherian ring. If M is a pure injective R-module, then M has a minimal $\mathcal{X}^\perp$-resolution $\cdots \to G_{n-2} \to G_{n-3} \to \cdots \to G_1 \to G_0 \to M \to 0$ with each G_i an injective module.*

Proof. By [7, Theorem 5.5], M has an $\mathcal{X}$-injective precover $\gamma\colon G_0 \to M$. Consider the exact sequence $0 \to G_0 \xrightarrow{i} E \to L \to 0$ with E an injective envelope of G_0. Since the exact sequence is pure and M is pure injective, there exists $h \in Hom(E, M)$ such that $\gamma = hi$. By $\mathcal{X}$-injective precover γ of M, there exists $\phi \in Hom(E, G)$ such that $\gamma\phi = h$. Hence, $\gamma\phi i = hi = \gamma$. This implies that ϕi is an isomorphism. Then, G is isomorphic to a direct summand of E and hence G is injective. Note that $\ker\gamma \in (\mathcal{X}^\perp)^\perp$. Hence, $\ker\gamma$ has an $\mathcal{X}$-injective precover $\gamma_1\colon G_1 \to \ker\gamma$ with G_1 an injective module, where $\ker\gamma_1 \in (\mathcal{X}^\perp)^\perp$. By continuing the above process, we get the minimal $\mathcal{X}^\perp$-resolution of M. □

Theorem 3.9. *Let R be a Noetherian ring. Consider the following conditions for a pure injective R-module N and an integer $n \geq 2$:*

(i) $Lcores.dim_{\mathcal{X}^\perp}(N) \leq n-2$;

(ii) $Fixt_{n+k}(M, N) = 0$ *for all R-modules M and all $k \geq -1$;*

(iii) $Fixt_{n+k}(M, N) = 0$ *for all R-modules M.*

Proof. $(1) \Rightarrow (2)$. By hypothesis, N has a left $\mathcal{X}^\perp$-resolution

$$0 \to G_{n-2} \to \cdots \to G_1 \to G_0 \to N \to 0.$$

Then, we have the following sequence

$$0 \to Hom(M, G_{n-2}) \to Hom(M, G_{n-3}) \to \cdots Hom(M, G_0) \to 0$$

for any R-module M. Hence, $Fixt_{n+k}(M, N) = 0$ for all $k \geq -1$.

$(2) \Rightarrow (3)$ is trivial.

$(3) \Rightarrow (1)$. By pure injectivity of N and Proposition 3.8, N has a minimal $\mathcal{X}^\perp$-resolution:

$$\cdots \to G_n \xrightarrow{\alpha} G_{n-1} \xrightarrow{\beta} G_{n-2} \xrightarrow{\gamma} G_{n-3} \xrightarrow{\mu} \cdots \to G_1 \to G_0 \to M \to 0$$

with each G_i an injective module. Let $i_\beta \in Hom(K, G_{n-1})$ be the inclusion and $p_\beta \in Hom(G_{n-1}, H)$ the canonical projection, where $K = \ker\beta$ and $H = G_{n-1}/K$. Then, there exists $\eta_1 \in Hom(G_n, K)$ such that $\alpha = i_\beta\eta_1$ and there exists a monomorphism $\eta_2 \in Hom(H, G_{n-2})$ such that $\beta = \eta_2 p_\beta$. Let $\eta_3 \in Hom(G_{n-2}, L)$ be the canonical projection, where $L = G_{n-2}/im(\eta_2.)$ Then, there exists a homomorphism $i_{\eta_2} \in Hom(L, G_{n-3})$ such that $\gamma = i_{\eta_2}\eta_3$. Then, we get the following commutative diagram:

$$\begin{tikzcd}
G_n \arrow[rr, "\alpha"] \arrow[dr, "\eta_1"] & & G_{n-1} \arrow[rr, "\beta"] \arrow[dr, "p_\beta"] & & G_{n-2} \arrow[rr, "\gamma"] \arrow[dr, "\eta_3"] & & G_{n-3} \\
& K \arrow[ur, "i_\beta"] \arrow[dr] & & H \arrow[ur, "\eta_2"] \arrow[dr] & & L \arrow[ur, "i_{\eta_2}"] \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0.
\end{tikzcd}$$

By (3), $Fixt_{n-1}(K,N)=0$. Then, the sequence

$$Hom(K,G_n) \xrightarrow{\alpha_*} Hom(K,G_{n-1}) \xrightarrow{\beta_*} Hom(K,G_{n-2})$$

is exact. Since $\beta_*(i_\beta)=\beta i_\beta=0, i_\beta\in\ker\beta_*=im\alpha_*$. Therefore $i_\beta=\alpha_*(t)=\alpha(t)$ for some $t\in Hom(K,G_n)$. By $\alpha=i_\beta\eta_1, i_\beta=i_\beta\eta_1 t$. Hence, $\eta_1 t=1_{G_n}$ since i_β is monic. So K is injective. It follows that H and L are injective. We claim that the complex

$$0\to L\overset{i_{\eta_2}}{\to} G_{n-3}\to\cdots\to G_1\to G_0\to N\to 0$$

is an $\mathcal{X}^\perp$-resolution of N. We only need to show that the complex

$$0\to Hom(G,L)\xrightarrow{(i_{\eta_2})_*} Hom(G,G_{n-3})\xrightarrow{\mu_*} Hom(G,G_{n-4})$$

is exact for any $\mathcal{X}$-injective R-module G. Note that we have the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
Hom(G,G_{n-1}) \arrow[rr, "\beta_*"] \arrow[dr, "(p_\beta)_*"'] & & Hom(G,G_{n-2}) \arrow[rr, "\gamma_*"] \arrow[dr, "(\eta_3)_*"'] & & Hom(G,G_{n-3}) \\
& Hom(G,H) \arrow[ur, "(\eta_2)_*"'] \arrow[dr] & & Hom(G,L) \arrow[ur, "(i_{\eta_2})_*"'] \arrow[dr] & \\
0 \arrow[ur] & & 0 & & 0.
\end{tikzcd}$$

Now $\ker((i_{\eta_2})_*(\eta_3)_*)=\ker(\gamma_*)=im(\beta_*)=im((\eta_2)_*(p_\beta)_*)=im((\eta_2)_*)=\ker((\eta_3)_*)$. Let $g\in\ker((i_{\eta_2})_*)$. Since $(\eta_3)_*$ is epic, $g=(\eta_3)_*(g_{n-2})$ for some $g_{n-2}\in Hom(G,G_{n-2})$. Therefore $(i_{\eta_2})_*(\eta_3)_*(g_{n-2})=0$. Hence, $(\eta_3)_*(g_{n-2})=0$. That is, $g=0$. It follows that $(i_{\eta_2})_*$ is monic. On the other hand $\ker(\mu_*)=im(\gamma_*)=im((i_{\eta_2})_*)$. Hence, we obtain the desired exact sequence. This completes the proof. □

Corollary 3.10. *Let R be a Noetherian ring and an integer $n\geq 2$. Consider the following conditions*

(i) $glLcores.dim_{\mathcal{X}^\perp}(\mathcal{M})\leq n-2$;

(ii) $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M})\leq n$;

(iii) $Lcores.dim_{\mathcal{X}^\perp}(N)\leq n-2$ *for all pure injective R-modules N*;

(iv) $Fixt_{n+k}(M,N)=0$ *for all R-modules M, all pure injective R-modules N and all $k\geq -1$*;

(v) $Fixt_{n-1}(M,N)=0$ *for all R-modules M and all pure injective R-modules N.*

Then, (1) ⇒ (2) ⇒ (3) ⇔ (4) ⇔ (5).

Proof. It follows from Corollary 3.7 and Theorem 3.9. □

Lemma 3.11. *Let R be a Noetherian ring and an integer $n\geq 1$. If M is pure injective R-module, then $id(M)\leq n$ if and only if for the minimal left $\mathcal{X}^\perp$-resolution $\cdots\to G_n\to G_{n-1}\to\cdots\to G_1\to G_0\to N\to 0$ of any pure injective R-module N, $Hom(M,G_n)\to Hom(M,K_n)$ is an epimorphism.*

Proof. We prove the result by induction on n. Let $n=0$. If M is injective, $Hom(M,G_0)\to Hom(M,K_0)$ is an epimorphism. Conversely, put $N=M$. Then, $Hom(M,G_0)\to Hom(M,M)$ is an epimorphism, and hence M is injective.

Let $n\geq 1$. Consider an exact sequence $0\to M\to E\to L\to 0$ with E an injective module. Then, we have the following diagrams

$$\begin{tikzcd}
Hom(E,G_n) \arrow[r] \arrow[d] & Hom(E,K_n) \arrow[r] \arrow[d] & 0 \\
Hom(M,G_n) \arrow[r] \arrow[d] & Hom(M,K_n) & \\
0 & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Hom(L,K_n) \arrow[r] \arrow[d] & Hom(L,G_{n-1}) \arrow[r] \arrow[d] & Hom(L,K_{n-1}) \arrow[d] & \\
0 \arrow[r] & Hom(E,K_n) \arrow[r] \arrow[d] & Hom(E,G_{n-1}) \arrow[r] \arrow[d] & Hom(E,K_{n-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Hom(M,K_n) \arrow[r] & Hom(M,G_{n-1}) \arrow[r] \arrow[d] & Hom(M,K_{n-1}). & \\
 & & 0 & &
\end{tikzcd}$$

are exact and commutative. By [3, Lemma 3.2.10], L is pure injective. Therefore $id(M) \leq n$ if and only if $id(L) \leq n-1$ if and only if $Hom(L,G_{n-1}) \to Hom(L,K_{n-1})$ is an epimorphism by induction on n if and only if $Hom(E,K_n) \to Hom(M,K_n)$ is an epimorphism by the second diagram if and only if $Hom(M,G_n) \to Hom(M,K_n)$ is an epimorphism by the first diagram. □

Theorem 3.12. *Let R be a Noetherian ring. Then, $Fixt_R^n(M,N) \cong Fixt_n^R(M,N)$ for $n \geq 0$.*

Proof. By Theorem [8, Theorem 2.2], M has a $\mathcal{X}^{\perp}$-coresolution. Let $0 \to M \to G^0 \to G^1 \to \cdots$ be an $\mathcal{X}^{\perp}$-coresolution of M. Since R is hereditary Noetherian, then there exists an $\mathcal{X}^{\perp}$-resolution $\cdots \to G_2 \to G_1 \to G_0 \to N \to 0$

$$\begin{tikzcd}
 & 0 & 0 & 0 & & \\
\cdots \arrow[r] & Hom(M,G_2) \arrow[r] \arrow[u] & Hom(M,G_1) \arrow[r] \arrow[u] & Hom(M,G_0) \arrow[r] \arrow[u] & 0 & \\
\cdots \arrow[r] & Hom(G^0,G_2) \arrow[r] \arrow[u] & Hom(G^0,G_1) \arrow[r] \arrow[u] & Hom(G^0,G_0) \arrow[r] \arrow[u] & Hom(G^0,N) \arrow[r] \arrow[u] & 0 \\
\cdots \arrow[r] & Hom(G^1,G_2) \arrow[r] \arrow[u] & Hom(G^1,G_1) \arrow[r] \arrow[u] & Hom(G^1,G_0) \arrow[r] \arrow[u] & Hom(G^1,N) \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Since G_i are $\mathcal{X}$-injective and $\mathbb{G}^{\bullet}$ is an $\mathcal{X}^{\perp}$-coresolution, all rows but the first are exact. Similarly all columns but the last are exact. Then, by chasing diagrams or by a spectral sequence argument,

$$H^n(Hom(M,\mathbb{G}^{\bullet}(N))) \cong H_n(Hom(\mathbb{G}_{\bullet}(M),N)$$

for each $n \geq 0$. This completes the proof. □

Theorem 3.13. *Let R be a Noetherian ring. Then, the following conditions are equivalent:*

(i) Every left R-module has an $\mathcal{X}$-injective cover with the unique mapping property;

(ii) $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq 2$;

(iii) $Fixt_1(M,N) = 0$ for all left R-modules M and N;

(iv) $Fixt_k(M,N) = 0$ for all left R-modules M and N and $k \geq 1$.

Proof. $(1) \Rightarrow (2)$. Let M be an R-module. Then, M has an $\mathcal{X}$-injective cover $\phi\colon U \to M$ with the unique mapping property. Hence, $0 \to U \to M \to 0$ is a left $\mathcal{X}$-resolution. Thus, $glLcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) = 0$. Hence, by Corollary 3.10, $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq 2$.

$(2) \Rightarrow (1)$. Let M be an R-module. By [7, Theorem 5.6], M has an $\mathcal{X}$-injective cover $h\colon U \to M$. It is enough to show that, if for any $\mathcal{X}$-injective left R-module U' and any homomorphism $j\colon U' \to U$ such that $hj = 0$ then $j = 0$. Consider the natural map $\pi\colon U \to U/imj$. Then, there exists $\overline{h}\colon U/imj \to M$ such that $\overline{h}\pi = h$ since $imj \subseteq \ker h$. By hypothesis, U/imj is $\mathcal{X}$-injective. Then there exists $\mu\colon U/imj \to U$ such that $\overline{h} = h\mu$. Hence, we get the following commutative diagram with exact row:

$$\begin{tikzcd} 0 \arrow[r] & \ker j \arrow[r, "i"] & U' \arrow[r, "j"] \arrow[dr, "0"'] & U \arrow[r, shift left, "\pi"] \arrow[d, "h"] & U/imj \arrow[l, shift left, "\mu"] \arrow[r] \arrow[dl, "\overline{h}"] & 0. \\ & & & M & & \end{tikzcd}$$

Thus, $h\mu\pi = h$, and hence $\mu\pi$ is an isomorphism. It follow that π is monic. Thus, $j = 0$.

$(2) \Leftrightarrow (3) \Leftrightarrow (4)$. It follows from Corollary 3.7. □

Remark 3.14. Let R be a Noetherian ring. Then, every left R-module has an $\mathcal{X}$-injective cover with the unique mapping property if and only if every pure injective R-module is $\mathcal{X}$-injective.

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