Palestine Journal of Mathematics

Vol. 2(2) (2013) , 287–299

Exponential Decay in Thermoelastic Systems with Internal Distributed Delay

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Communicated by Salim Messaoudi

MSC 2010: 35B37, 35L55, 74D05, 93D15, 93D20

Keywords and phrases: Thermoelasticity, distributed delay, Stability, exponential decay

Abstract. In this paper, we consider a thermoelastic system with an internal distributed delay. We use the energy method to prove, under a suitable assumption on the weight of the delay, that the damping effect through heat conduction is strong enough to uniformly stabilize the system even in the presence of time delay.

1 Introduction

In this paper, we are concerned with the following problem

$$\begin{cases} au_{tt}(x,t) - du_{xx}(x,t) + \beta\theta_x(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\ b\theta_t(x,t) - k_1\theta_{xx}(x,t) - \int_{\tau_1}^{\tau_2} k_2(s)\theta_{xx}(x,t-s)ds + \beta u_{xt}(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x), & x \in (0,L) \\ \theta_x(x,-t) = f_0(x,t), \quad x \in (0,L), & t \in (0,\tau_2), \end{cases} \tag{1.1}$$

a thermoelastic system with a distributed delay associated with initial data u_0, u_1, θ_0 and history function f_0 in suitable function spaces. Here $a, b, d, k_1, \beta, \tau_2$ are positive constants, τ_1 is a non-negative constant with $\tau_1 < \tau_2$, u is the displacement, θ is the temperature difference from a reference value, and $k_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function. We consider boundary conditions of the type

$$u_x(0,t) = u_x(L,t) = \theta(0,t) = \theta(L,t) = 0 \tag{1.2}$$

or

$$u(0,t) = u(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0 \tag{1.3}$$

and study the asymptotic behavior of the solution in each case and look for sufficient conditions on k_2 that guarantee the uniform stability of these systems.

Time delays arise in many applications because, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [1, 31], and references therein. In many cases it was shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. The stability issue of systems with delay is, therefore, of theoretical and practical importance.

In the absence of delay, it is well-known (see [3, 10, 21]) that the one dimensional linear thermoelastic system associated with various types of boundary conditions decays to zero exponentially. For the multi-dimensional case, we have the pioneering work of Daformos [5], in which he proved an asymptotic stability result; but no rate of decay has been given. The uniform rate of decay for the solution in two or three dimensional space was obtained by Jiang, Rivera and Racke [13] in special situation like radial symmetry. Lebeau and Zuazua [18] proved that the decay rate is never uniform when the domain is convex. Thus, in order to solve this problem, additional damping mechanisms are necessary. In this aspect, Pereira and Menzala [28] introduced a linear internal damping effective in the whole domain, and established the uniform decay rate. A similar result was obtained by Liu [20] for a linear boundary velocity feedback acting on the elastic component of the system, and by Liu and Zuazua [22] for a nonlinear boundary feedback. Oliveira and Charao [27] improved the result in [28] by including a weak localized dissipative term effective only in a neighborhood of part of the boundary and proved an exponential decay result when the damping term is linear and a polynomial decay result for a nonlinear damping term. For more literature on the subject, we refer the reader to books by Jiang and Racke [12] and Zheng [34].

Introducing a delay term in the internal feedback of the thermoelastic system makes the problem different from that considered in the literature. The presence of a delay term may turn such well-behaved system into a wild one. For instance, contrary to the exponential stability of the thermoelastic system

$$\begin{cases} au_{tt}(x,t) - du_{xx}(x,t-\tau_1) + \beta\theta_x(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\ b\theta_t(x,t) - k\theta_{xx}(x,t-\tau_2) + \beta u_{xt}(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\ u(0,t) = u(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0, & t \ge 0 \end{cases}$$

when $\tau_1 = \tau_2 = 0$, Racke [29] proved that, for any constant delays $\tau_1 > 0$ or $\tau_2 > 0$, this system is instable. In [6] and [7], the authors also showed that a small delay in a boundary control of certain hyperbolic systems could be a source of instability, and stabilizing these systems, involving input delay terms, requires additional control terms. In this aspect, Datko, Lagnese and Polis [7] examined the following problem:

$$u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + a^2u(x,t) = 0, \qquad \text{in } (0,1)\times(0,\infty)$$

$$u(0,t) = 0, \qquad u_x(1,t) = -ku_t(1,t-\tau), \qquad t > 0$$

with a, k, τ positive real numbers. Through a careful spectral analysis, they showed that, for any $a > 0$ and any k satisfying

$$0 < k < \frac{1-e^{-2a}}{1+e^{-2a}},$$

the spectrum of this system lies in $\operatorname{Re} w \le -\beta$, where β is a positive constant depending on the delay τ. Consequently, the uniform stability of the system is obtained.

Regarding the systems of wave equations with linear frictional damping term and internal constant delay

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) + a_0u_t(x,t) + au_t(x,t-\tau) = 0, & \text{in } \Omega\times(0,\infty) \\ u(x,t) = 0, & x\in\Gamma_0, t>0 \\ \frac{\partial u}{\partial v}(x,t) = 0, & x\in\Gamma_1, t>0 \end{cases} \tag{1.4}$$

or with boundary constant delay

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) = 0, & \text{in } \Omega\times(0,\infty) \\ u(x,t) = 0, & x\in\Gamma_0, t>0 \\ \frac{\partial u}{\partial v}(x,t) = -a_0u_t(x,t) - au_t(x,t-\tau), & x\in\Gamma_1, t>0 \end{cases} \tag{1.5}$$

it is well-known, in the absence of delay ($a = 0, a_0 > 0$), that these systems are exponentially stable, see [15, 16, 17, 19, 35, 36]. In the presence of delay ($a > 0$), Nicaise and Pignotti [23] examined systems (1.4) and (1.5) and proved under the assumption $a < a_0$ (which means that the weight of the delay is smaller than the one of the feedback) that the energy is exponentially stable. Otherwise, they produced a sequence of delays for which the corresponding solution is instable. The main approach used there is an observability inequality combined with a Carleman estimate. See also [2] for treatment to these problems in more general abstract form and [25] for analogous results in the case of boundary time-varying delay. We also recall the result by Yung, Xu, and Li [33], where the authors proved the same result as in [23] for the one space dimension by adopting the spectral analysis approach. Said-Houari and Laskri [30] also imposed the same condition ($a < a_0$) to establish the exponential stability of the following Timoshenko system with constant delay

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \quad (0,1)\times\mathbb{R}_+ \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + a_0\psi_t(x,t) + a\psi_t(x,t-\tau) = 0. \end{cases} \tag{1.6}$$

This result was recently extended to the case of time-varying delay by Kirane, Said-Houari and Anwar [14]. In [9], Guesmia studied an abstract evolution equation with time delay in the presence of viscoelastic damping and proved an exponential decay result.

When the delay term in (1.4) or (1.5) is replaced by the distributed delay

$$\int_{\tau_1}^{\tau_2} a(s)u_t(x,t-s)ds,$$

exponential stability results have been obtained in [24] under the condition

$$\int_{\tau_1}^{\tau_2} |a(s)|\, ds < a_0.$$

The heat equation with internal or boundary delay was also treated in [4, 8, 11, 26, 32]. In particular, for the case of internal time-varying delay, the problem

$$\begin{cases} u_t(x,t) - a_0u_{xx}(x,t) - au_{xx}(x,t-\tau(t)) = 0, \qquad \text{in } (0,\pi)\times(0,\infty) \\ u(0,t) = u(\pi,t) = 0, \qquad t>0 \end{cases}$$

was studied in [4] and the case of boundary time-varying delay, the problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0, \qquad \text{in } (0,\pi)\times(0,\infty) \\ u(0,t) = 0, \qquad t>0 \\ u_x(\pi,t) = -a_0u(\pi,t) - au(\pi,t-\tau(t)), \qquad t>0 \end{cases}$$

was studied in [26]. In both situations, the authors assumed that $\tau(t)$ satisfies, for some positive constants K, M, d and for any $t > 0$,

$$K \le \tau(t) \le M \quad \text{and} \quad \tau'(t) \le d < 1$$

and showed, using two different methods, that the condition

$$|a| < \sqrt{1-d}a_0$$

is sufficient in both cases to obtain exponential stability.

Our aim in this work is to investigate (1.1) with the boundary conditions (1.2) or (1.3) and establish exponential decay results under suitable assumption on the delay term. In fact, the uniform stability is obtained provided $\int_{\tau_1}^{\tau_2} |k_2(s)|\, ds < k_1$. We also discuss the case of constant delay and give sufficient condition for the exponential decay. The paper is organized as follows. In section 2, we present some notations and material needed for our work. Then, in section 3, we use the multiplier method and adopt some arguments from [24] to state and prove our main results.

2 Preliminaries

Let us introduce the following new variable

$$z(x,\rho,s,t)=\theta_x(x,t-\rho s), \qquad (x,\rho,s,t)\in(0,L)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty). \tag{2.1}$$

Then, system (1.1) is equivalent to

$$\begin{cases} au_{tt}(x,t)-du_{xx}(x,t)+\beta\theta_x(x,t)=0, \qquad \text{in } (0,L)\times(0,\infty) \\ b\theta_t(x,t)-k_1\theta_{xx}(x,t)-\int_{\tau_1}^{\tau_2}k_2(s)z_x(x,1,s,t)ds+\beta u_{xt}(x,t)=0, \\ sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0, \qquad \text{in } (0,L)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty) \\ z(x,0,s,t)=\theta_x(x,t), \qquad \text{in } (0,L)\times(\tau_1,\tau_2)\times(0,\infty) \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \theta(x,0)=\theta_0(x), \qquad x\in(0,L) \\ z(x,p,s,0)=f_0(x,\rho s), \qquad (x,\rho,s)\in(0,L)\times(0,1)\times(\tau_1,\tau_2) \end{cases} \tag{2.2}$$

In the sequel, we consider (u,θ,z) to be a solution of system (2.2) and the boundary conditions (1.2) (or (1.3)) with the regularity needed to justify the calculations in this paper (see [24] for the well-posedness of a similar problem). We assume that

$$\int_{\tau_1}^{\tau_2}|k_2(s)|\,ds<k_1 \tag{2.3}$$

and show that this condition is sufficient to prove the exponential decay of the energy, defined by

$$E(t)=\frac{1}{2}\int_0^L(au_t^2+du_x^2+b\theta^2)dx+\frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}s(|k_2(s)|+\xi)z^2dsd\rho dx$$

where ξ is a positive constant satisfying

$$\int_{\tau_1}^{\tau_2}|k_2(s)|\,ds+\frac{\xi(\tau_2-\tau_1)}{2}<k_1. \tag{2.4}$$

The main idea is to construct a Lyapunov functional $\mathcal{L}$ equivalent to E satisfying

$$a_1E(t)\le\mathcal{L}(t)\le a_2E(t),$$

for two positive constants a_1,a_2, and

$$\mathcal{L}'(t)\le-\alpha\mathcal{L}(t),$$

for some $\alpha>0$. A suitable choice of the multipliers and the sequence of estimates in the energy method will give the desired result. We will use c, throughout this paper, to denote a generic positive constant.

3 Uniform Stability

3.1 The system (2.2) with (1.2)

In this subsection we state and prove our decay result for the energy of the system (2.2) with the boundary conditions

$$u_x(0,t)=u_x(L,t)=\theta(0,t)=\theta(L,t)=0, \qquad t\ge0. \tag{3.1}$$

For this purpose, we establish several lemmas.

Lemma 3.1. *The the energy functional E satisfies, for some positive constant m,*

$$E'(t) \leq -m\left[\int_0^L \theta_x^2 dx + \int_0^L \int_{\tau_1}^{\tau_2} z^2(x,1,s,t) ds dx\right]. \tag{3.2}$$

Proof. Using equations (2.2) and (3.1) and integrating by parts yield

$$\begin{aligned} E'(t) &= -k_1 \int_0^L \theta_x^2 dx - \int_0^L \theta_x \int_{\tau_1}^{\tau_2} k_2(s) z(x,1,s,t) ds dx \\ &\quad - \int_0^L \int_{\tau_1}^{\tau_2} (|k_2(s)| + \xi) \int_0^1 z(x,\rho,s,t) z_\rho(x,\rho,s,t) d\rho ds dx \\ &= -k_1 \int_0^L \theta_x^2 dx - \int_0^L \theta_x \int_{\tau_1}^{\tau_2} k_2(s) z(x,1,s,t) ds dx \\ &\quad + \frac{1}{2}\left[\int_{\tau_1}^{\tau_2} |k_2(s)|\, ds + \xi(\tau_2 - \tau_1)\right] \int_0^L \theta_x^2 dx - \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} (|k_2(s)| + \xi) z^2(x,1,s,t) ds dx. \end{aligned}$$

By exploiting Young's inequality, we obtain

$$E'(t) \leq -\left[k_1 - \int_{\tau_1}^{\tau_2} |k_2(s)|\, ds - \frac{\xi(\tau_2 - \tau_1)}{2}\right] \int_0^L \theta_x^2 dx - \frac{\xi}{2}\int_0^L \int_{\tau_1}^{\tau_2} z^2(x,1,s,t) ds dx$$

Then, using (2.4), our conclusion follows. ∎

From the first equation of (2.2) and (3.1), we conclude that

$$\int_0^L u(x,t) dx = pt + q$$

where $p = \int_0^L u_1(x) dx$ and $q = \int_0^L u_0(x) dx$. This implies that

$$\overline{u}(x,t) := u(x,t) - \frac{1}{L}(pt + q)$$

satisfies

$$\int_0^L \overline{u}(x,t) dx = \int_0^L \overline{u}_t(x,t) dx = 0. \tag{3.3}$$

In addition, (u, θ, z) and $(\overline{u}, \theta, z)$ satisfy the same differential equations and boundary conditions. Therefore, for this subsection, we work with $\overline{u}$ but we denote it by u for simplicity.

Lemma 3.2. *The functional*

$$F_1(t) := a \int_0^L u u_t dx$$

satisfies, for some positive constant m_0, the estimate

$$F_1'(t) \leq -m_0 \int_0^L u_x^2 dx + a \int_0^L u_t^2 dx + c \int_0^L \theta_x^2 dx. \tag{3.4}$$

Proof. Direct computations, using (2.2), yield

$$F_1'(t) = a\int_0^L u_t^2 dx - d\int_0^L u_x^2 dx - \beta\int_0^L u\theta_x dx.$$

We use Young's and Poincaré's inequalities to get

$$F_1'(t) \le a\int_0^L u_t^2 dx - d\int_0^L u_x^2 dx + c\varepsilon\int_0^L u_x^2 dx + C_\varepsilon\int_0^L \theta_x^2 dx.$$

Then, choosing ε small enough, (3.4) is established. ∎

Lemma 3.3. *The functional*

$$F_2(t) := -ab\int_0^L \theta\left(\int_0^x u_t(y,t)dy\right)dx$$

satisfies, for some positive constant m_1 *and for any* $0<\delta<1$*, the estimate*

$$F_2'(t) \le -m_1\int_0^L u_t^2 dx + \delta\int_0^L u_x^2 dx + C_\delta\int_0^L \theta_x^2 dx + c\int_0^L\int_{\tau_1}^{\tau_2} z^2(x,1,s,t)dsdx. \tag{3.5}$$

Proof. By exploiting (2.2), (3.1), (3.3) and integrating by parts, we have

$$\begin{aligned}
F_2'(t) &= -a\int_0^L\left(k_1\theta_{xx} + \int_{\tau_1}^{\tau_2} k_2(s)z_x(x,1,s,t)ds - \beta u_{xt}\right)\left(\int_0^x u_t(y,t)dy\right)dx \\
&\quad -b\int_0^L \theta\left(\int_0^x (du_{xx} - \beta\theta_x)dy\right)dx \\
&= -a\beta\int_0^L u_t^2 dx + ak_1\int_0^L u_t\theta_x dx + a\int_0^L u_t\left(\int_{\tau_1}^{\tau_2} k_2(s)z(x,1,s,t)ds\right)dx \\
&\quad -bd\int_0^L \theta u_x dx + b\beta\int_0^L \theta^2 dx.
\end{aligned}$$

Then, use of Young's inequality gives

$$\begin{aligned}
F_2'(t) &\le -a\beta\int_0^L u_t^2 dx + \varepsilon\int_0^L u_t^2 dx + \delta\int_0^L u_x^2 dx \\
&\quad + C_\varepsilon\left[\int_0^L \theta_x^2 dx + \int_0^L\int_{\tau_1}^{\tau_2} z^2(x,1,s,t)dsdx\right] + C_\delta\int_0^L \theta^2 dx + b\beta\int_0^L \theta^2 dx.
\end{aligned}$$

Again, the choice of ε small enough and use of Poincaré's inequality yield (3.5). ∎

Lemma 3.4. *The functional*

$$F_3(t) := \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}(|k_2(s)| + \xi)z^2(x,\rho,s,t)dsd\rho dx$$

satisfies, for some positive constant γ_0*, the estimate*

$$F_3'(t) \le -\gamma_0\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s(|k_2(s)| + \xi)z^2 dsd\rho dx + c\int_0^L \theta_x^2 dx. \tag{3.6}$$

Proof. We use the third equation of (2.2) to find that

$$\begin{aligned}
F_3'(t) &= -2\int_0^L \int_{\tau_1}^{\tau_2} (|k_2(s)|+\xi) \int_0^1 e^{-s\rho} z z_\rho d\rho ds dx \\
&= -\int_0^L \int_{\tau_1}^{\tau_2} (|k_2(s)|+\xi) \int_0^1 e^{-s\rho} \frac{\partial}{\partial \rho} z^2 d\rho ds dx \\
&= -\int_0^L \int_{\tau_1}^{\tau_2} (|k_2(s)|+\xi) \left[e^{-s} z^2(x,1,s,t) - z^2(x,0,s,t) + s\int_0^1 e^{-s\rho} z^2 d\rho \right] ds dx \\
&\leq c\int_0^L \theta_x^2 dx - \gamma_0 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s(|k_2(s)|+\xi) z^2 ds d\rho dx
\end{aligned}$$

which gives (3.6). ∎

We are now ready to prove the following result.

Theorem 3.5. *Assume that* $\int_{\tau_1}^{\tau_2} |k_2(s)|\, ds < k_1$. *Then, there exist positive constants* c_0, c_1 *such that the energy functional for system (2.2) and (3.1) satisfies*

$$E(t) \leq c_0 e^{-c_1 t}. \tag{3.7}$$

Proof. For $N_1, N_2 > 1$, let

$$\mathcal{L}(t) := N_1 E(t) + F_1(t) + N_2 F_2(t) + F_3(t).$$

By combining (3.2) and (3.4)-(3.6), we obtain

$$\mathcal{L}'(t) \leq -(m_1 N_2 - a)\int_0^L u_t^2 dx - (m_0 - \delta N_2)\int_0^L u_x^2 dx - (mN_1 - 2c - C_\delta N_2)\int_0^L \theta_x^2 dx$$

$$-\gamma_0 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s(|k_2(s)|+\xi) z^2 ds d\rho dx - (mN_1 - cN_2)\int_0^L \int_{\tau_1}^{\tau_2} z^2(x,1,s,t) ds dx.$$

At this point, we choose N_2 large enough so that

$$\gamma_1 := (m_1 N_2 - a) > 0,$$

then δ small enough so that

$$\gamma_2 := (m_0 - \delta N_2) > 0.$$

Next, we choose N_1 large enough so that

$$\gamma_3 := (mN_1 - 2c - C_\delta N_2) > 0$$

and

$$(mN_1 - cN_2) > 0.$$

So, we arrive at

$$\mathcal{L}'(t) \leq -\gamma_1 \int_0^L u_t^2 dx - \gamma_2 \int_0^L u_x^2 dx - \gamma_3 \int_0^L \theta_x^2 dx - \gamma_0 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s(|k_2(s)|+\xi) z^2 ds d\rho dx$$

which, using Poincaré's inequality, yields

$$\mathcal{L}'(t) \leq -c' E(t) \tag{3.8}$$

for some constant $c' > 0$. On the other hand, we find that

$$
\begin{aligned}
|\mathcal{L}(t) - N_1 E(t)| &\leq |F_1(t)| + N_2 |F_2(t)| + |F_3(t)| \\
\leq\ & a\int_0^L |uu_t|\, dx + abN_2 \int_0^L \left|\theta\left(\int_0^x u_t(s,t)ds\right)\right| dx \\
& + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}(|k_2(s)| + \xi) z^2(x,\rho,s,t)dsd\rho dx \\
\leq\ & c\int_0^L (u_x^2 + u_t^2 + \theta^2)dx + c\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s(|k_2(s)| + \xi)z^2(x,\rho,s,t)dsd\rho dx \\
\leq\ & cE(t).
\end{aligned}
$$

Therefore, we can choose N_1 even larger (if needed) so that

$$\mathcal{L}(t) \sim E(t). \tag{3.9}$$

Hence, (3.8) and (3.9) lead to

$$\mathcal{L}'(t) \leq -\alpha \mathcal{L}(t)$$

for some $\alpha > 0$. A simple integration on $(0, t)$, then again use of (3.9) give (3.7). ∎

3.2 The system (2.2) with (1.3)

Let us consider system (2.2) with the boundary conditions

$$
\begin{aligned}
u(0,t) = u(L,t) = 0, &\qquad t \geq 0 \\
\theta_x(0,t) = \theta_x(L,t) = 0, &\qquad t \geq -\tau_2.
\end{aligned} \tag{3.10}
$$

From the second equation of (2.2) and (3.10), we conclude that

$$\int_0^L \theta(x,t)dx = \int_0^L \theta_0(x)dx$$

which means that

$$\bar{\theta}(x,t) := \theta(x,t) - \frac{1}{L}\int_0^L \theta_0(x)dx$$

satisfies

$$\int_0^L \bar{\theta}(x,t)dx = 0. \tag{3.11}$$

In addition, (u, θ, z) and $(u, \bar{\theta}, z)$ satisfy the same differential equations and boundary conditions. Therefore, as in the previous subsection, we work with $\bar{\theta}$ but we denote it by θ for simplicity.

One can similarly show that the energy $E(t)$ of system (2.2) and (3.10) is decreasing and the inequality (3.2) holds as well as the inequalities (3.4) and (3.6) for the derivatives of the functionals F_1 and F_3 defined above. We only need to change the definition of F_2 to become

$$\overline{F}_2(t) := ab\int_0^L u_t \left(\int_0^x \theta(y,t)dy\right) dx.$$

Lemma 3.6. *The functional $\overline{F}_2$ satisfies, for some positive constant m_1 and for any $0 < \delta < 1$, the estimate*

$$\overline{F}_2'(t) \leq -m_1 \int_0^L u_t^2 dx + \delta \int_0^L u_x^2 dx + C_\delta \int_0^L \theta_x^2 dx + c\int_0^L \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)dsdx. \tag{3.12}$$

Proof. We exploit (2.2), (3.10), (3.11), integrate by parts, and use Young's inequality to get

$$
\begin{aligned}
\overline{F}_2'(t) &= b\int_0^L (du_{xx}-\beta\theta_x)\left(\int_0^x \theta(y,t)dy\right)dx \\
&\quad +a\int_0^L u_t\left(\int_0^x (k_1\theta_{xx}+\int_{\tau_1}^{\tau_2} k_2(s)z_x(y,1,s,t)ds-\beta u_{xt})dy\right)dx \\
&= -bd\int_0^L u_x\theta dx + b\beta\int_0^L \theta^2 dx + ak_1\int_0^L u_t\theta_x dx \\
&\quad +a\int_0^L u_t\left(\int_{\tau_1}^{\tau_2} k_2(s)z(x,1,s,t)ds\right)dx - a\beta\int_0^L u_t^2 dx \\
&\leq -a\beta\int_0^L u_t^2 dx + \varepsilon\int_0^L u_t^2 dx + \delta\int_0^L u_x^2 dx \\
&\quad +C_\varepsilon[\int_0^L \theta_x^2 dx + \int_0^L\int_{\tau_1}^{\tau_2} z^2(x,1,s,t)dsdx] + C_\delta\int_0^L \theta^2 dx + b\beta\int_0^L \theta^2 dx.
\end{aligned}
$$

Then, choosing ε small enough and using Poincaré's inequality, (3.12) is obtained. ∎

Making simple modifications in the proof of Theorem 3.5, we also obtain the following stability result.

Theorem 3.7. *Assume that $\int_{\tau_1}^{\tau_2}|k_2(s)|\,ds < k_1$. Then, there exist positive constants d_0, d_1 such that the energy functional for system (2.2) and (3.10) satisfies*

$$
E(t) \leq d_0 e^{-d_1 t}. \tag{3.13}
$$

3.3 Constant delay

Here, we consider the following system with constant time delay τ and constant coefficient k_2

$$
\begin{cases}
au_{tt}(x,t) - du_{xx}(x,t) + \beta\theta_x(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\
b\theta_t(x,t) - k_1\theta_{xx}(x,t) - k_2\theta_{xx}(x,t-\tau) + \beta u_{xt}(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x), & x\in(0,L) \\
\theta_x(x,-t) = f_0(x,t), \quad x\in(0,L), & t\in(0,\tau).
\end{cases} \tag{3.14}
$$

By introducing the variable

$$
z(x,\rho,t) = \theta_x(x,t-\tau\rho), \qquad (x,\rho,t)\in(0,L)\times(0,1)\times(0,\infty)
$$

we obtain the equivalent system

$$
\begin{cases}
au_{tt}(x,t) - du_{xx}(x,t) + \beta\theta_x(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\
b\theta_t(x,t) - k_1\theta_{xx}(x,t) - k_2 z_x(x,1,t) + \beta u_{xt}(x,t) = 0, & \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } (0,L)\times(0,1)\times(0,\infty) \\
z(x,0,t) = \theta_x(x,t), & \text{in } (0,L)\times(0,\infty) \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x), & x\in(0,L) \\
z(x,p,0) = f_0(x,\tau\rho), & (x,\rho)\in(0,L)\times(0,1)
\end{cases} \tag{3.15}
$$

which we associate, as before, with boundary conditions of type (3.1) or (3.10). Now, our assumption becomes

$$|k_2| < k_1. \tag{3.16}$$

In this case, we take ξ any positive constant satisfying

$$|k_2| < \xi < k_1 \tag{3.17}$$

and define the energy functional, for each type of boundary conditions, by

$$E(t) = \frac{1}{2}\int_0^L (au_t^2 + du_x^2 + b\theta^2)dx + \frac{\tau\xi}{2}\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx.$$

Using Young's inequality and (3.17), we easily get

$$\begin{aligned} E'(t) &= -k_1\int_0^L \theta_x^2 dx - k_2\int_0^L \theta_x z(x,1,t)dx - \xi\int_0^L\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx \\ &= -k_1\int_0^L \theta_x^2 dx - k_2\int_0^L \theta_x z(x,1,t)dx - \frac{\xi}{2}\int_0^L z^2(x,1,t)dx + \frac{\xi}{2}\int_0^L \theta_x^2 dx \\ &\le -(k_1 - \frac{|k_2|}{2} - \frac{\xi}{2})\int_0^L \theta_x^2 dx - (\frac{\xi}{2} - \frac{|k_2|}{2})\int_0^L z^2(x,1,t)dx \\ &\le -\gamma\{\int_0^L \theta_x^2 dx + \int_0^L z^2(x,1,t)dx\} \end{aligned} \tag{3.18}$$

for some positive constant γ.
Using (3.18), the same functionals F_1, F_2 (or $\overline{F}_2$) defined above, the functional

$$\overline{F}_3(t) := \tau\int_0^L\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx,$$

and following the same arguments, we similarly arrive at the following result.

Theorem 3.8. *Assume that $|k_2| < k_1$. Then, there exist positive constants k_0, k_1 such that the energy functional for system (3.15) and (3.1) (or (3.10)) satisfies*

$$E(t) \le k_0 e^{-k_1 t}.$$

Remark. It is evident that the condition (2.3) is sufficient to guarantee the uniform stability of the heat equation

$$u_t(x,t) - k_1 u_{xx}(x,t) - \int_{\tau_1}^{\tau_2} k_2(s)u_{xx}(x,t-s)ds = 0, \qquad \text{in } (0,L)\times(0,\infty)$$

with distributed delay and Diriclet or Neumann boundary conditions . Similarly, the exponential decay is obtained for the heat equation with constant delay provided that the condition (3.16) is satisfied.

Acknowledgment. The authors thank King Fahd University of Petroleum and Minerals (KFUPM) for its continuous support. This work has been funded by KFUPM under Project # IN 101002.

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Received: January 6, 2013

Accepted: July 8, 2013