

DERIVED FUNCTORS USING \mathcal{X} -INJECTIVE CORESOLUTIONS AND DIMENSIONS

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Abstract In this paper, we take \mathcal{X} is the class of all pure projective modules and introduce a right derived functor $\text{Fixt}_R^n(-, -)$ using \mathcal{X} -injective coresolutions. It is shown that $\text{Fixt}_R^1(M, N) \rightarrow \text{Ext}_R^1(M, N)$ is an isomorphism for all R modules N if and only if M is \mathcal{X}^\perp -projective. If R is a Noetherian ring, then we show that $\text{glLcores.dim}_{\mathcal{X}^\perp}(\mathcal{M}) \leq n - 2$ if and only if $\text{Fcores.dim}_{\mathcal{X}^\perp}(\mathcal{M}) \leq n$. Finally, we show that every left R -module has an \mathcal{X} -injective cover with the unique mapping property if and only if every pure injective R -module is \mathcal{X} -injective.

1 Introduction

The notions of covers and envelopes of modules were introduced by Enochs in [1]. Let \mathcal{C} be a class of left R -modules. Following [1], we say that a map $f \in \text{Hom}_R(C, M)$ with $C \in \mathcal{C}$ is a \mathcal{C} -precover of M , if the group homomorphism $\text{Hom}_R(C', f) : \text{Hom}_R(C', C) \rightarrow \text{Hom}_R(C', M)$ is surjective for each $C' \in \mathcal{C}$. A \mathcal{C} -precover $f \in \text{Hom}_R(C, M)$ of M is called a \mathcal{C} -cover of M if f is right minimal. That is, if $fg = f$ implies that g is an automorphism for each $g \in \text{End}_R(C)$. $\mathcal{C} \subseteq R\text{-Mod}$ is a precovering class (covering class) provided that each module has a \mathcal{C} -precover (\mathcal{C} -cover). Dually, we have the definition of \mathcal{C} preenvelope (\mathcal{C} envelope).

Given a class \mathcal{C} of left R -modules, we write

$$\begin{aligned}\mathcal{C}^\perp &= \{N \in R\text{-Mod} \mid \text{Ext}_R^1(M, N) = 0, \forall M \in \mathcal{C}\} \\ {}^\perp\mathcal{C} &= \{N \in R\text{-Mod} \mid \text{Ext}_R^1(N, M) = 0, \forall M \in \mathcal{C}\}.\end{aligned}$$

A \mathcal{C} -precover f of M is said to be *special* if f is an epimorphism and $\ker f \in \mathcal{C}^\perp$.

A \mathcal{C} -preenvelope f of M is said to be *special* if f is a monomorphism and $\text{coker } f \in {}^\perp\mathcal{C}$.

A module is said to be *pure projective* [5] if it is projective with respect to pure exact sequence.

The notions of FP -injective modules and FP -injective dimensions of modules and rings were first introduced by Stenström in [9]. The FP -injective dimension of an R -module M , denoted by $FP\text{-id}(M)$, is defined to be the smallest nonnegative integer n such that M has an FP -injective resolution of length n . L. Mao and N. Ding in [4] introduced by \mathcal{X} -injective modules. An R -module M is called \mathcal{X} -injective if $\text{Ext}_R^1(A, M) = 0$ for all R -modules $A \in \mathcal{X}$.

In this paper, we take \mathcal{X} is the class of all pure projective modules. \mathcal{X}^\perp is the class of all \mathcal{X} -injective modules. An R -module M is \mathcal{X}^\perp -projective if $\text{Ext}_R^1(M, U) = 0$ for all R -modules $U \in \mathcal{X}^\perp$. ${}^\perp(\mathcal{X}^\perp)$ is the class of all \mathcal{X}^\perp -projective modules. Clearly, $({}^\perp(\mathcal{X}^\perp), \mathcal{X}^\perp)$ is a cotorsion theory.

Throughout this paper, R denotes a non-trivial associative ring with identity and \mathcal{M} denotes a category of left R -modules. Clearly, \mathcal{M} is an abelian category with enough injectives. The class \mathcal{X}^\perp of \mathcal{X} -injective modules is a full subcategory which is closed under isomorphisms. Similarly, a subcategory of a subcategory \mathcal{X}^\perp of \mathcal{M} always means a full subcategory of \mathcal{X}^\perp which is closed under isomorphisms. By $\mathcal{I}(\mathcal{M})$ we denote the classes of all injective objects of a category \mathcal{M} .

If \mathcal{X} is the class of all pure projective modules, then every left R -module has an \mathcal{X}^\perp -coresolution over an arbitrary ring by [8, Theorem 2.2]. Then, the functor $\text{Hom}(-, -)$ is right balanced on $R\text{-Mod} \times R\text{-Mod}$ by $\mathcal{X}^\perp \times \mathcal{X}^\perp$. In [7], if \mathcal{X} is the class of all pure projective modules, then every R -module has an \mathcal{X} -injective precover over a Noetherian ring R . It follows

that every R -module has an \mathcal{X}^\perp -resolution. Hence, if R is a Noetherian ring, then $Hom(-, -)$ is left balanced.

Let $Fixt^n(-, -)$ denote the n th right derived functor of $Hom(-, -)$ with respect to the pair $\mathcal{X}^\perp \times \mathcal{X}^\perp$. Then, for two left R -modules M and N , $Fixt^n(M, N)$ can be computed using a \mathcal{X}^\perp -resolution of M or a \mathcal{X}^\perp -coresolution of N . Also we denote $Fixt_n(-, -)$ is the n th left derived functor of $Hom(-, -)$ with respect to the pair $\mathcal{X}^\perp \times \mathcal{X}^\perp$. Then, for two left R -modules M and N , $Fixt_n(M, N)$ can be computed using a \mathcal{X}^\perp -coresolution of M or a \mathcal{X}^\perp -resolution of N .

The left \mathcal{X}^\perp -dimension of a left R -module M , denoted by $Lcores.dim_{\mathcal{X}^\perp}(M)$, is defined as $\inf\{n: \text{there is a } \mathcal{X}^\perp\text{-resolution of the form } 0 \rightarrow G_n \rightarrow \dots \rightarrow G_0 \rightarrow M \rightarrow 0 \text{ of } M\}$. If there is no such n , set $Lcores.dim_{\mathcal{X}^\perp}(M) = \infty$. The global left \mathcal{X}^\perp -dimension of \mathcal{M} , denoted by $glLcores.dim_{\mathcal{X}^\perp}(\mathcal{M})$, is defined to be $\sup\{Lcores.dim_{\mathcal{X}^\perp}(M): M \in \mathcal{M}\}$ and is infinite otherwise. The right versions can be defined similarly. We denote by $Rcores.dim_{\mathcal{X}^\perp}(M)$ the right \mathcal{X}^\perp -dimension of a left R -module M and we denote by $glRcores.dim_{\mathcal{X}^\perp}(\mathcal{M})$ the global right \mathcal{X}^\perp -dimension of \mathcal{M} .

This paper is organized as follows: In Section 2, we take \mathcal{X} is the class of all pure projective modules and we introduce a right derived functor $Fixt_R^n(-, -)$ using \mathcal{X}^\perp -injective coresolutions. It is shown that $Fixt_R^1(M, N) \rightarrow Ext_R^1(M, N)$ is an isomorphism for all R modules N if and only if M is \mathcal{X}^\perp -projective.

In Section 3, we investigate the \mathcal{X} -injective dimension of modules and rings and the left derived functors $Fixt_n(-, -)$. Let R be a Noetherian ring. We prove that $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq 1$ if and only if the canonical map $\mu: Fixt_0({}_R R, N) \rightarrow Hom({}_R R, N)$ is a monomorphism for any left R -module N . Then, it is shown that $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq n (n \geq 2)$ if and only if $Fixt_{n+k}(M, N) = 0$ for all left R -modules N and all $k \geq -1$. Moreover, $glLcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq n - 2 (n \geq 2)$ if and only if $Fixt_{n+k}(M, N) = 0$ for all pure injective R -modules N and all $k \geq -1$. Finally, every left R -module has an \mathcal{X}^\perp -injective cover with the unique mapping property if and only if $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq 2$.

2 The Right Derived Functors using \mathcal{X} -injective coresolutions:

In this section, we investigate right derived functors using \mathcal{X}^\perp -coresolution. By [8, Theorem 2.2], every R -module has an \mathcal{X}^\perp -coresolution. Let $0 \rightarrow N \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$ be an \mathcal{X}^\perp -coresolution of N . This complex is unique up to homotopy. This leads us to new derived functors $Fixt_R^n(-, -)$, which are well defined. Applying the functor $Hom_R(M, -)$, we obtain the deleted complex

$$\mathbb{G}^\bullet: 0 \rightarrow Hom_R(M, G^0) \rightarrow Hom_R(M, G^1) \rightarrow \dots$$

Then, we can define $Fixt_R^n(M, N) = H^n(Hom(M, \mathbb{G}^\bullet))$.

Theorem 2.1. *For $M, N \in Obj(\mathcal{M})$, then the $Fixt_R^n(M, N)$ are well defined.*

Proof. For any R -module N , there is an \mathcal{X}^\perp -coresolution of M

$$0 \rightarrow N \xrightarrow{\epsilon} G^0 \xrightarrow{\alpha_1} G^1 \xrightarrow{\alpha_2} \dots$$

and an R -module \bar{N} , there is an \mathcal{X}^\perp -coresolution of \bar{N}

$$0 \rightarrow \bar{N} \xrightarrow{\epsilon'} \bar{G}^0 \xrightarrow{\alpha'_1} \bar{G}^1 \xrightarrow{\alpha'_2} \dots$$

Let $\gamma \in Hom_R(N, \bar{N})$. We only need to show that there is a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & N & \xrightarrow{\epsilon} & G^0 & \xrightarrow{\alpha_1} & G^1 & \xrightarrow{\alpha_2} & \dots & \longrightarrow & G^n & \xrightarrow{\alpha_{n+1}} & G^{n+1} & \longrightarrow & \dots \\ & & \gamma \downarrow & & \gamma_0 \downarrow & & \gamma_1 \downarrow & & & & \gamma_n \downarrow & & \gamma_{n+1} \downarrow & & \\ 0 & \longrightarrow & \bar{N} & \xrightarrow{\epsilon'} & \bar{G}^0 & \xrightarrow{\alpha'_1} & \bar{G}^1 & \xrightarrow{\alpha'_2} & \dots & \longrightarrow & \bar{G}^n & \xrightarrow{\alpha'_{n+1}} & \bar{G}^{n+1} & \longrightarrow & \dots \end{array}$$

and the associated map of \mathcal{X}^\perp -coresolutions is unique up to homotopy.

Every R -module has a special \mathcal{X}^\perp -preenvelope. Then, N has an \mathcal{X}^\perp -preenvelope G^0 and \bar{N} has an \mathcal{X}^\perp -preenvelope \bar{G}^0 such that $\gamma_0 \in Hom_R(G^0, \bar{G}^0), \epsilon' \in Hom_R(\bar{N}, \bar{G}^0)$ and $\gamma_0 \epsilon = \epsilon' \gamma$. Then the following diagram is commutative

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N & \xrightarrow{\epsilon} & G^0 & \longrightarrow & L^1 \longrightarrow 0 \\
 & & \gamma \downarrow & & \downarrow \gamma_0 & & \downarrow l_1 \\
 0 & \longrightarrow & \overline{N} & \xrightarrow{\epsilon'} & \overline{G^0} & \longrightarrow & \overline{L^1} \longrightarrow 0,
 \end{array}$$

where L^1 and $\overline{L^1}$ are cokernel of ϵ and ϵ' . Let G^1 and $\overline{G^1}$ are \mathcal{X}^\perp -preenvelop of L^1 and $\overline{L^1}$, respectively. Then, the diagram

$$\begin{array}{ccc}
 0 & \longrightarrow & L^1 \longrightarrow G^1 \\
 & & \downarrow l_1 \quad \downarrow \gamma_1 \\
 0 & \longrightarrow & \overline{L^1} \longrightarrow \overline{G^1}
 \end{array}$$

is commutative. Continuing this process, there is $\gamma_{n-1} \in \text{Hom}_R(G^{n-1}, \overline{G^{n-1}})$ such that the following diagram

$$\begin{array}{ccccccccccccccc}
 0 & \longrightarrow & N & \xrightarrow{\epsilon} & G^0 & \xrightarrow{\alpha_1} & G^1 & \longrightarrow & \dots & \longrightarrow & G^{n-1} & \longrightarrow & L_n & \longrightarrow & 0 \\
 & & \gamma \downarrow & & \downarrow \gamma_0 & & \downarrow \gamma_1 & & & & \downarrow \gamma_{n-1} & & \downarrow l_n & & \\
 0 & \longrightarrow & \overline{N} & \xrightarrow{\epsilon'} & \overline{G^0} & \xrightarrow{\alpha'_1} & \overline{G^1} & \longrightarrow & \dots & \longrightarrow & \overline{G^{n-1}} & \longrightarrow & \overline{L^n} & \longrightarrow & 0,
 \end{array}$$

is commutative, where G^i and $\overline{G^i}$ are \mathcal{X} -injective for each $i \in \{0, 1, \dots, n-1\}$, L^n and $\overline{L^n}$ are cokernels. Let G^n and $\overline{G^n}$ are \mathcal{X}^\perp -preenvelop of L^n and $\overline{L^n}$, respectively. Then, there exists $\gamma_n \in \text{Hom}_R(G^n, \overline{G^n})$ such that the diagram

$$\begin{array}{ccc}
 0 & \longrightarrow & L^n \longrightarrow G^n \\
 & & \downarrow l_n \quad \downarrow \gamma_n \\
 0 & \longrightarrow & \overline{L^n} \longrightarrow \overline{G^n}
 \end{array}$$

is commutative. It follows that we can complete the diagram.

We are now to prove the uniqueness up to homotopy, that is, to prove that from the following diagram

$$\begin{array}{ccccccccccccccc}
 0 & \longrightarrow & N & \xrightarrow{\alpha_0} & G^0 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_n} & G^n & \xrightarrow{\alpha_{n+1}} & G^{n+1} & \longrightarrow & \dots \\
 & & \parallel & \swarrow s_{-1} & \downarrow \gamma_0 & \downarrow \gamma'_0 & & \swarrow s_{n-1} & \downarrow \gamma_n & \downarrow \gamma'_n & \swarrow s_n & \downarrow \gamma_{n+1} & \downarrow \gamma'_{n+1} \\
 0 & \longrightarrow & N & \xrightarrow{\alpha'_0} & \overline{G^0} & \xrightarrow{\alpha'_1} & \dots & \xrightarrow{\alpha'_{n+1}} & \overline{G^n} & \xrightarrow{\alpha'_{n+1}} & \overline{G^{n+1}} & \longrightarrow & \dots
 \end{array}$$

there exist s_0, \dots, s_n, \dots , with $s_n : G^{n+1} \rightarrow \overline{G^n}$ such that $\gamma_n - \gamma'_n = \alpha'_n \circ s_{n-1} + s_n \circ \alpha_{n+1}$, where $s_{-1} = 0$. We know that $\gamma_0 \alpha_0 = \alpha'_0 = \gamma'_0 \alpha_0$. It follows that $(\gamma_0 - \gamma'_0) \alpha_0 = 0$. Then, we have the diagram

$$\begin{array}{ccccccc}
 N & \longrightarrow & G^0 & \longrightarrow & G^0/N & \longrightarrow & G^1 \\
 \parallel & & \downarrow \gamma_0 - \gamma'_0 & & \downarrow & \swarrow & \\
 N & \longrightarrow & \overline{G^0} & \longrightarrow & \overline{G^0}/\overline{N} & &
 \end{array}$$

which can be completed since G^1 is an \mathcal{X}^\perp -preenvelope, that is, there exists $s_0 \in Hom_R(G^1, \overline{G^0})$ such that $\gamma_0 - \gamma'_0 = s_0 \circ \alpha_1$.

We will show that the following diagram is commutative, that is, a map on G^0 which is zero.

$$\begin{array}{ccccccc}
 N & \longrightarrow & G^0 & \xrightarrow{\alpha_1} & G^1 & \xrightarrow{\alpha_2} & G^2 \\
 \parallel & & \downarrow \gamma_0 - \gamma'_0 & \swarrow s_0 & \downarrow \gamma_1 - \gamma'_1 & \swarrow s_1 & \dashrightarrow \\
 N & \longrightarrow & \overline{G^0} & \xrightarrow{\alpha'_1} & \overline{G^1} & \xrightarrow{\alpha'_2} & \overline{G^2}
 \end{array}$$

Let s_1 be the map which completes the following diagram

$$\begin{array}{ccc}
 G^1/im\alpha_1 & \longrightarrow & G^2 \\
 \downarrow \gamma_1 - \gamma'_1 - \alpha'_1 s_0 & \swarrow s_1 & \\
 \overline{G^1} & &
 \end{array}$$

Hence,

$$\begin{aligned}
 (\gamma_1 - \gamma'_1 - \alpha'_1 s_0)\alpha_1 &= (\gamma_1 - \gamma'_1)\alpha_1 - (\alpha'_1 s_0)\alpha_1 \\
 &= (\gamma_1 - \gamma'_1)\alpha_1 - \alpha'_1(\gamma_0 - \gamma'_0) \\
 &= 0,
 \end{aligned}$$

as desired. Continuing this process, we can find s_2, \dots, s_{n-1} . Define s_n as the completion of the following diagram

$$\begin{array}{ccc}
 G^1/im\alpha_1 & \longrightarrow & G^2 \\
 \downarrow \gamma_1 - \gamma'_1 - \alpha'_1 s_0 & \swarrow s_1 & \\
 \overline{G^1} & &
 \end{array}$$

It follows that we can get a following commutative diagram

$$\begin{array}{ccccccc}
 & & G^{n-1} & \xrightarrow{\alpha_n} & G_n & \xrightarrow{\alpha_{n+1}} & G^{n+1} \\
 & \swarrow s_{n-2} & \downarrow \gamma_{n-1} - \gamma'_{n-1} & \swarrow s_{n-1} & \downarrow \gamma_n - \gamma'_n & \swarrow s_n & \dashrightarrow \\
 \overline{G^{n-2}} & \xrightarrow{\alpha'_{n-1}} & \overline{G^{n-1}} & \xrightarrow{\alpha'_n} & \overline{G^n} & \xrightarrow{\alpha'_{n+1}} & \overline{G^{n+1}}
 \end{array}$$

Since the previous diagram has exact rows, then

$$\begin{aligned}
 (\gamma_n - \gamma'_n - \alpha'_n s_{n-1})\alpha_n &= (\gamma_n - \gamma'_n)\alpha_n - \alpha'_n(s_{n-1}\alpha_n) \\
 &= (\gamma_n - \gamma'_n)\alpha_n - \alpha'_n(\gamma_{n-1} - \gamma'_{n-1} - \alpha'_{n-1}s_{n-1}) \\
 &= (\gamma_n - \gamma'_n)\alpha_n - \alpha'_n(\gamma_{n-1} - \gamma'_{n-1}) + (\alpha'_{n-1}s_{n-1}) \\
 &= 0.
 \end{aligned}$$

Hence, the \mathcal{X}^\perp -coresolution is unique up to homotopy. □

From the homology groups of this \mathcal{X}^\perp -coresolution gives a well defined derived functor which we will call $Fixt_R^n(M, N)$. Let $0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$ be an \mathcal{X}^\perp -coresolution of M and $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ an $\mathcal{I}(\mathcal{M})$ -coresolution of M . Since I_0 is injective, there exists $g_0 \in Hom_R(G^0, I^0)$ such that $\alpha' = g_0 \circ \alpha$. By the injectivity of I^1 , there exists $g_1 \in Hom_R(G^1, I^1)$ such that $g_1\alpha_0 = \alpha'_0 g_0$. Continuing this process and using analog proof of Theorem 2.1, we can complete the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M & \longrightarrow & G^0 & \xrightarrow{\alpha_1} & G^1 \xrightarrow{\alpha_2} \dots \\
 & & \parallel & & \downarrow g_0 & & \downarrow g_1 \\
 0 & \longrightarrow & M & \xrightarrow{\alpha'} & I^0 & \xrightarrow{\alpha'_1} & I^1 \xrightarrow{\alpha'_2} \dots
 \end{array}$$

to a commutative diagram uniquely, up to homotopy. Now applying $Hom_R(M, -)$ to the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & G^0 & \longrightarrow & G^1 & \xrightarrow{\alpha_2} & \dots \\
 & & \downarrow g_0 & & \downarrow g_1 & & \\
 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots
 \end{array}$$

gives natural maps $Fixt_R^n(M, G) \rightarrow Ext_R^n(M, G)$ for all $n \geq 0$.

Proposition 2.2. For $M, G \in Obj(\mathcal{M})$, $Fixt_R^0(M, G) \cong Hom_R(M, G)$.

Proof. Let $0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$ be a \mathcal{X}^\perp -coresolution of M . Then, the homotopy groups of the complex

$$0 \rightarrow Hom_R(M, G^0) \rightarrow Hom_R(M, G^1) \rightarrow \dots$$

gives us the groups $Fixt_R^n(M, G)$. Hence, $Fixt_R^0(M, G)$ is the kernel of $Hom_R(M, G^0) \rightarrow Hom_R(M, G^1)$. But the functor $Hom_R(M, -)$ is left exact. So if the sequence $0 \rightarrow G \rightarrow G^0 \rightarrow G^1$ is exact, then the sequence $0 \rightarrow Hom_R(M, G^0) \rightarrow Hom_R(M, G^1) \rightarrow \dots$ is exact. Now $Hom_R(M, G)$ is isomorphic to the kernel of $Hom_R(M, G^0) \rightarrow Hom_R(M, G^1)$, that is, $Hom_R(M, G) \cong Fixt_R^0(M, G)$, as desired. \square

Proposition 2.3. For $M, G \in Obj(\mathcal{M})$, $Fixt_R^1(M, N) \rightarrow Ext_R^1(M, N)$ is injective.

Proof. Let $0 \rightarrow N \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$ be an \mathcal{X}^\perp -coresolution of N and $0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$ be an $\mathcal{I}(\mathcal{M})$ -coresolution of N . Consider the exact sequences $0 \rightarrow N \rightarrow G^0 \rightarrow G^0/N \rightarrow 0$ and $0 \rightarrow N \rightarrow E^0 \rightarrow E^0/N \rightarrow 0$. From the following diagram with exact rows

$$\begin{array}{ccccccc}
 & & & & M & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & N & \longrightarrow & G^0 & \longrightarrow & G^0/N \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N & \longrightarrow & E^0 & \longrightarrow & E^0/N \longrightarrow 0,
 \end{array}$$

we get the following diagram

$$\begin{array}{ccc}
 & M & \\
 & \downarrow & \\
 & L & \\
 & \downarrow & \\
 E^0 & \longrightarrow & E^0/N
 \end{array}$$

is commutative. Then, the diagram

$$\begin{array}{ccc}
 & & M \\
 & \swarrow & \downarrow \\
 G^0 & \longrightarrow & G^0/N
 \end{array}$$

can also be completed, as desired. □

Theorem 2.4. *For $M \in \text{Obj}(\mathcal{M})$, M is \mathcal{X}^\perp -projective if and only if $\text{Fixt}_R^1(M, N) \rightarrow \text{Ext}_R^1(M, N)$ is an isomorphism for all R -modules N .*

Proof. Suppose M is \mathcal{X}^\perp -projective. By Theorem [8, Theorem 2.2], N has an \mathcal{X} -injective coresolution, $0 \rightarrow N \xrightarrow{f} G^0 \rightarrow G^1 \rightarrow \dots$. Consider the short exact sequence $0 \rightarrow N \xrightarrow{f} G^0 \rightarrow G^0/\text{im}f \rightarrow 0$. Then, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 \text{Hom}_R(M, G^0) & \longrightarrow & \text{Hom}_R(M, G^0/\text{im}f) & \longrightarrow & \text{Fixt}_R^1(M, N) & \longrightarrow & \text{Fixt}_R^1(M, G^0) = 0 \\
 \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \\
 \text{Hom}_R(M, G^0) & \longrightarrow & \text{Hom}_R(M, G^0/\text{im}f) & \longrightarrow & \text{Ext}_R^1(M, N) & \longrightarrow & \text{Ext}_R^1(M, G^0) = 0.
 \end{array}$$

Hence, $\text{Fixt}_R^1(M, N) \cong \text{Ext}_R^1(M, N)$ for any R -module N . Conversely, we may assume that $\text{Fixt}_R^1(M, N) \rightarrow \text{Ext}_R^1(M, G)$ is an isomorphism for all R -modules N . Then, $\text{Ext}_R^1(M, G) = 0$ for all \mathcal{X} -injective R -modules G since $\text{Fixt}_R^1(M, G) = 0$ for all \mathcal{X} -injective R -modules G and R -modules M . Thus, M is \mathcal{X}^\perp -projective. □

3 \mathcal{X} -injective dimensions and the left derived functors of Hom

In this section, we deal with the \mathcal{X} -injective dimensions of modules and the left derived functors $\text{Fixt}_n(-, -)$.

For $M \in \text{Obj}\mathcal{M}$, there exists an \mathcal{X} -injective coresolution of M such that $0 \rightarrow M \xrightarrow{f} G^0 \xrightarrow{g_0} G^1 \rightarrow \dots$. Then, we obtain the deleted complex

$$\mathbb{G}_\bullet: \dots \rightarrow \text{Hom}(G^i, N) \xrightarrow{g_0^*} \text{Hom}(G^0, N) \rightarrow 0$$

when we apply the functor $\text{Hom}(-, N)$. It follows that $\text{Fixt}_n(M, N)$ is exactly n th homology of the complex, that is, $\text{Fixt}_n(M, N) = H_n(\text{Hom}(\mathbb{G}_\bullet, N))$. Then, there is a canonical map

$$\mu: \text{Fixt}_0(M, N) = \text{Hom}(G^0, N)/\text{im}(g_0^*) \rightarrow \text{Hom}(M, N)$$

defined by $\mu(h + \text{im}(g_0^*)) = hg_0$ for $h \in \text{Hom}(G^0, N)$.

Proposition 3.1. *Let R be a Noetherian ring. For $M \in \text{Obj}\mathcal{M}$, M is \mathcal{X} -injective if and only if the canonical map $\mu: \text{Fixt}_0(M, N) \rightarrow \text{Hom}(M, N)$ is an epimorphism for any left R -module N .*

Proof. The direct implication is clear if $G^0 = M$. Conversely, if $N = M$ then there exists $h \in \text{Hom}(G^0, M)$ such that $\mu(h + \text{im}(g_0^*)) = hg_0 = 1_M$. Thus, M is isomorphic to a direct summand of G^0 . Since the direct summand of \mathcal{X} -injective module is \mathcal{X} -injective, M is \mathcal{X} -injective. □

Corollary 3.2. *Let R be a Noetherian ring. Then, the following conditions are equivalent:*

- (i) ${}_R R$ is \mathcal{X} -injective;
- (ii) The canonical map $\mu: \text{Fixt}_0({}_R R, N) \rightarrow \text{Hom}({}_R R, N)$ is an epimorphism for any left R -module N ;
- (iii) The canonical map $\mu: \text{Fixt}_0({}_R R, {}_R R) \rightarrow \text{Hom}({}_R R, {}_R R)$ is an epimorphism;
- (iv) Every R -module has a special \mathcal{X} -injective cover.

Proof. (1) \Leftrightarrow (2) \Leftrightarrow (3) follows from Proposition 3.1.

(1) \Rightarrow (4). For any left R -module M , M has an \mathcal{X} -injective precover. Consider the exact sequence $F \rightarrow M \rightarrow 0$ with F a free module. By hypothesis, F is \mathcal{X} -injective.

(4) \Rightarrow (1). Let $\phi: M \rightarrow {}_R R$ be a special \mathcal{X} -injective cover. Then, ${}_R R$ is isomorphic to a direct summand of M . Thus, ${}_R R$ is \mathcal{X} -injective. □

Proposition 3.3. *Let R be a Noetherian ring. Then, the following are equivalent:*

- (i) $Rcores.dim_{\mathcal{X}^\perp}(M) \leq 1$;
- (ii) *The canonical map $\mu: Fict_0(M, N) \rightarrow Hom(M, N)$ is a monomorphism for any left R -module N .*

Proof. (1) \Rightarrow (2). By hypothesis, M has a \mathcal{X}^\perp -coresolution $0 \rightarrow G^0 \rightarrow G^1 \rightarrow 0$. It follows that we get an exact sequence $0 \rightarrow Hom(G^1, N) \rightarrow Hom(G^0, N) \rightarrow Hom(M, N)$ for any R -module N . Hence, μ is a monomorphism.

(2) \Rightarrow (1). Consider the exact sequence $0 \rightarrow M \rightarrow G^0 \rightarrow L^1 \rightarrow 0$, where $M \rightarrow G^0$ is an \mathcal{X} -injective preenvelope. We only need to show that L^1 is \mathcal{X} -injective. By [2, Theorem 8.2.3], we have the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 Fict_0(L^1, L^1) & \longrightarrow & Fict_0(G^0, L^1) & \longrightarrow & Fict_0(M, L^1) & \longrightarrow & 0 \\
 \mu_1 \downarrow & & \mu_2 \downarrow & & \mu_3 \downarrow & & \\
 0 & \longrightarrow & Hom(L^1, L^1) & \longrightarrow & Hom(G^0, L^1) & \longrightarrow & Hom(M, L^1).
 \end{array}$$

Note that α_2 is an epimorphism by Proposition 3.1 and μ_3 is a monomorphism by hypothesis. Hence, μ_1 is an epimorphism by the Snake Lemma [6, Theorem 6.5]. By Proposition 3.1, L^1 is \mathcal{X} -injective. □

Lemma 3.4. *Let R be a Noetherian ring. Then, $Rcores.dim_{\mathcal{X}^\perp}(M) = cores.dim_{\mathcal{X}^\perp}(M)$ for any left R -module M .*

Proof. Clearly, $cores.dim_{\mathcal{X}^\perp}(M) \leq Rcores.dim_{\mathcal{X}^\perp}(M)$. Conversely, we assume that $cores.dim_{\mathcal{X}^\perp}(M) = n < \infty$. Consider the partial \mathcal{X}^\perp -coresolution of M , $0 \rightarrow G^0 \rightarrow G^1 \rightarrow \dots \rightarrow G^{n-1}$. Then, we get an exact sequence $0 \rightarrow G^0 \rightarrow G^1 \rightarrow \dots \rightarrow G^{n-1} \rightarrow L \rightarrow 0$. Hence, L is \mathcal{X} -injective, and so $Rcores.dim_{\mathcal{X}^\perp}(M) \leq n$, as desired. □

The finitistic \mathcal{X}^\perp -coresolution of dimension, denoted by $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M})$, is defined as $\sup\{cores.dim_{\mathcal{X}^\perp}(M) : M \in Obj(\mathcal{M})\}$.

Proposition 3.5. *Let R be a Noetherian ring. Then, the following are equivalent:*

- (i) $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq 1$;
- (ii) *The canonical map $\mu: Fict_0({}_R R, N) \rightarrow Hom({}_R R, N)$ is a monomorphism for any left R -module N .*

Proof. (1) \Leftrightarrow (2). It follows by Proposition 3.3 and Lemma 3.4. □

Proposition 3.6. *Let R be a Noetherian ring and an integer $n \geq 2$. Then, the following conditions are equivalent:*

- (i) $Rcores.dim_{\mathcal{X}^\perp}(M) \leq n$;
- (ii) $Fict_{n+k}(M, N) = 0$ for all R -modules N and all $k \geq -1$;
- (iii) $Fict_{n-1}(M, N) = 0$ for all R -modules N .

Proof. (1) \Rightarrow (2). Note that $Fict_{n+k}(M, N) = 0$ for all $k \geq 1$. We only need to show that $Fict_{n+k}(M, N) = 0$ for all $k \in \{-1, 0\}$. Consider \mathcal{X}^\perp -coresolution of M , $0 \rightarrow G^0 \rightarrow G^1 \rightarrow \dots \rightarrow G^n \rightarrow 0$. Which induces an exact sequence $0 \rightarrow Hom(G^n, N) \rightarrow Hom(G^{n-1}, N) \rightarrow Hom(G^{n-2}, N)$ for any R -module N . Hence, $Fict_n(M, N) = Fict_{n-1}(M, N) = 0$.

(2) \Rightarrow (3) is trivial.

(3) \Rightarrow (1). Consider the \mathcal{X}^\perp -coresolution of M , $0 \rightarrow G^0 \rightarrow G^1 \rightarrow \dots \rightarrow G^n \rightarrow 0$. with $L^n = coker(G^{n-2} \rightarrow G^{n-1})$. We have the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M & \longrightarrow & G^0 & \longrightarrow & \dots G^{n-2} \xrightarrow{\alpha} G^{n-1} \xrightarrow{\beta} G^n \longrightarrow \dots \\
 & & & & & & \searrow \gamma \qquad \nearrow \eta \\
 & & & & & & L^n \\
 & & & & & & \nearrow \qquad \searrow \\
 & & & & 0 & & 0.
 \end{array}$$

Since $\text{Fixt}_{n-1}(M, L^n) = 0$, the sequence

$$\text{Hom}(G^n, L^n) \xrightarrow{\beta^*} \text{Hom}(G^{n-1}, L^n) \xrightarrow{\alpha^*} \text{Hom}(G^{n-2}, L^n)$$

is exact. Now $\alpha^*(\gamma) = \gamma\alpha = 0$. It follows that $\gamma \in \ker(\alpha^*) = \text{im}(\beta^*)$. Then, there exists $\nu \in \text{Hom}(G^n, L^n)$ such that $\gamma = \beta^*(\nu) = \nu\beta = \nu\eta\gamma$, and hence $\nu\eta = 1$ since γ is epic. Therefore L^n is \mathcal{X} -injective, as desired. \square

Corollary 3.7. *Let R be a Noetherian ring and an integer $n \geq 2$. Then, the following are equivalent:*

- (i) $\text{cores.dim}_{\mathcal{X}^\perp}(\mathcal{M}) \leq n$;
- (ii) $\text{Fixt}_{n+k}(M, N) = 0$ for all R -modules M, N and all $k \geq -1$;
- (iii) $\text{Fixt}_{n-1}(M, N) = 0$ for all R -modules M and N .

Proposition 3.8. *Let R be a Noetherian ring. If M is a pure injective R -module, then M has a minimal \mathcal{X}^\perp -resolution $\cdots \rightarrow G_{n-2} \rightarrow G_{n-3} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ with each G_i an injective module.*

Proof. By [7, Theorem 5.5], M has an \mathcal{X} -injective precover $\gamma: G_0 \rightarrow M$. Consider the exact sequence $0 \rightarrow G_0 \xrightarrow{i} E \rightarrow L \rightarrow 0$ with E an injective envelope of G_0 . Since the exact sequence is pure and M is pure injective, there exists $h \in \text{Hom}(E, M)$ such that $\gamma = hi$. By \mathcal{X} -injective precover γ of M , there exists $\phi \in \text{Hom}(E, G)$ such that $\gamma\phi = h$. Hence, $\gamma\phi i = hi = \gamma$. This implies that ϕi is an isomorphism. Then, G is isomorphic to a direct summand of E and hence G is injective. Note that $\ker \gamma \in (\mathcal{X}^\perp)^\perp$. Hence, $\ker \gamma$ has an \mathcal{X} -injective precover $\gamma_1: G_1 \rightarrow \ker \gamma$ with G_1 an injective module, where $\ker \gamma_1 \in (\mathcal{X}^\perp)^\perp$. By continuing the above process, we get the minimal \mathcal{X}^\perp -resolution of M . \square

Theorem 3.9. *Let R be a Noetherian ring. Consider the following conditions for a pure injective R -module N and an integer $n \geq 2$:*

- (i) $\text{Lcores.dim}_{\mathcal{X}^\perp}(N) \leq n - 2$;
- (ii) $\text{Fixt}_{n+k}(M, N) = 0$ for all R -modules M and all $k \geq -1$;
- (iii) $\text{Fixt}_{n+k}(M, N) = 0$ for all R -modules M .

Proof. (1) \Rightarrow (2). By hypothesis, N has a left \mathcal{X}^\perp -resolution

$$0 \rightarrow G_{n-2} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow N \rightarrow 0.$$

Then, we have the following sequence

$$0 \rightarrow \text{Hom}(M, G_{n-2}) \rightarrow \text{Hom}(M, G_{n-3}) \rightarrow \cdots \rightarrow \text{Hom}(M, G_0) \rightarrow 0$$

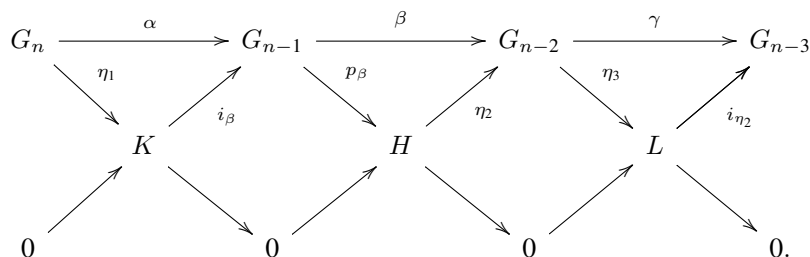
for any R -module M . Hence, $\text{Fixt}_{n+k}(M, N) = 0$ for all $k \geq -1$.

(2) \Rightarrow (3) is trivial.

(3) \Rightarrow (1). By pure injectivity of N and Proposition 3.8, N has a minimal \mathcal{X}^\perp -resolution:

$$\cdots \rightarrow G_n \xrightarrow{\alpha} G_{n-1} \xrightarrow{\beta} G_{n-2} \xrightarrow{\gamma} G_{n-3} \xrightarrow{\mu} \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$$

with each G_i an injective module. Let $i_\beta \in \text{Hom}(K, G_{n-1})$ be the inclusion and $p_\beta \in \text{Hom}(G_{n-1}, H)$ the canonical projection, where $K = \ker \beta$ and $H = G_{n-1}/K$. Then, there exists $\eta_1 \in \text{Hom}(G_n, K)$ such that $\alpha = i_\beta \eta_1$ and there exists a monomorphism $\eta_2 \in \text{Hom}(H, G_{n-2})$ such that $\beta = \eta_2 p_\beta$. Let $\eta_3 \in \text{Hom}(G_{n-2}, L)$ be the canonical projection, where $L = G_{n-2}/\text{im}(\eta_2)$. Then, there exists a homomorphism $i_{\eta_2} \in \text{Hom}(L, G_{n-3})$ such that $\gamma = i_{\eta_2} \eta_3$. Then, we get the following commutative diagram:



By (3), $\text{Filt}_{n-1}(K, N) = 0$. Then, the sequence

$$\text{Hom}(K, G_n) \xrightarrow{\alpha_*} \text{Hom}(K, G_{n-1}) \xrightarrow{\beta_*} \text{Hom}(K, G_{n-2})$$

is exact. Since $\beta_*(i_\beta) = \beta i_\beta = 0, i_\beta \in \ker \beta_* = \text{im} \alpha_*$. Therefore $i_\beta = \alpha_*(t) = \alpha(t)$ for some $t \in \text{Hom}(K, G_n)$. By $\alpha = i_\beta \eta_1, i_\beta = i_\beta \eta_1 t$. Hence, $\eta_1 t = 1_{G_n}$ since i_β is monic. So K is injective. It follows that H and L are injective. We claim that the complex

$$0 \rightarrow L \xrightarrow{i_{\eta_2}} G_{n-3} \rightarrow \dots \rightarrow G_1 \rightarrow G_0 \rightarrow N \rightarrow 0$$

is an \mathcal{X}^\perp -resolution of N . We only need to show that the complex

$$0 \rightarrow \text{Hom}(G, L) \xrightarrow{(i_{\eta_2})_*} \text{Hom}(G, G_{n-3}) \xrightarrow{\mu_*} \text{Hom}(G, G_{n-4})$$

is exact for any \mathcal{X} -injective R -module G . Note that we have the following commutative diagram:

$$\begin{array}{ccccc} \text{Hom}(G, G_{n-1}) & \xrightarrow{\beta_*} & \text{Hom}(G, G_{n-2}) & \xrightarrow{\gamma_*} & \text{Hom}(G, G_{n-3}) \\ & \searrow (p_\beta)_* & \nearrow (\eta_2)_* & \searrow (\eta_3)_* & \nearrow (i_{\eta_2})_* \\ & & \text{Hom}(G, H) & & \text{Hom}(G, L) \\ & \nearrow & \searrow & \nearrow & \searrow \\ 0 & & 0 & & 0 \end{array}$$

Now $\ker((i_{\eta_2})_*(\eta_3)_*) = \ker(\gamma_*) = \text{im}(\beta_*) = \text{im}((\eta_2)_*(p_\beta)_*) = \text{im}((\eta_2)_*) = \ker((\eta_3)_*)$. Let $g \in \ker((i_{\eta_2})_*)$. Since $(\eta_3)_*$ is epic, $g = (\eta_3)_*(g_{n-2})$ for some $g_{n-2} \in \text{Hom}(G, G_{n-2})$. Therefore $(i_{\eta_2})_*(\eta_3)_*(g_{n-2}) = 0$. Hence, $(\eta_3)_*(g_{n-2}) = 0$. That is, $g = 0$. It follows that $(i_{\eta_2})_*$ is monic. On the other hand $\ker(\mu_*) = \text{im}(\gamma_*) = \text{im}((i_{\eta_2})_*)$. Hence, we obtain the desired exact sequence. This completes the proof. \square

Corollary 3.10. *Let R be a Noetherian ring and an integer $n \geq 2$. Consider the following conditions*

- (i) $\text{glLcores.dim}_{\mathcal{X}^\perp}(\mathcal{M}) \leq n - 2$;
- (ii) $\text{Fcores.dim}_{\mathcal{X}^\perp}(\mathcal{M}) \leq n$;
- (iii) $\text{Lcores.dim}_{\mathcal{X}^\perp}(N) \leq n - 2$ for all pure injective R -modules N ;
- (iv) $\text{Filt}_{n+k}(M, N) = 0$ for all R -modules M , all pure injective R -modules N and all $k \geq -1$;
- (v) $\text{Filt}_{n-1}(M, N) = 0$ for all R -modules M and all pure injective R -modules N .

Then, (1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5).

Proof. It follows from Corollary 3.7 and Theorem 3.9. \square

Lemma 3.11. *Let R be a Noetherian ring and an integer $n \geq 1$. If M is pure injective R -module, then $\text{id}(M) \leq n$ if and only if for the minimal left \mathcal{X}^\perp -resolution $\dots \rightarrow G_n \rightarrow G_{n-1} \rightarrow \dots \rightarrow G_1 \rightarrow G_0 \rightarrow N \rightarrow 0$ of any pure injective R -module N , $\text{Hom}(M, G_n) \rightarrow \text{Hom}(M, K_n)$ is an epimorphism.*

Proof. We prove the result by induction on n . Let $n = 0$. If M is injective, $\text{Hom}(M, G_0) \rightarrow \text{Hom}(M, K_0)$ is an epimorphism. Conversely, put $N = M$. Then, $\text{Hom}(M, G_0) \rightarrow \text{Hom}(M, M)$ is an epimorphism, and hence M is injective.

Let $n \geq 1$. Consider an exact sequence $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$ with E an injective module. Then, we have the following diagrams

$$\begin{array}{ccccc} \text{Hom}(E, G_n) & \longrightarrow & \text{Hom}(E, K_n) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \text{Hom}(M, G_n) & \longrightarrow & \text{Hom}(M, K_n) & & \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

and

$$\begin{array}{ccccccc}
 & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathbf{0} & \longrightarrow & \text{Hom}(L, K_n) & \longrightarrow & \text{Hom}(L, G_{n-1}) & \longrightarrow & \text{Hom}(L, K_{n-1}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathbf{0} & \longrightarrow & \text{Hom}(E, K_n) & \longrightarrow & \text{Hom}(E, G_{n-1}) & \longrightarrow & \text{Hom}(E, K_{n-1}) \longrightarrow \mathbf{0} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathbf{0} & \longrightarrow & \text{Hom}(M, K_n) & \longrightarrow & \text{Hom}(M, G_{n-1}) & \longrightarrow & \text{Hom}(M, K_{n-1}) \\
 & & & & \downarrow & & \\
 & & & & \mathbf{0} & &
 \end{array}$$

are exact and commutative. By [3, Lemma 3.2.10], L is pure injective. Therefore $id(M) \leq n$ if and only if $id(L) \leq n - 1$ if and only if $\text{Hom}(L, G_{n-1}) \rightarrow \text{Hom}(L, K_{n-1})$ is an epimorphism by induction on n if and only if $\text{Hom}(E, K_n) \rightarrow \text{Hom}(M, K_n)$ is an epimorphism by the second diagram if and only if $\text{Hom}(M, G_n) \rightarrow \text{Hom}(M, K_n)$ is an epimorphism by the first diagram. \square

Theorem 3.12. *Let R be a Noetherian ring. Then, $\text{Fixt}_R^n(M, N) \cong \text{Fixt}_n^R(M, N)$ for $n \geq 0$.*

Proof. By Theorem [8, Theorem 2.2], M has a \mathcal{X}^\perp -coresolution. Let $0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$ be an \mathcal{X}^\perp -coresolution of M . Since R is hereditary Noetherian, then there exists an \mathcal{X}^\perp -resolution $\dots \rightarrow G_2 \rightarrow G_1 \rightarrow G_0 \rightarrow N \rightarrow 0$

$$\begin{array}{ccccccc}
 & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \dots & \longrightarrow & \text{Hom}(M, G_2) & \longrightarrow & \text{Hom}(M, G_1) & \longrightarrow & \text{Hom}(M, G_0) \longrightarrow \mathbf{0} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \dots & \longrightarrow & \text{Hom}(G^0, G_2) & \longrightarrow & \text{Hom}(G^0, G_1) & \longrightarrow & \text{Hom}(G^0, G_0) \longrightarrow \text{Hom}(G^0, N) \longrightarrow \mathbf{0} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \dots & \longrightarrow & \text{Hom}(G^1, G_2) & \longrightarrow & \text{Hom}(G^1, G_1) & \longrightarrow & \text{Hom}(G^1, G_0) \longrightarrow \text{Hom}(G^1, N) \longrightarrow \mathbf{0} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Since G_i are \mathcal{X} -injective and \mathbb{G}^\bullet is an \mathcal{X}^\perp -coresolution, all rows but the first are exact. Similarly all columns but the last are exact. Then, by chasing diagrams or by a spectral sequence argument,

$$H^n(\text{Hom}(M, \mathbb{G}^\bullet(N))) \cong H_n(\text{Hom}(\mathbb{G}_\bullet(M), N))$$

for each $n \geq 0$. This completes the proof. \square

Theorem 3.13. *Let R be a Noetherian ring. Then, the following conditions are equivalent:*

- (i) Every left R -module has an \mathcal{X} -injective cover with the unique mapping property;
- (ii) $\text{Fcores.dim}_{\mathcal{X}^\perp}(\mathcal{M}) \leq 2$;
- (iii) $\text{Fixt}_1(M, N) = 0$ for all left R -modules M and N ;
- (iv) $\text{Fixt}_k(M, N) = 0$ for all left R -modules M and N and $k \geq 1$.

Proof. (1) \Rightarrow (2). Let M be an R -module. Then, M has an \mathcal{X} -injective cover $\phi: U \rightarrow M$ with the unique mapping property. Hence, $0 \rightarrow U \rightarrow M \rightarrow 0$ is a left \mathcal{X} -resolution. Thus, $glCores.dim_{\mathcal{X}^\perp}(\mathcal{M}) = 0$. Hence, by Corollary 3.10, $Fcores.dim_{\mathcal{X}^\perp}(\mathcal{M}) \leq 2$.

(2) \Rightarrow (1). Let M be an R -module. By [7, Theorem 5.6], M has an \mathcal{X} -injective cover $h: U \rightarrow M$. It is enough to show that, if for any \mathcal{X} -injective left R -module U' and any homomorphism $j: U' \rightarrow U$ such that $hj = 0$ then $j = 0$. Consider the natural map $\pi: U \rightarrow U/imj$. Then, there exists $\bar{h}: U/imj \rightarrow M$ such that $\bar{h}\pi = h$ since $imj \subseteq \ker h$. By hypothesis, U/imj is \mathcal{X} -injective. Then there exists $\mu: U/imj \rightarrow U$ such that $\bar{h} = h\mu$. Hence, we get the following commutative diagram with exact row:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker j & \xrightarrow{i} & U' & \xrightarrow{j} & U & \xrightarrow{\pi} & U/imj & \longrightarrow & 0 \\
 & & & & \searrow & & \downarrow h & & \swarrow \bar{h} & & \\
 & & & & & & M & & & &
 \end{array}$$

Thus, $h\mu\pi = h$, and hence $\mu\pi$ is an isomorphism. It follows that π is monic. Thus, $j = 0$.

(2) \Leftrightarrow (3) \Leftrightarrow (4). It follows from Corollary 3.7. □

Remark 3.14. Let R be a Noetherian ring. Then, every left R -module has an \mathcal{X} -injective cover with the unique mapping property if and only if every pure injective R -module is \mathcal{X} -injective.

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