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ON ϕ-2-ABSORBING ELEMENTS IN MULTIPLICATIVE LATTICES

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Abstract In this paper, we introduce the concept of ϕ-2-absorbing elements in multiplicative lattices. Let $\phi : L \to L \cup \{\emptyset\}$ be a function. We will say a proper element q of L to be a ϕ-2-absorbing element of L if whenever $a, b, c \in L$ with $abc \leq q$ and $abc \nleq \phi(q)$ implies either $ab \leq q$ or $ac \leq q$ or $bc \leq q$. We give some basic properties and establish some characterizations of ϕ-2-absorbing elements in some special lattices.

1 Introduction

Several authors have studied various extensions of prime and primary ideals. A. Badawi [6] introduced the concept of 2-absorbing ideals in a commutative ring with identity, which is a generalization of prime ideals. A. Badawi and A.Y. Darani [5] studied weakly 2-absorbing ideals which are generalizations of weakly prime ideals [3]. Weakly prime elements in multiplicative lattices are studied in [10]. The concepts of 2-absorbing primary and weakly 2-absorbing primary ideals of commutative rings are studied in [7] and [8]. The concepts of 2-absorbing, weakly 2-absorbing, 2-absorbing primary and weakly 2-absorbing primary elements in multiplicative lattices are studied in [16] and [11] as generalizations of prime and weakly prime elements. Later, the concepts of ϕ-prime, ϕ-primary ideals are recently introduced in [12], [9], and generalizations of these are studied in [17]. In this work, our aim is to extend the concepts of 2-absorbing elements to ϕ-2-absorbing elements and investigate some characterizations in some special lattices.

Throughout this paper R denotes a commutative ring with identity and $L(R)$ denotes the lattice of all ideals of R. An element a of L is said to be compact if whenever $a \leq \bigvee_{\alpha \in I} a_\alpha$ implies $a \leq \bigvee_{\alpha \in I_0} a_\alpha$ for some finite subset I_0 of I. A *multiplicative lattice*, we mean a complete lattice L with the least element 0_L and compact greatest element 1_L, on which there is defined a commutative, associative, completely join distributive product for which 1_L is a multiplicative identity. Throughout this paper L denotes a multiplicative lattice and L_* denotes the set of all compact elements of L. By a *C-lattice* we mean a (not necessarily modular) multiplicative lattice which is generated under joins by a multiplicatively closed subset C of compact elements. We note that in a C-lattice, a finite product of compact elements is again compact. An element $a \in L$ is said to be *idempotent* if $a = a^2$. For any $a \in L$, $L/a = \{b \in L : a \leq b\}$ is a multiplicative lattice with the multiplication $c \circ d = cd \vee a$. An element $a \in L$ is said to be *proper* if $a < 1_L$. A proper element p of L is said to be prime if $ab \leq p$ implies either $a \leq p$ or $b \leq p$. C-lattices can be localized. For any prime element p of L, L_p denotes the localization at $F = \{x \epsilon C : x \nleq p\}$. If 0_L is prime, then L is said to be a *domain*. A proper element p is called as *ϕ-prime* if $ab \leq p$ and $ab \nleq \phi(p)$ implies either $a \leq p$ or $b \leq p$ for $a, b \in L$. In a C-lattice, an element p is ϕ-prime if and only if $ab \leq p$ and $ab \nleq \phi(p)$ implies either $a \leq p$ or $b \leq p$ for all $a, b \in L_*$ by [17]. An element $m < 1_L$ is said to be *maximal* in L if $m < x \leq 1_L$ implies $x = 1_L$. It can be easily shown that maximal elements are prime. For $a, b \in L$, we denote $(a : b) = \vee\{x \in L : xb \leq a\}$. For $a \in L$, we define $\sqrt{a} = \wedge\{p \in L : p$ is prime and $a \leq p\}$. Note that in a C-lattice L, $\sqrt{a} = \wedge\{p \in L : a \leq p$ is a minimal prime over $a\} = \vee\{x \in L_* : x^n \leq a$ for some $n \in \mathbb{Z}^+\}$. A proper element q is said to be *primary* if $ab \leq q$ implies either $a \leq q$ or $b \leq \sqrt{q}$ for every pair of elements $a, b \in L$. A proper element q is said to be ϕ-*primary* if for every $a, b \in L$ with $ab \leq q$ and $ab \nleq \phi(q)$ implies either $a \leq q$ or $b \leq \sqrt{q}$. A proper element q of L is said to be a 2-absorbing element if whenever $a, b, c \in L$ with $abc \leq q$ implies either $ab \leq q$ or $bc \leq q$ or $ac \leq q$.

A multiplicative lattice is called a *Noether lattice* if it is modular, principally generated (every element is a join of some principal elements) which satisfies the ascending chain condition. A Noether lattice L is local if it contains precisely one maximal prime. If L is a *Noether lattice* and 0_L is prime, then L is said to be a *Noether domain*. In [18], J. F. Wells studied the restricted cancellation law of a Noether lattice. An element a in a Noether lattice L satisfies the restricted cancellation law if $ab = ac \neq 0_L$ implies $b = c$ for any $a, b, c \in L$.

2 ϕ-2-absorbing elements

Definition 2.1. Let $\phi : L \to L \cup \{\emptyset\}$ be a function and $q \in L$ be a proper element. Then q is said to be a ϕ-2-absorbing element of L whenever if $a, b, c \in L$ with $abc \leq q$ and $abc \nleq \phi(q)$ implies either $ab \leq q$ or $ac \leq q$ or $bc \leq q$.

We can define the following special functions ϕ_α as follows: Let q be a ϕ_α-2-absorbing element of L. Then we say

$\phi_\emptyset(q) = \emptyset \quad \Rightarrow \quad q$ is a 2-absorbing element,
$\phi_0(q) = 0 \quad \Rightarrow \quad q$ is a weakly 2-absorbing element,
$\phi_2(q) = q^2 \quad \Rightarrow \quad q$ is an almost 2-absorbing element,
...
$\phi_n(q) = q^n \quad \Rightarrow \quad q$ is an n-almost 2-absorbing element for $n > 2$,
$\phi_\omega(q) = \wedge_{n=1}^{\infty} q^n \quad \Rightarrow \quad q$ is a ω-2-absorbing element.

Throughout this paper, ϕ denotes a function defined from L to $L \cup \{\emptyset\}$. Since for an element $a \in L$ with $a \leq q$ but $a \nleq \phi(q)$ implies that $a \nleq q \wedge \phi(q)$, there is no loss generality in assuming that $\phi(q) \leq q$. We henceforth make this assumption. For any two functions $\psi_1, \psi_2 : L \to L \cup \{\emptyset\}$, we say $\psi_1 \leq \psi_2$ if $\psi_1(a) \leq \psi_2(a)$ for each $a \in L$. Thus clearly we have the following order: $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq ... \leq \phi_{n+1} \leq \phi_n \leq ... \leq \phi_2 \leq \phi_1$.

Lemma 2.2. *Let q be a proper element of L and $\psi_1, \psi_2 : L \to L \cup \{\emptyset\}$ be two functions with $\psi_1 \leq \psi_2$. If q is a ψ_1-2-absorbing element of L, then q is a ψ_2-2-absorbing element of L.*

Proof. Suppose that q is a ψ_1-2-absorbing element of L and $a, b, c \in L$ such that $abc \leq q$ and $abc \nleq \psi_2(q)$. Since $abc \leq q$ and $abc \nleq \psi_1(q)$, we are done. □

Hence we have the following relations among the concepts mentioned in Definition 2.1:

Theorem 2.3. *Let q be a proper element of L. Then*

(i) q is a 2-absorbing element of $L \Rightarrow q$ is a weakly 2-absorbing element of $L \Rightarrow q$ is a ω-2-absorbing element of $L \Rightarrow q$ is an $(n+1)$-almost 2-absorbing element of $L \Rightarrow q$ is an n-almost 2-absorbing element of L for all $n \geq 2 \Rightarrow q$ is an almost 2-absorbing element of L.

(ii) q is a ϕ-prime element of $L \Rightarrow q$ is a ϕ-2-absorbing element of L.

(iii) A proper element q of L is an idempotent element $\Rightarrow q$ is a ω-2-absorbing element of L and q is an n-almost 2-absorbing element of L for all $n \geq 2$.

(iv) q is an n-almost 2-absorbing element of L for all $n \geq 2 \Leftrightarrow q$ is a ω-2-absorbing element of L.

Proof. (i) It is clear from Lemma 2.2.

(ii) Suppose that $a, b, c \in L$ with $abc \leq q$, $abc \nleq \phi(q)$ and $ab \nleq q$. Hence $c \leq q$ as q is a ϕ-prime element of L. Thus $ac \leq q$ or $bc \leq q$, we are done.

(iii) Suppose that q is an idempotent element of L. Then $q = q^n$ for all $n > 0$, and so $\phi_\omega(q) = \wedge_{n=1}^{\infty} q^n = q$. Thus q is a ω-2-absorbing element of L. Finally, q is an n-almost 2-absorbing element for all $n \geq 2$ from (i).

(iv) Let $a,b,c \in L$ with $abc \leq q$ but $abc \nleq \wedge_{n=1}^{\infty} q^n$. Hence $abc \leq q$ but $abc \nleq q^m$ for some $m \geq 2$. Since q is n-almost 2-absorbing for all $n \geq 2$, this implies either $ab \leq q$ or $bc \leq q$ or $ac \leq q$, we are done. The converse is clear from (i). □

Theorem 2.4. *Let q be a ϕ-2-absorbing element of L. If $\phi(q)$ is a 2-absorbing element of L, then q is 2-absorbing.*

Proof. Let $abc \leq q$ for some $a,b,c \in L$. If $abc \nleq \phi(q)$, then we have either $ab \leq q$ or $ac \leq q$ or $bc \leq q$ as q is ϕ-2-absorbing. Suppose that $abc \leq \phi(q)$. Hence we conclude that either $ab \leq \phi(q)$ or $ac \leq \phi(q)$ or $bc \leq \phi(q)$. Since $\phi(q) \leq q$, we are done. □

Definition 2.5. Let q be a ϕ-2-absorbing element of L and $a,b,c \in L$. If $abc \leq \phi(q)$ but $ab \nleq q$, $bc \nleq q$, $ac \nleq q$, then (a,b,c) is called a ϕ-triple zero of q.

Remark 2.6. If q is a ϕ-2-absorbing element of L which is not 2-absorbing, then there exists (a,b,c) a ϕ-triple zero of q for some $a,b,c \in L$.

Lemma 2.7. *Let q be a ϕ-2-absorbing element of L and suppose that (a,b,c) is a ϕ-triple zero of q for some $a,b,c \in L$. Then*

(i) $abq, bcq, acq \leq \phi(q)$.

(ii) $aq^2, bq^2, cq^2 \leq \phi(q)$.

(iii) $q^3 \leq \phi(q)$.

Proof. (i) Suppose that $abq \nleq \phi(q)$. Then $ab(c \vee q) \nleq \phi(q)$. Since $ab \nleq q$ and q is ϕ-2-absorbing, we have $a(c \vee q) \leq q$ or $b(c \vee q) \leq q$. So $ac \leq q$ or $bc \leq q$, which contradicts with our hypothesis. Thus $abq \leq \phi(q)$. Similarly one can easily show that $bcq \leq \phi(q)$ and $acq \leq \phi(q)$.

(ii) Suppose that $aq^2 \nleq \phi(q)$. Hence we have $a(b \vee q)(c \vee q) \nleq \phi(q)$ by (i). So we conclude either $a(b \vee q) \leq q$ or $a(c \vee q) \leq q$ or $(b \vee q)(c \vee q) \leq q$. Thus either $ab \leq q$ or $ac \leq q$ or $bc \leq q$, a contradiction. Therefore $aq^2 \leq \phi(q)$. Similarly it can be easily verified that $bq^2, cq^2 \leq \phi(q)$.

(iii) Assume that $q^3 \nleq \phi(q)$. Then we have $(a \vee q)(b \vee q)(c \vee q) \leq q$ but $(a \vee q)(b \vee q)(c \vee q) \nleq \phi(q)$ by (i) and (ii). Since q is ϕ-2-absorbing, $(a \vee q)(b \vee q) \leq q$ or $(a \vee q)(c \vee q) \leq q$ or $(b \vee q)(c \vee q) \leq q$, so we conclude $ab \leq q$ or $ac \leq q$ or $bc \leq q$, a contradiction. Thus $q^3 \leq \phi(q)$. □

Now we can give a condition for a ϕ-2-absorbing element to be a 2-absorbing element of L.

Corollary 2.8. *Let q be a proper element of L. Then the following statements hold:*

(i) If q is a ϕ-2-absorbing element of L such that $q^3 \nleq \phi(q)$, then q is a 2-absorbing element of L.

(ii) Let L be a C-lattice. If q is a ϕ-2-absorbing element of L that is not a 2-absorbing, then $\sqrt{q} = \sqrt{\phi(q)}$.

Proof. (i) The proof is clear by Remark 2.6 and Lemma 2.7 (iii).

(ii) Since q is not a 2-absorbing element of L, $q^3 \leq \phi(q)$ by Lemma 2.7 (iii). Hence $\sqrt{q} \leq \sqrt{\phi(q)}$. Since $\phi(q) \leq q$ is always hold, we get $\sqrt{q} = \sqrt{\phi(q)}$. □

Recall from [13] that an element $e \in L$ is said to be *principal*, if it satisfies the dual identities (i) $a \wedge be = ((a : e) \wedge b)e$ and (ii) $((ae \vee b) : e) = (b : e) \vee a$. Elements satisfying the identity (i) are called *meet principal* and elements satisfying the identity (ii) are called *join principal*. If the both identities are satisfied, then e is said to be a principal element of L. Note that by [13, Lemma 3.3 and Lemma 3.4], a finite product of principal elements of L is again principal. If every element of L can be written as a join of some principal elements of L, then L is said to be join principally generated lattice.

Theorem 2.9. *Let L be a join principally generated C-lattice and a,b,c be proper join principal elements of L. Then abc is a ϕ-2-absorbing element of L if and only if $abc = \phi(abc)$.*

Proof. Suppose that abc is a ϕ-2-absorbing element of L. Assume that $abc \neq \phi(abc)$. Then we have either $ab \leq abc$ or $ac \leq abc$ or $bc \leq abc$. Without loss generality we may assume that $ab \leq abc$. Since ab is principal, we conclude that $1_L = (abc : ab) = c \vee (0_L : ab)$. Observe that $(0_L : ab) \neq 1_L$. Indeed, if $(0_L : ab) = 1_L$, then $abc = 0_L \leq \phi(abc)$, a contradiction. Since $(0_L : ab) \neq 1_L$, $c \leq J(L)$ we conclude that $1_L \neq c \vee (0_L : ab)$, a contradiction. Thus $abc = \phi(abc)$. The converse part is clear. □

Theorem 2.10. *Let L be a local Noether domain. If q is an ϕ_n-2-absorbing element of L for all $n \geqslant 2$, then q is a 2-absorbing element of L.*

Proof. Let $abc \leq q$ for some $a, b, c \in L$. If $abc \nleq \phi_n(q)$, then we have either $ab \leq q$ or $bc \leq q$ or $ac \leq q$ as q is ϕ_n-2-absorbing. So suppose that $abc \leq \phi_n(q)$. Since $\wedge_{n=1}^{\infty} q^n = 0_L$, from Corollary 3.3 of [13], we conclude that $abc \leq 0_L$. Thus $a \leq 0_L$ or $b \leq 0_L$ or $c \leq 0_L$ as L is a domain, so clearly $ab \leq q$ or $bc \leq q$ or $ac \leq q$. □

We remind to the reader that for any $a \in L$, $L/a = \{b \in L : a \leq b\}$ is a multiplicative lattice with multiplication $c \circ d = cd \vee a$.

Theorem 2.11. *Let q be a proper element of L. Then the following statements hold:*

(i) q is a ϕ-2-absorbing element of L if and only if q is a weakly 2-absorbing element of $L/\phi(q)$.

(ii) q is a ϕ-prime element of L if and only if q is a weakly prime element of $L/\phi(q)$.

(iii) q is a ϕ-primary element of L if and only if q is a weakly primary element of $L/\phi(q)$.

Proof. (i) If $\phi(q) = \emptyset$, then there is nothing to prove. Thus assume that $\phi(q) \neq \emptyset$. Let $\phi(q) \neq (a \vee \phi(q)) \circ (b \vee \phi(q)) \circ (c \vee \phi(q)) = abc \vee \phi(q) \leq q$ for some $a, b, c \in L$. Then $abc \leq q$, but $abc \nleq \phi(q)$. Hence either $ab \leq q$ or $bc \leq q$ or $ac \leq q$. So $(a \vee \phi(q)) \circ (b \vee \phi(q)) \leq q$ or $(b \vee \phi(q)) \circ (c \vee \phi(q)) \leq q$ or $(a \vee \phi(q)) \circ (c \vee \phi(q)) \leq q$. Therefore q is a weakly 2-absorbing element of $L/\phi(q)$.

Conversely, let $abc \leq q$ and $abc \nleq \phi(q)$ for some $a, b, c \in L$. Then $\phi(q) \neq (a \vee \phi(q)) \circ (b \vee \phi(q)) \circ (c \vee \phi(q)) \leq q$. Hence $(a \vee \phi(q)) \circ (b \vee \phi(q)) \leq q$ or $(b \vee \phi(q)) \circ (c \vee \phi(q)) \leq q$ or $(a \vee \phi(q)) \circ (c \vee \phi(q)) \leq q$. Thus $ab \leq q$ or $bc \leq q$ or $ac \leq q$. Similarly one can easily prove (ii) and (iii). □

Corollary 2.12. *Let q be a proper element of L and $n \geq 2$. Then*

(i) q is a ϕ_n-2-absorbing element of L if and only if q is a weakly 2-absorbing element of L/q^n.

(ii) q is a ϕ_n-prime element of L if and only if q is a weakly prime element of L/q^n.

(iii) q is a ϕ_n-primary element of L if and only if q is a weakly primary element of L/q^n.

Proof. Since $\phi_n(q) = q^n$, the proof is clear by Theorem 2.11. □

Corollary 2.13. *Let q be a ϕ-2-absorbing element of L such that $\phi \leq \phi_3$. Then*

(i) q is a ϕ_n-2-absorbing element of L for every $n \geq 3$.

(ii) q is a ϕ_ω-2-absorbing element of L.

Proof. Suppose that q is a 2-absorbing element of L. Hence (i) and (ii) are clear.

(i) Assume that q is not a 2-absorbing element of L. Thus $q^3 \leq \phi(q)$ by Lemma 2.7 (iii)**.** Then we have $q^3 \leq \phi(q) \leq q^3$ as $\phi \leq \phi_3$. This follows $q^3 = q^n = \phi(q)$ for every $n \geq 3$, so we are done.

(ii) Let $abc \leq q$ and $abc \nleq \wedge_{n=1}^{\infty} q^n$. Then $abc \nleq q^n$ for some $n \geq 2$. If $n \geq 3$, then it is clear from (i). So suppose that $n = 2$. Hence $abc \nleq q^2$ which implies that $abc \nleq q^3$, so from (i) the result is obtained. □

Theorem 2.14. *Let x and y be two proper elements of L such that $x \leq y$ and let $n \geq 2$. If y is a ϕ_n-2-absorbing element of L, then y is a ϕ_n-2-absorbing element of L/x.*

Proof. Suppose that y is a ϕ_n-2-absorbing element of L. Assume that $(a\vee x)\circ(b\vee x)\circ(c\vee x) = abc\vee x \leq y$ and $(a\vee x)\circ(b\vee x)\circ(c\vee x) = abc\vee x \nleq y^n$ for some $a,b,c\in L$. As $y\in L/x$, then $y^n = y\circ y\circ y\circ ...\circ y = y^n\vee x$. Since $x\leq y$ and $abc\vee x\nleq y^n = y^n\vee x$, then we have $abc\leq y$ and $abc\nleq y^n$. Thus $ab\leq y$ or $ac\leq y$ or $bc\leq y$. Hence $(a\vee x)\circ(b\vee x)\leq y$ or $(a\vee x)\circ(c\vee x)\leq y$ or $(b\vee x)\circ(c\vee x)\leq y$ which means that y is a ϕ_n-2-absorbing primary element of L/x. □

Corollary 2.15. *Let x and y be two proper elements of L such that $x\leq y$. If y is a ϕ_ω-2-absorbing element of L, then y is a ϕ_ω-2-absorbing element of L/x.*

Proof. Similar to the proof of Theorem 2.14. □

Definition 2.16. Let x be a proper element of L/q such that $q\leq x$. Then x is called a ϕ_q-2-absorbing element of L/q if whenever $a,b,c\in L/q$ with $abc\leq x$ and $abc\nleq \phi(x)\vee q$ implies $ab\leq x$ or $ac\leq x$ or $bc\leq x$.

Theorem 2.17. *Let p and q be two elements of L with $q\leq p<1$. If p is a ϕ-2-absorbing element of L, then p is a ϕ_q-2-absorbing element of L/q.*

Proof. Let $(a\vee q)\circ(b\vee q)\circ(c\vee q)\leq p$ and $abc\vee q = (a\vee q)\circ(b\vee q)\circ(c\vee q)\nleq \phi(p)\vee q$ for some $a,b,c\in L$. Hence $abc\leq p$ and $abc\nleq\phi(p)$. Since p is ϕ-2-absorbing element of L, we conclude that $ab\leq p$ or $ac\leq p$ or $bc\leq p$. So we get $(a\vee q)\circ(b\vee q)\leq p$ or $(a\vee q)\circ(c\vee q)\leq p$ or $(b\vee q)\circ(c\vee q)\leq p$. □

Theorem 2.18. *Let p and q be two proper elements of L such that $q\leq\phi(p)$. Then the following statements are equivalent:*

(i) p is a ϕ-2-absorbing element of L.

(ii) p is a ϕ_q-2-absorbing element of L/q.

(iii) p is a ϕ_{q^n}-2-absorbing element of L/q^n.

Proof. (i)⇒(ii): It is clear by Theorem 2.17.

(ii)⇒(iii): Let $n\geq 1$. Since $q\leq\phi(p)$, we have $q^n\leq q\leq\phi(p)$. Suppose that $(a\vee q^n)\circ(b\vee q^n)\circ(c\vee q^n)\leq p$ and $(a\vee q^n)\circ(b\vee q^n)\circ(c\vee q^n)\nleq\phi(p)\vee q^n$ for some $a,b,c\in L$. Hence $abc\nleq\phi(p)$. Since $q\leq\phi(p)$ and $abc\nleq\phi(p)$, we have $abc\nleq q$. Thus $(a\vee q)\circ(b\vee q)\circ(c\vee q)\leq p$ and $(a\vee q)\circ(b\vee q)\circ(c\vee q)\nleq\phi(p)\vee q$. Since p is a ϕ_q- 2-absorbing element of L/q, one can conclude that $ab\leq p$ or $ac\leq p$ or $bc\leq p$. Thus $ab\vee q^n\leq p$ or $ac\vee q^n\leq p$ or $bc\vee q^n\leq p$ (in L/q^n).

(iii)⇒(i): Suppose that $abc\leq p$ and $abc\nleq\phi(p)$ for some $a,b,c\in L$. Since $q^n\leq\phi(p)$, we have $abc\nleq q^n$. As $q^n\leq\phi(p)\leq p$, we get $(a\vee q^n)\circ(b\vee q^n)\circ(c\vee q^n) = abc\vee q^n\leq p$ and $(a\vee q^n)\circ(b\vee q^n)\circ(c\vee q^n)\nleq\phi(p)\vee q^n$. Since p is a ϕ_{q^n}-2-absorbing element of L/q^n, one can conclude that $ab\leq p$ or $ac\leq p$ or $bc\leq p$. □

Corollary 2.19. *Let p and q be two proper elements of L. Suppose that q is not a weakly 2-absorbing element of L. The following statements are equivalent:*

(i) p is a ϕ-2-absorbing element of L.

(ii) p is a ϕ_{p^3}-2-absorbing element of L/p^3.

(iii) p is a ϕ_{p^n}-2-absorbing element of L/p^n for every $n\geq 3$.

Proof. Suppose that p is not a weakly 2-absorbing element of L. Hence p is not a 2-absorbing element of L. So we conclude $p^3\leq\phi(q)$ by Lemma 2.7. Thus we are done by Theorem 2.18. □

Definition 2.20. Let q be a proper element of L and $n\geq 2$. We call q as an n-potent 2-absorbing if whenever $a,b,c\in L$ with $abc\leq q^n$, then $ab\leq q$ or $bc\leq q$ or $ac\leq q$.

Theorem 2.21. *Let q be an n-almost 2-absorbing element for some $n\geq 2$. If q is k-potent 2-absorbing for some $k\leq n$, then q is a 2-absorbing element.*

Proof. Suppose that q is an n-almost 2-absorbing element. Let $abc \leq q$ for some $a, b, c \in L$. If $abc \nleq q^k$, then $abc \nleq q^n$. It implies either $ab \leq q$ or $bc \leq q$ or $ac \leq q$ as q is an n-almost 2-absorbing element. If $abc \leq q^k$, then we obtain the same result as q is k-potent 2-absorbing, so we are done. □

In the following theorems, we obtain some conditions under which a ϕ-2-absorbing element of L is a 2-absorbing element of L.

Let $J(L) = \wedge\{m \in L : m$ is a maximal element of $L\}$.

Theorem 2.22. *Let L be a Noether domain. Then an element q of L with $q \leq J(L)$ is a 2-absorbing element of L if and only if q is a ϕ_n-2-absorbing element of L for all $n \geq 2$.*

Proof. If q is 2-absorbing, then q is ϕ_n-2-absorbing by Theorem 2.3. Conversely, suppose that q is ϕ_n-2-absorbing for all $n \geq 2$ and let $a, b, c \in L$ with $abc \leq q$. If $abc \nleq q^k$ for some $k \geq 2$, we have either $ab \leq q$ or $bc \leq q$ or $ac \leq q$. So suppose that $abc \leq q^n$ for all $n \geq 2$. Since L is a Noether domain, we conclude $abc \leq \wedge_{n=1}^{\infty} q^n = 0_L$ by Corollary 1.4 in [4]. Since 0_L is prime, we get either $a = 0_L$ or $b = 0_L$ or $c = 0_L$. Without loss generality suppose that $a = 0_L$. This implies that $ab = 0_L \leq q$ which completes the proof. □

Theorem 2.23. *Let L be a Noether lattice. Let q be a non-zero non-nilpotent proper element of L satisfying the restricted cancellation law. Then q is a ϕ-2-absorbing element of L for some $\phi \leq \phi_n$ and for all $n \geq 2$ if and only if q is a 2-absorbing element of L.*

Proof. Assume that q is a 2-absorbing element of L. Then q is a ϕ-2-absorbing element of L for all ϕ. Thus q is ϕ-2-absorbing for some $\phi \leq \phi_n$ and for all $n \geq 2$.

Conversely assume that q is a ϕ-2-absorbing element of L for some $\phi \leq \phi_n$ for all $n \geq 2$. Hence q is a ϕ_n-2-absorbing element of L for all $n \geq 2$ by Lemma 2.2. Let $abc \leq q$ for some a, b, c in L. Here there are two cases:

Case 1: Let $abc \nleq q^n$ for some $n \geq 2$. Then by hypothesis we get $ab \leq q$ or $bc \leq q$ or $ac \leq q$. Case 2: Let $abc \leq q^n$ for all $n \geq 2$. We have that $a(b \vee q)(c \vee q) = abc \vee abq \vee acq \vee aq^2 \leq q$. If $a(b \vee q)(c \vee q) \nleq q^n$, then $a(b \vee q) \leq q$ or $a(c \vee q) \leq q$ or $(b \vee q)(c \vee q) \leq q$. It follows that either $ab \leq q$ or $bc \leq q$ or $ac \leq q$. If $a(b \vee q)(c \vee q) \leq q^n$, then $a(b \vee q)(c \vee q) = abc \vee abq \vee acq \vee aq^2 \leq q^n \leq q^2$. By [18, Lemma 1.11], we get $ab \leq q$ and $ac \leq q$. Thus q is a 2-absorbing element of L. □

Remark 2.24. Let $L = L_1 \times L_2 \times ...L_n$ where $L_1, L_2, ..., L_n$ are multiplicative lattices $(n \geq 1)$ and let $\phi = \psi_1 \times \psi_2 \times ... \times \psi_n$ where $\psi_i : L_i \to L_i \cup \{\emptyset\}$ $(i = 1, ..., n)$ be a function. Let $a = (a_1, a_2, ..., a_n)$ be an element of L. Observe that if $\psi_i(a_i) = \emptyset$ for some $i = 1, ..., n$, then there is no element of $\phi(a)$ and vice versa. Thus $\phi(a) = \emptyset$ if and only if $\psi_i(a_i) = \emptyset$ for some $i = 1, ..., n$.

Lemma 2.25. *Let $L = L_1 \times L_2$ where L_1, L_2 are two multiplicative lattices. Let $\phi = \psi_1 \times \psi_2$, where $\psi_i : L_i \to L_i \cup \{\emptyset\}$ $(i = 1, 2)$ is a function. Then q_1 is a 2-absorbing element of L_1 if and only if $q = (q_1, 1_{L_2})$ is a 2-absorbing element of L.*

Proof. Suppose that q_1 is a 2-absorbing element of L_1 and $(a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \leq q$ for some elements $(a, 1_{L_2}), (b, 1_{L_2}), (c, 1_{L_2})$ of L. Then $abc \leq q_1$ which implies that either $ab \leq q_1$ or $bc \leq q_1$ or $ac \leq q_1$. It follows $(a, 1_{L_2})(b, 1_{L_2}) \leq q$ or $(b, 1_{L_2})(c, 1_{L_2}) \leq q$ or $(a, 1_{L_2})(c, 1_{L_2}) \leq q$. Thus q is a 2-absorbing element of L.

Conversely suppose that $q = (q_1, 1_{L_2})$ is a 2-absorbing element of L but assume that q_1 is not a 2-absorbing element of L_1. Hence there exists $a, b, c \in L_1$ with $abc \leq q_1$ but neither $ab \leq q_1$ nor $bc \leq q_1$ nor $ac \leq q_1$. Thus we conclude $(a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \leq q$ but $(a, 1_{L_2})(b, 1_{L_2}) \nleq q$ and $(b, 1_{L_2})(c, 1_{L_2}) \nleq q$ and $(a, 1_{L_2})(c, 1_{L_2}) \nleq q$, a contradiction. □

Theorem 2.26. *Let $L = L_1 \times L_2$ where L_1, L_2 are two multiplicative lattices. Let $\phi = \psi_1 \times \psi_2$, where $\psi_i : L_i \to L_i \cup \{\emptyset\}$ $(i = 1, 2)$ is a function. Then the following statements hold:*

(i) If q_i is a proper element of L_i with $\psi_i(q_i) = q_i$ $(i = 1, 2)$, then $q = (q_1, q_2)$ is a ϕ-2-absorbing element of L.

(ii) If q_1 is a ψ_1-2-absorbing element of L_1, and $\psi_2(1_{L_2}) = 1_{L_2}$, then $q = (q_1, 1_{L_2})$ is a ϕ-2-absorbing element of L.

(iii) If q_2 is a ψ_2-2-absorbing element of L_2 and $\psi_1(1_{L_1}) = 1_{L_1}$, then $q = (1_{L_1}, q_2)$ is a ϕ-2-absorbing element of L.

Proof. (i) If $\psi_1(q_1) = q_1$ and $\psi_2(q_2) = q_2$, then there is no such an element (a, b) which satisfies $(a, b) \leq (q_1, q_2)$ and $(a, b) \nleq \phi(q_1, q_2) = (q_1, q_2)$, so we are done.

(ii) Suppose that $\psi_1(q) = \emptyset$. Then $q = (q_1, 1_{L_2})$ is a ϕ-2-absorbing element of L by Lemma 2.25. So assume that $\psi_1(q) \neq \emptyset$ and q_1 is ψ_1-2-absorbing element of L_1. Let $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$ such that $abc \leq q$ and $abc \nleq \phi(q)$. Hence $a_1b_1c_1 \leq q_1$ and $a_1b_1c_1 \nleq \psi_1(q_1)$, this implies that either $a_1b_1 \leq q_1$ or $b_1c_1 \leq q_1$ or $a_1c_1 \leq q_1$. Thus either $ab \leq q$ or $bc \leq q$ or $ac \leq q$.

(iii) This can be easily obtained similar to (ii). □

Theorem 2.27. *Let $L = L_1 \times L_2$ where L_1, L_2 are two multiplicative lattices and $\phi = \psi_1 \times \psi_2$, where $\psi_i : L_i \longrightarrow L_i \cup \{\emptyset\}$ $(i = 1, 2)$ is a function such that $\psi_2(1_{L_2}) \neq 1_{L_2}$. Let q_1 be a proper element of L_1 and $q = (q_1, 1_{L_2})$. Then the following statements are equivalent:*

(i) $(q_1, 1_{L_2})$ is a ϕ-2-absorbing element of L.

(ii) $(q_1, 1_{L_2})$ is a 2-absorbing element of L.

(iii) q_1 is a 2-absorbing element of L_1.

Proof. If $\psi_1(q_1) = \emptyset$ and $\psi_2(1_{L_2}) = \emptyset$, then $\phi(q) = \emptyset$ by Remark 2.24. So we are done from Lemma 2.25. Thus assume that $\psi_1(q_1) \neq \emptyset$ or $\psi_2(1_{L_2}) \neq \emptyset$.

(i)⇒(ii): Assume that $q = (q_1, 1_{L_2})$ is a ϕ-2-absorbing element of L. Then q_1 is a ψ_1-2-absorbing element of L_1. Indeed, if q_1 is not a ψ_1-2-absorbing element of L_1, then there exist a, b, c in L_1 such that $abc \leq q_1$ and $abc \nleq \psi_1(q_1)$ but $ab \nleq q_1$ and $bc \nleq q_1$ and $ac \nleq q_1$. Then $(abc, 1_{L_2}) = (a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \leq q$ and $(abc, 1_{L_2}) = (a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \nleq (\psi_1(p_1), \psi_2(1_{L_2})) = \phi(q)$. This implies $(ab, 1_{L_2}) = (a, 1_{L_2})(b, 1_{L_2}) \leq q$ or $(bc, 1_{L_2}) = (b, 1_{L_2})(c, 1_{L_2}) \leq q$ or $(ac, 1_{L_2}) = (a, 1_{L_2})(c, 1_{L_2}) \leq q$, which means $ab \leq q_1$ or $bc \leq q_1$ or $ac \leq q_1$, a contradiciton. Thus q_1 is a ψ_1-2-absorbing element of L_1.

If q_1 is a 2-absorbing element of L_1, then it is clear. Assume that q_1 is not a 2-absorbing element of L_1. Hence q_1 has a ψ_1-triple-zero (x, y, z) for some x, y, z in L_1 by Remark 2.6. Since $\psi_2(1_{L_2}) \neq 1_{L_2}$, then we get $(xyz, 1_{L_2}) = (x, 1_{L_2})(y, 1_{L_2})(z, 1_{L_2}) \leq q$ and $(xyz, 1_{L_2}) = (x, 1_{L_2})(y, 1_{L_2})(z, 1_{L_2}) \nleq \phi(q)$. Therefore $(x, 1_{L_2})(y, 1_{L_2}) \leq q$ or $(y, 1_{L_2})(z, 1_{L_2}) \leq q$ or $(x, 1_{L_2})(z, 1_{L_2}) \leq q$. So we get $xy \leq q_1$ or $yz \leq q_1$ or $xz \leq q_1$, a contradiction. Thus q_1 is a 2-absorbing element of L_1. Consequently, $(q_1, 1_{L_2})$ is a 2-absorbing element of L.

(ii)⇒ (iii): It can be easily shown similar to the argument in $(i) \Rightarrow$ (ii).

(iii)⇒(i): It is clear. □

Lemma 2.28. *Let $L = L_1 \times L_2 \times L_3$ where L_1, L_2, L_3 are C-lattices. Let $\phi = \psi_1 \times \psi_2 \times \psi_3$, where $\psi_i : L_i \longrightarrow L_i \cup \{\emptyset\}$ $(i = 1, 2, 3)$ is a function with $\psi_i(1_{L_i}) \neq 1_{L_i}$. If $q = (q_1, q_2, q_3)$ is a ϕ-2-absorbing element of L, then either $q = \phi(q)$ or q is a 2-absorbing element of L.*

Proof. If $\phi(q) = \emptyset$, then we are done. So assume $\phi(q) \neq \emptyset$. Suppose that $q \neq \phi(q)$. Hence there is an element $(a, b, c) \in L$ with $(a, b, c) \leq q$ but $(a, b, c) \nleq \phi(q)$. So $(a, b, c) = (a, 1_{L_2}, 1_{L_3})(1_{L_1}, b, 1_{L_3})(1_{L_1}, 1_{L_2}, c) \leq q$ implies that either $(a, 1_{L_2}, 1_{L_3})(1_{L_1}, b, 1_{L_3}) \leq q$ or $(1_{L_1}, b, 1_{L_3})(1_{L_1}, 1_{L_2}, c) \leq q$ or $(a, 1_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, c) \leq q$. Without loss of generality assume that $(a, 1_{L_2}, 1_{L_3})(1_{L_1}, b, 1_{L_3}) \leq q$. Then $q_3 = 1_{L_3}$ which means that $q^3 \nleq \phi(q)$. Thus q is a 2-absorbing element of L by Corollary 2.8. □

Theorem 2.29. *Let $L = L_1 \times L_2 \times L_3$ where L_1, L_2, L_3 are C-lattices. Let $\phi = \psi_1 \times \psi_2 \times \psi_3$, where $\psi_i : L_i \longrightarrow L_i \cup \{\emptyset\}$ $(i = 1, 2, 3)$ is a function with $\psi_i(1_{L_i}) \neq 1_{L_i}$. If $q \neq \phi(q)$, then the followings are equivalent:*

(i) q is a ϕ-2-absorbing element of L.

(ii) q is a 2-absorbing element of L.

(iii) q is in one of the following type:

I) $q = (1_{L_1}, q_2, q_3)$, where q_2 is a prime element of L_2 and q_3 is a prime element of L_3.

II) $q = (q_1, 1_{L_2}, q_3)$, where q_1 is a prime element of L_1 and q_3 is a prime element of L_3.

III) $q = (q_1, q_2, 1_{L_3})$, where q_1 is a prime element of L_1 and q_2 is a prime element of L_2.

IV) For some $i \in \{1,2,3\}$, q_i is a 2-absorbing element of L_i and $q_j = 1_{l_j}$ for every $j \in \{1,2,3\} \backslash \{i\}$.

Proof. (i) $\Rightarrow$ (ii): If $\phi(q) = \emptyset$ and q is a ϕ-2-absorbing element, then obviously q is a 2-absorbing element of L. So assume that $\phi(q) \neq \emptyset$. Let $q = (q_1, q_2, q_3)$ be a ϕ-2-absorbing element of L, then q is a 2-absorbing element of L by Lemma 2.28.

(ii) $\Rightarrow$ (iii): Suppose that q is a 2-absorbing element of L. Since $q \neq \phi(q)$, there is a compact element of L such that $(a_1, a_2, a_3) \leq q$ and $(a_1, a_2, a_3) \nleq \phi(q)$. Since $(a_1, a_2, a_3) = (a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, a_2, 1_{L_3})(1_{L_1}, 1_{L_2}, a_3)$ and q is ϕ-2-absorbing, we have $(a_1, a_2, 1_{L_3}) \leq q$ or $(1_{L_1}, a_2, a_3) \leq q$ or $(a_1, 1_{L_2}, a_3) \leq q$. This means that either $q_1 = 1_{L_1}$ or $q_2 = 1_{L_2}$ or $q_3 = 1_{L_3}$.

Case I. Suppose that $q = (1_{L_1}, q_2, q_3)$ where $q_2 \neq 1_{L_2}$ and $q_3 \neq 1_{L_3}$. We show that q_2 is a prime element of L_2. Let $xy \leq q_2$. Hence $(1_{L_1}, x, 1_{L_3})(1_{L_1}, 1_{L_2}, q_3)(1_{L_1}, y, 1_{L_3}) \leq q$ and it implies that $(1_{L_1}, x, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \leq q$ or $(1_{L_1}, x, 1_{L_3})(1_{L_1}, y, 1_{L_3}) \leq q$ or $(1_{L_1}, 1_{L_2}, 0_{L_3})(1_{L_1}, y, 1_{L_3}) \leq q$. Since q_3 is proper, we get $(1_{L_1}, xy, 1_{L_3}) = (1_{L_1}, x, 1_{L_3})(1_{L_1}, y, 1_{L_3}) \nleq q$. Thus $x \leq q_2$ or $y \leq q_2$, which shows that q_2 is prime. By the similar argument one can easily show that q_3 is a prime element of L_3.

Case II. $q = (q_1, 1_{L_2}, q_3)$, where $q_1 \neq 1_{L_1}$ and $q_3 \neq 1_{L_3}$ and Case III. $q = (q_1, q_2, 1_{L_3})$, where $q_1 \neq 1_{L_1}$ and $q_2 \neq 1_{L_2}$ can be easily obtained similar to Case I.

Case IV. Without loss of generality suppose that $q = (q_1, 1_2, 1_{L_3})$ where q_1 is a proper element of L_1. Let $x_1x_2x_3 \leq q_1$ for some $x_1, x_2, x_3 \in L_1$. Then $(x_1x_2x_3, 1_{L_2}, 0_{L_3}) = (x_1, 1_{L_2}, 0_{L_3})(x_2, 1_{L_2}, 0_{L_3})(x_3, 1_{L_2}, 0_{L_3}) \leq q$ and $(x_1x_2x_3, 1_{L_2}, 0_{L_3}) \nleq \phi(q)$. Since q is ϕ-2-absorbing, we have either $(x_1x_2, 1_{L_2}, 0_{L_3}) \leq q$ or $(x_2x_3, 1_{L_2}, 0_{L_3}) \leq q$ or $(x_1x_3, 1_{L_2}, 0_{L_3}) \leq q$. So $x_1x_2 \leq q_1$ or $x_2x_3 \leq q_1$ or $x_2x_3 \leq q_1$.

(iii)$\Rightarrow$ (i): Suppose that q_2 and q_3 are prime elements of L_2 and L_3, respectively and $q = (1_{L_1}, q_2, q_3)$. Let (a_1, a_2, a_3), (b_1, b_2, b_3), $(c_1, c_2, c_3) \in L$ such that $(a_1, a_2, a_3)(b_1, b_2, b_3)(c_1, c_2, c_3) \leq q$ and $(a_1, a_2, a_3)(b_1, b_2, b_3)(c_1, c_2, c_3) \nleq \phi(q)$. Assume that $(a_1, a_2, a_3)(b_1, b_2, b_3) \nleq q$. Hence $a_2b_2 \nleq q_2$ or $a_3b_3 \nleq q_3$. Without loss of generality we may suppose that $a_2b_2 \nleq q_2$ and $a_3b_3 \leq q_3$. Since q_2 is prime, we have $c_2 \leq q_2$, which implies that $(a_1, a_2, a_3)(c_1, c_2, c_3) \leq q$, we are done.

If $q = (q_1, 1_2, 1_{L_3})$ where q_1 is a 2-absorbing element of L_1, then it can be seen that q is a 2-absorbing element of L. Thus q is a ϕ-2-absorbing element of L. □

Theorem 2.30. *Let $L = L_1 \times L_2 \times L_3$ where L_1, L_2, L_3 are multiplicative lattices. Let $\phi = \psi_1 \times \psi_2 \times \psi_3$, where $\psi_i : L_i \longrightarrow L_i \cup \{\emptyset\}$ $(i = 1,2,3)$ is a function. If a proper $a = (a_1, a_2, a_3) \in L$ is a ϕ-2-absorbing element, then $\psi_i(a_i) = \emptyset$ or $\psi_i(a_i) = a_i$ $(i = 1,2,3)$ for every proper element a_i of L_i.*

Proof. Assume on the contrary that $\psi_1(a_1) \neq a_1$ and $\psi_1(a_1) \neq \emptyset$ for some proper element $a_1 \in L_1$. Put $a = (a_1, 0_{L_2}, 0_{L_3})$. Hence $(a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 0_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \leq a$, but $(a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 0_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \nleq \phi(a)$. Since a is a ϕ-2-absorbing element, we conclude either $(a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 0_{L_2}, 1_{L_3}) \leq a$ or $(1_{L_1}, 0_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \leq a$ or $(a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \leq a$. It follows $1_{L_3} \leq a_3$ or $1_{L_1} \leq a_1$ or $1_{L_2} \leq a_2$, which are contradictions. Thus $\psi_i(a_i) = a_i$ $(i = 1,2,3)$ for every proper element a_i of L_i. □

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