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ON SOME SMARANDACHE DETERMINANT SEQUENCES

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Abstract In [1], Murthy introduced the concept of the Smarandache Cyclic Determinant Natural Sequence, the Smarandache Cyclic Arithmetic Determinant Sequence, the Smarandache Bisymmetric Determinant Natural Sequence, and the Smarandache Bisymmetric Arithmetic Determinant Sequence and in [2], Majumdar derived the n-th terms of these four sequences. In this paper, we present some of the results found by Majumdar in [2] but of different approach..

1 Introduction

An $n \times n$ matrix C of the form

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{bmatrix}$$

is called a circulant matrix with circulant vector $\vec{c} = (c_0, c_1, c_2, \ldots, c_{n-1})$. It can be seen easily that the given matrix is structured as follows:

(i) Each row is a right cyclic shift of the row above it. Thus, C is determined by the first row $(c_0, c_1, c_2, \ldots, c_{n-1})$.

(ii) $c_k = c_{ij}$, whenever $j - i \equiv k (\text{mod } n)$ for all $i, j = 1, 2, \ldots, n$ and $k = 0, 1, 2, \ldots, n-1$.

In his paper, Murthy [2] defined the Smarandache cyclic determinant natural sequence and the Smarandache cyclic arithmetic determinant sequence as follows:

Definition 1.1. The Smarandache cyclic determinant natural sequence, $\{SCDNS(n)\}$ is

$$\left\{ |1|, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}, \ldots \right\}.$$

Definition 1.2. The Smarandache cyclic arithmetic determinant sequence, $\{SCADS(n)\}$ is

$$\left\{ |a|, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a \\ a+2d & a & a+2d \end{vmatrix}, \ldots \right\}.$$

Majumdar derive the explicit expressions of the n-th terms of the two determinant sequences: Smarandache cyclic determinant natural sequence and Smarandache cyclic arithmetic determinant sequence. In this paper, we present the same results shown by Majumdar and provide a much simple proof of the formulas for the n-th terms of the two determinant sequences.

2 Preliminaries

In this section, we state some important results found by Bahsi and Solak in [3] as part of the proof of our main results presented in section 3. We also state some formulas involving eigenvalues.

Lemma 2.1. *The eigenvalues of the $n \times n$ matrix $C_{a,d}$ are*

$$\lambda_0 = na + \frac{n(n-1)d}{2},\ \lambda_m = \frac{nd}{e^{\frac{2\pi i m}{n}} - 1},$$

where $m = 1, 2, \ldots, n-1$.

Proof. *see* Theorem 2.1 in [3]. □

Here, the circulant matrix $C_{a,d} = [c_{ij}]$ where $c_{ij} \equiv a + ((j-i) \bmod n)d$, and a and d are real numbers as defined by Bahsi and Solak in [3].

Lemma 2.2. *For any natural number $n \geq 2$ we have,*

$$\prod_{m=1}^{n-1}\left(e^{\frac{2\pi i m}{n}} - 1\right) = (-1)^{n-1}n.$$

Proof. Let $\epsilon = e^{\frac{2\pi i}{n}}$ be the n-th root of unity and consider the polynomial $X^n - 1$. Then, it is clear that $1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1}$ are exactly n distinct roots of $X^n - 1$. Hence, we can express $X^n - 1$ as follows:

$$X^n - 1 = (X-1)(X-\epsilon)(X-\epsilon^2)\cdots(X-\epsilon^{n-1}) = \prod_{m=0}^{n-1}(X-\epsilon^m).$$

But, $X^n - 1 = (X-1)(X^{n-1} + X^{n-2} + \ldots + X^2 + X + 1)$. It follows that,

$$(X-1)\prod_{m=1}^{n-1}(X-\epsilon^m) = (X-1)(X^{n-1} + X^{n-2} + \ldots + X^2 + X + 1).$$

And so we have,

$$\begin{aligned}
\prod_{m=1}^{n-1}(\epsilon^m - X) &= \prod_{m=1}^{n-1}(-1)(X-\epsilon^m) \\
&= (-1)^{n-1}\prod_{m=1}^{n-1}(X-\epsilon^m) \\
&= (-1)^{n-1}(X^{n-1} + X^{n-2} + \ldots + X^2 + X + 1).
\end{aligned}$$

Letting $X = 1$, we will obtain $\prod_{m=1}^{n-1}(\epsilon^m - 1) = (-1)^{n-1}n$. □

Lemma 2.3. *For any natural numbers m and n we have*

$$\sum_{i=1}^{m}\left\lfloor\frac{n+(m-1)-i}{m}\right\rfloor = n-1 = \sum_{i=1}^{m}\left\lfloor\frac{i+n-2}{m}\right\rfloor.$$

Proof. *see* Lemma 2.5 in [3]. □

3 Main Results

We now provide a much simple proof of the following theorem.

Theorem 3.1. *The n-th term of the Smarandache cyclic determinant natural sequence, $SCDNS(n)$ is*

$$SCDNS(n) = \begin{vmatrix}
1 & 2 & 3 & \ldots & n-2 & n-1 & n \\
2 & 3 & 4 & \ldots & n-1 & n & 1 \\
3 & 4 & 5 & \ldots & n & 1 & 2 \\
\vdots & & & & & & \\
n-1 & n & 1 & \ldots & n-4 & n-3 & n-2 \\
n & 1 & 2 & \ldots & n-3 & n-2 & n-1
\end{vmatrix} = (-1)^{\lfloor\frac{n}{2}\rfloor}\frac{n+1}{2}n^{n-1}.$$

Proof. First we let $[d_{ij}]$ be an $n \times n$ matrix with entries $d_{ij} = k+1$, whenever $j - i = k(\mathrm{mod}\ n)$ for all $i, j = 1, 2, \ldots, n$ and $k = 0, 1, 2, \ldots, n-1$. Obviously, $|d_{ij}| = |C_{1,1}|$. Moreover, $|d_{ij}| = SCDNS(n)$ for $n < 3$. Now, for $n \geq 3$ we fixed the first row of $SCDNS(n)$ and apply the row operation $R_i \leftrightarrow R_{n+2-i}$ for $2 \leq i \leq n$. If n is even, $n = 2l$ for some $l = 0, 1, 2, \ldots$. Hence, we apply the row operation $R_i \leftrightarrow R_{2(l+1)-i}$ for $1 \leq i-1 \leq 2l-1$. On the other hand, if n is odd, we have $n = 2l+1$ for some $l = 0, 1, 2, \ldots$. So, we apply the row operation $R_i \leftrightarrow R_{2(l+1)+1-i}$ for $1 \leq i-1 \leq 2l$. It follows that $(-1)^{\lfloor \frac{n-1}{2} \rfloor} SCDNS(n) = |d_{ij}|$. Then, from Lemma 2.1 and 2.2 we have,

$$\begin{aligned}
(-1)^{\lfloor \frac{n-1}{2} \rfloor} SCDNS(n) &= |C_{1,1}| \\
&= \prod_{m=0}^{n-1} \lambda_m \\
&= \left(n + \frac{n(n-1)}{2}\right) \prod_{m=1}^{n-1} \left(\frac{n}{e^{\frac{2\pi i m}{n}} - 1}\right) \\
&= \left(\frac{n(n+1)}{2}\right) \frac{(-1)^{n-1} n^{n-1}}{n}.
\end{aligned}$$

Using Lemma (2.3) we'll obtain,

$$\begin{aligned}
SCDNS(n) &= (-1)^{n-1-\lfloor \frac{n-1}{2} \rfloor} \left(\frac{n+1}{2}\right) n^{n-1} \\
&= (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{n+1}{2} n^{n-1}.
\end{aligned}$$

This completes the proof of the theorem. □

We now proceed to a more general form of the Smarandache cyclic determinant natural sequence.

Theorem 3.2. *The n-th term of the Smarandache cyclic arithmetic determinant sequence, is*

$$SCADS(n) = \begin{vmatrix}
a & a+d & \ldots & a+(n-2)d & a+(n-1)d \\
a+d & a+2d & \ldots & a+(n-1)d & a \\
a+2d & a+3d & \ldots & a & a+d \\
\vdots & & & & \\
a+(n-2)d & a+(n-1)d & \ldots & a+(n-4)d & a+(n-3)d \\
a+(n-1)d & a & \ldots & a+(n-3)d & a+(n-2)d
\end{vmatrix}$$

$$= (-1)^{\lfloor \frac{n}{2} \rfloor} \left(a + \frac{n-1}{2} d\right) (nd)^{n-1}.$$

Proof. The proof is similar to the previous theorem. Consider the circulant matrix $C_{a,d} = [c_{ij}]$ with entries $c_{ij} = a + ((j-i) \mod n)d$ for all $i, j = 1, 2, \ldots, n$ and $k = 0, 1, 2, \ldots, n-1$. It can be seen easily that $C_{a,d} = SCADS(n)$ for $n < 3$. Now, for $n \geq 3$ we fixed the first row of $SCADS(n)$ and apply the row operation $R_i \leftrightarrow R_{n+2-i}$ for $2 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$ obtaining $(-1)^{\lfloor \frac{n-1}{2} \rfloor} SCADS(n) = |c_{ij}|$.

Then, from Lemma (2.1) and (2.2) we have,

$$\begin{aligned}
(-1)^{\lfloor \frac{n-1}{2} \rfloor} SCADS(n) &= |C_{a,d}| \\
&= \prod_{m=0}^{n-1} \lambda_m \\
&= \left(na + \frac{n(n-1)d}{2}\right) \prod_{m=1}^{n-1} \left(\frac{nd}{e^{\frac{2\pi i m}{n}} - 1}\right) \\
&= \left(na + \frac{n(n-1)d}{2}\right) \frac{(-1)^{n-1} (nd)^{n-1}}{n}.
\end{aligned}$$

Using Lemma (2.3) we'll obtain,

$$\begin{aligned} SCADS(n) &= (-1)^{n-1-\lfloor \frac{n-1}{2} \rfloor} \left(a + \frac{(n-1)d}{2}\right)(nd)^{n-1} \\ &= (-1)^{\lfloor \frac{n}{2} \rfloor} \left(a + \frac{(n-1)d}{2}\right)(nd)^{n-1}. \end{aligned}$$

This completes the proof of the theorem. □

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