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On Generalized Jordan Higher Homomorphisms

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Dedicated to Patrick Smith and John Clark on their $70^{th.}$ birthday.

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Abstract. In this paper we extend to the higher homomorphism a well known result proved by Herstein concerning homomorphism in prime rings. We prove results which imply that every Jordan triple higher homomorphism (*resp. generalized Jordan triple higher homomorphism*) of a ring R onto a 2-torsion-free prime rings R' is of the form $\Phi_i = \pm\Psi_i$ for all $i \in \mathbb{N}$, where $\Psi_i \in \Psi = (\Psi_i)_{i\in N}$ which Ψ is either higher homomorphism or higher anti-homomorphism (*resp.* Ψ *is either generalized higher homomorphism or generalized higher anti-homomorphism*) of R onto R'.

1 Introduction

The study of additive mappings from one ring R into another ring R' which preserve squares was initiated by Ancochea[1], [2] in connection with problem arising in projective geometry. Hua [9] and Kaplansky [12] took the subject up where Ancochea had left off and pushed his results further. Jacobson and Rickart [10] then proceeded to carry out an extensive study of such functions.

An additive mapping Θ of a ring R into a 2-torsion-free ring R' is called a *Jordan homomorphism* if $\Theta(ab + ba) = \Theta(a)\Theta(b) + \Theta(b)\Theta(a)$ for all $a, \in R$. A well known result due to Herstein [6] states that every Jordan homomorphism of a ring R onto a prime ring R' of characteristic different from 2 and 3 is either a homomorphism or an anti-homomorphism. Later Smilley [14] provided a brief proof of this result and also same time removed the requirement that the characteristic be different from 3. Suppose R' contains ideals U' and V' with null intersection. Let Φ: $R \to U'$ be a homomorphism and Ψ: $R \to V'$ be an anti-homomorphism. A mapping $\Theta = \Phi + \Psi$ is a so-called *direct sum* of mappings Φ and Ψ. Obviously, Θ is a Jordan homomorphism. According to this construction we see that Herstein's result [6] does not hold in semiprime rings. Moreover, Baxter and Martindale [3] showed by an example that a Jordan homomorphism Θ of a ring R onto a semiprime ring R' is not necessarily a direct sum of a homomorphism and an anti-homomorphism. Although they proved that there always exists an essential ideal E of R such that the restriction of Θ to E is a direct sum of a homomorphism Φ: $E \to R'$ and an anti-homomorphism Ψ: $E \to R'$ [3, Theorem 2.7]. In 1989, Brešar [4, Theorem 2.3] extended this result by showing that E can be chosen so that it is a sum of ideals U and V of R such that Φ vanishes on V and Ψ vanishes on U. Even more, for each $x \in R$ we have $\Theta(ux) = \Theta(u)\Theta(x)$ for all $u \in U$ and $\Theta(vx) = \Theta(v)\Theta(x)$ for all $v \in V$.

An additive mapping Θ of ring R into a ring R' which satisfies $\Theta(aba) = \Theta(a)\Theta(b)\Theta(a)$ for all $a, b \in R$ will be called a *Jordan homomorphism.* An easy computation shows that every Jordan homomorphism is also a Jordan triple homomorphism (see for example [8, Lemma 3.1]). In [7] Herstein proved that a Jordan triple homomorphism Θ of a ring R into a prime ring R' of characteristic diferent from 2 and 3 is of the form $\Theta = \pm\Phi$, where Φ is a homomorphism or an anti-homomorphism of R onto R'. In his paper Brešar [4, Theorem 3.3] generalized this result by removing the requirement that the characteristic be different from 3. In the same paper, Brešar [4] obtained a more general result where Θ is a Jordan triple homomorphism of a ring R onto a 2-torsion-free semiprme ring R'.

An additive mapping F: $R \to R'$ is said to be a *generalized Jordan homomorphism* if there exists a Jordan homomorphism Θ: $R \to R'$ such that $F(ab + ba) = F(a)\Theta(b) + F(b)\Theta(a)$ for all $a, b \in R$ and Θ is called the *relating Jordan homomorphism*. An additive mapping F: $R \to R'$ is said to be a *generalized Jordan triple homomorphism* if there exists a Jordan triple homomorphism Θ: $R \to R'$ such that $F(aba) = F(a)\Theta(b)\Theta(a)$, for all $a, b \in R$ and Θ is called the*relating Jordan triple homomorphism*.

In the present paper our objective is to study the concept of higher homomorphism, Jordan higher homomorphism, Jordan triple higher homomorphism and generalized Jordan triple higher homomorphism.

The rings R and R' used in the whole paper will be all associative ones with center $Z(R)$, and for each $x, y \in R$, the symbol $[x, y]$ will represent the usual commutator $xy - yx$.

2 Preliminary results

Our goal is to study the case of higher homomorphisms. Let $\mathbb{N}$ be the set of all natural numbers and let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a family of additive mappings of a ring R into a ring R'. Then Θ is called:

(a). a *Higher Homomorphism* (HH, for short) if for each $n \in \mathbb{N}$, $\Phi_n(ab) = \sum_{i=1}^{n} \Phi_i(a)\Phi_i(b)$, for all $a, b \in R$;

(b). a *Jordan Higher Homomorphism* (JHH, for short) if for each $n \in \mathbb{N}$, $\Phi_n(ab + ba) = \sum_{i=1}^{n} \Phi_i(a)\Phi_i(b) + \Phi_i(b)\Phi_i(a)$, for all $a, b \in R$. When R' is 2-torsion-free we define a JHH by merely insisting that $\Phi_n(a^2) = \sum_{i=1}^{n} (\Phi_i(a))^2$, for all $a \in R$, $n \in \mathbb{N}$;

(c). a *Jordan Triple Higher Homomorphism* (JTHH, for short) if for each $n \in \mathbb{N}$, $\Phi_n(aba) = \sum_{i=1}^{n} \Phi_i(a)\Phi_i(b)\phi_i(a)$, for all $a, b \in R$;

(d). a *Higher Anti-Homomorphism* (HAH, for short) if for each $n \in \mathbb{N}$, $\Phi_n(ab) = \sum_{i=1}^{n} \Phi_i(b)\Phi_i(a)$, for all $a, b \in R$.

Remark 2.1. Every Higher Homomorphism is a Jordan Higher Homomorphism, but the converse need not be true in general, following example shows:

Example 2.1. Let S by any ring with involution $*$, set $R = S \oplus S$ and $a \in S$ such that $a \in Z(S)$ and $s_1 a s_2 = 0$ for all $s_1, s_2 \in S$. Let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a family of mappings of R into itself defined by

$$\Phi_n(s,t) = \begin{cases} ((2 - n_a s, (n-1)t^*), & n = 1, 2, \quad \text{for all } (s,t) \in R; \\ 0, & n \geq 3. \end{cases}$$

Then it is easy to see that Θ is JHH. Since

$$\begin{aligned} (0, (t_1, t_2)^* &= \Phi_2(s_1 s_2, t_1 t_2) = \Phi_2((s_1, t_1), (s_2, t_2)) \neq \\ &\neq \Phi_1(s_1, t_1)\Phi_1(s_2, t_2) + \Phi_2(s_1, t_1)\Phi_2(s_2, t_2) = \\ &= 0 + (0, t_1^*)(0, t_2^*) = (0, t_1^* t_2^*), \end{aligned}$$

therefore Θ is not HH.

To facilitate our discussion, we shall begin with the following known results:

Lemma 2.1. [4, Lemma 1.1] Let R be a semiprime ring. If $a, b \in R$ are such that $axb + bxa = 0$, for all $x \in R$, then $axb = bxa = 0$, for all $x \in R$. If R is semiprime, then $axb = 0$, for all $x \in R$, implies $bxa = ab = ba = 0$, too.

Lemma 2.2. [4, Lemma 1.2] Let $G_1, G_2, \cdots, G_n$ be additive groups, R a 2-torsion-free semiprime ring. Suppose that mappings S: $G_1 \times G_2 \times \cdots \times G_n \longrightarrow R$ and T: $G_1 \times G_2 \times \cdots \times G_n \longrightarrow R$ are addtitive in each argument. If $S(a_1, a_2, \cdots, a_n)xT(a_1, a_2, \cdots, a_n) = 0$, for all $x \in R$, $a_i \in G_i$, $i = 1, 2, \cdots n$, then $S(a_1, a_2, \cdots, a_n)xT(b_1, b_2, \cdots, b_n) = 0$, for all $x \in R$, $a_i, b_i \in G_i$ $i = 1, 2, \cdots n$.

Lemma 2.3. [4, Theorem 2.3] Let Θ be a Jordan homomorphism of a ring R onto a 2-torsion-free semiprime ring R'. Then there exist ideals U and V of R such that, for all $x \in R$, $\Theta(ux) =$

$\Theta(u)\Theta(x)$, for all $u \in U$ and $\Theta(vx) = \Theta(v)\Theta(x)$, for all $v \in V$. The ideal $U + V$ is an essential ideal of R, $U \cap V = \ker(\Theta)$, $\Theta(U)$ and $\Theta(V)$ are ideals of R', $\Theta(U) \cap \Theta(V) = 0$, and $\Theta(U) \oplus \Theta(V)$ is an essential ideal of R'. If $U = \ker(\Theta)$, then Θ is an anti-homomorphism, and $V = \ker(\Theta)$ if and only if Θ is a homomorphism.

Lemma 2.4. [4, Lemma 3.1] Let Θ be a Jordan triple homomorphism of a ring R onto a ring R'. Then $S(a,b,c)\Theta(x)T(a,b,c) + T(a,b,c)\Theta(x)S(a,b,c) = 0$, for all $a,b,c,x \in R$, where $S(a,b,c) = \Theta(abc) - \Theta(a)\Theta(b)\Theta(c)$ and $T(a,b,c) = \Theta(abc) - \Theta(c)\Theta(b)\Theta(a)$.

Lemma 2.5. [4, Theorem 3.3] Let Θ be a Jordan triple homomorphism of a ring R onto a prime ring R' of characteristic different from 2. Then $\Theta = \pm\Phi$, where Φ is a homomorphism or anti-homomorphism of R onto R'.

Lemma 2.6. [13] Let F be a generalized Jordan triple homomorphism of a ring R onto ring R' and let Θ: $R \to R'$ be the relating Jordan triple homomorphism. Then $S^*(a,b,c)\Theta(x)T(a,b,c)+ T^*(a,b,c)\Theta(x)S(a,b,c) = 0$, for all $a,b,c,x \in R$, where $S^*(a,b,c) = F(abc) - F(a)\Theta(b)\Theta(c)$ and $T^*(a,b,c) = F(abc) - F(c)\Theta(b)\Theta(a)$.

Lemma 2.7. [13] Let F be a generalized Jordan triple homomorphism of a ring R onto a 2-torsion-free prime ring R'. Then $F = \pm\Psi$, where Ψ is either a generalized homomorphism or a generalized anti-homomorphism of R onto R'.

We are going to continue our discussion with the following results which are essential for developing the proof of our main theorem.

Lemma 2.8. Let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a Jordan Higher Homomorphism of R into R' which is a 2-torsion-free ring. Then Θ is a Jordan Triple Higher Homomorphism.

Proof. We have

$$\Phi_n(ab+ba) = \sum_{i=1}^{n} \Phi_i(a)\Phi_i(b) + \Phi_i(b)\Phi_i(a), \text{ for all } a,b \in R. \tag{2.1}$$

Replacing b by $ab+ba$ in (2.1) and using (2.1), we get

$$\begin{aligned}\Phi_n(a(ab+ba)+(ab+ba)a) &= \sum_{i=1}^{n} \Phi_i(a)\left(\sum_{l=1}^{i} \Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a)\right) + \\ &\quad + \sum_{i=1}^{n}\left(\sum_{l}^{i} \Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a)\right)\Phi_i(a) = \\ &= \sum_{i=1}^{n}\sum_{l=1}^{i} \Phi_i(a)\Phi_l(a)\Phi_l(b) + \sum_{i=1}^{n}\sum_{l=1}^{i} \Phi_i(a)\Phi_l(b)\Phi_l(a) + \\ &\quad + \sum_{i=1}^{n}\sum_{l=1}^{i} \Phi_l(a)\Phi_l(b)\Phi_i(a) + \sum_{i=1}^{n}\sum_{l=1}^{i} \Phi_l(b)\Phi_l(a)\Phi_i(a),\end{aligned}$$

and hence

$$\Phi_n(a(ab+ba)+(ab+ba)a) = \sum_{i=1}^{n}(\Phi_i(a))^2\Phi_i(b) + 2\sum_{i=1}^{n} \Phi_i(a)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^{n} \Phi_i(b)(\Phi_i(a))^2.$$

On the other hand,

$$\Phi_n(a(ab+ba)+(ab+ba)a) = \Phi_n(a^2b+ba^2) + 2\Phi_n(aba) = \sum_{i=1}^{n}\left(\Phi_i(a^2)\Phi_i(b) + \Phi_i(b)\Phi_i(a^2)\right) + 2\Phi_n(aba).$$

On comparing the last two equations, we get the required result. ∎

If we linearize $\Phi_n(aba) = \sum_{i=1}^{n} \Phi_i(a)\Phi_i(b)\Phi_i(a)$ on a, we get the following

Corollary 2.1. Let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a Jordan Triple Higher Homomorphism of a ring R into ring R' which is a 2-torsion-free. Then $\phi_n(abc+cba) = \sum_{i=1}^{n} (\Phi_i(a)\Phi_i(b)\Phi_i(c) + \Phi_i(c)\Phi_i(b)\Phi_i(a))$, for all $a,b,c \in R$, $n \in \mathbb{N}$.

Let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a Jordan Triple Higher Homomorphism (JTHH) of a ring R into ring R' for every fixed $n \in \mathbb{N}$ and for each $a,b,c \in R$. For the purpose of this section, we shall write:

(i). $(a^b)^n = \Phi_n(ab) - \sum_{i=1}^{n} \Phi_i(a)\Phi_i(b)$ and $(a_b)^n = \Phi_n(ab) - \sum_{i=1}^{n} \Phi_i(b)\Phi_i(a)$;

(ii). $S_n(a,b,c) = \Phi_n(abc) - \sum_{i=1}^{n} \Phi_i(a)\Phi_i(b)\Phi_i(c)$ and $T_n(a,b,c) = \Phi_n(abc) - \sum_{i=1}^{n} \Phi_i(c)\Phi_i(b)\Phi_i(a)$.

Lemma 2.9. Let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a Jordan Triple Higher Homomorphism of R into a ring R'. Then $(a^b)^n\Phi(x)(a_b)^n + (a_b)^n\Phi(x)(a^b)^n$, for all $x, a, b \in R$, $n \in \mathbb{N}$.

Proof. Since $(a^b)\Phi(x)(a_b) + (a_b)\Phi(x)(a^b)$, for all $x, a, b \in R$, by induction we can assume that $(a^b)^m\Phi_m(x)(a_b)^m + (a_b)^m\Phi_m(x)(a^b)^m$, for all $x, a, b \in R$, $m, n \in \mathbb{N}$, $m < n$.

Let $W = abxba + baxab$. Then

$$\begin{aligned}
\Phi_n(W) &= \Phi_n(a(bxb)a + b(axa)b) = \sum_{i=1}^{n}\Phi_i(a)\Phi_i(bxb)\Phi_i(a) + \sum_{i=1}^{n}\Phi_i(b)\Phi_i(axa)\Phi_i(b) = \\
&= \sum_{i=1}^{n}\Phi_i(a)\left(\sum_{l=1}^{i}\Phi_l(b)\Phi_l(x)\Phi_l(b)\right)\Phi_i(a) + \sum_{i=1}^{n}\Phi_i(b)\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(x)\Phi_l(a)\right)\Phi_i(b) = \\
&= \sum_{i=1}^{n}\Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^{n}\Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(a)\Phi_i(b) = \\
&= \left(\sum_{i=1}^{n}\Phi_i(a)\Phi_i(b)\Phi_i(x)\right)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) + \left(\sum_{i=1}^{n}\Phi_i(b)\Phi_i(a)\Phi_i(x)\right)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) = \\
&= \Phi_n(a)\Phi_n(b)\Phi_n(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) + \left(\sum_{i=1}^{n-1}\Phi_i(a)\Phi_i(b)\Phi_i(x)\right)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) + \\
&\quad + \Phi_n(b)\Phi_n(a)\Phi_n(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) + \left(\sum_{i=1}^{n-1}\Phi_i(b)\Phi_i(a)\Phi_i(x)\right)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b).
\end{aligned}$$

On the other hand, $\Phi_n(W) = \Phi_n((ab)x(ba) + (ba)x(ab))$. Thus, by Corollary 2.1 and the definition of HH, we get

$$\begin{aligned}
\Phi_n(W) &= \sum_{i=1}^{n}\Phi_i(ab)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a) - \Phi_i(ab)\right) + \\
&\quad + \sum_{i=1}^{n}\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a) - \Phi_i(ab)\right)\Phi_i(x)\Phi_i(ab) = \\
&= \sum_{i=1}^{n}\Phi_i(ab)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) + \sum_{i=1}^{n}\Phi_i(ab)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) - \sum_{i=1}^{n}\Phi_i(ab)\Phi_i(x)\Phi_i(ab) + \\
&\quad + \sum_{i=1}^{n}\Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \sum_{i=1}^{n}\Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab) - \sum_{i=1}^{n}\Phi_i(ab))\Phi_i(x)\Phi_i(ab) = \\
&= -\sum_{i=1}^{n}\Phi_i(ab)\Phi_i(x)\left(\Phi_i(ab) - \sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right) - \sum_{i=1}^{n}\Phi_i(ab)\Phi_i(x)\left(\Phi_i(ab) - \sum_{l=1}^{i}\Phi_l(b)\Phi_l(a)\right) + \\
&\quad + \sum_{i=1}^{n}\Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \sum_{i=1}^{n}\Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab) = \\
&= -\sum_{i=1}^{n}\Phi_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^{n}\Phi_i(ab)\Phi_i(x)(a_b)^i + \\
&\quad + \sum_{i=1}^{n}\Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \sum_{i=1}^{n}\Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab) = \\
&= -\Phi_n(ab)\Phi_n(x)(a^b)^n - \sum_{i=1}^{n-1}\Phi_i(ab)\Phi_i(x)(a^b)^i - \Phi_n(ab)\Phi_n(x)(a_b)^n - \sum_{i=1}^{n}\Phi_i(ab)\Phi_i(x)(a_b)^i + \\
&\quad + \Phi_n(a)\Phi_n(b)\Phi_n(x)\Phi_n(ab) + \sum_{i=1}^{n-1}\Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \Phi_n(b)\Phi_n(a)\Phi_n(x)\Phi_n(ab) + \\
&\quad + \sum_{i=1}^{n-1}\Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab).
\end{aligned}$$

On comparing the right hand side of $\Phi_n(W)$, we find that

$$\begin{aligned}
0 &= -\Phi_n(ab)\Phi_n(x)\Phi_n(a^b)^n - \Phi_n(ab)\Phi_n(x)(a_b)^n + \Phi_n(a)\Phi_n(b)\Phi_n(x)\left(\Phi_n(ab) - \sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)\right) + \\
&\quad + \Phi_n(b)\Phi_n(a)\Phi_n(x)\left(\Phi_n(ab) - \sum_{l=1}^{n}\Phi_l(a)\Phi_l(b)\right) - \sum_{i=1}^{n-1}\Phi_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^{n-1}\Phi_i(ab)\Phi_i(x)(a_b)^i + \\
&\quad + \sum_{i=1}^{n-1}\Phi_i(a)\Phi_i(b)\Phi_i(x)\left(\Phi_i(ab) - \sum_{l=1}^{i}\Phi_l(b)\Phi_l(a)\right) + \sum_{i=1}^{n-1}\Phi_i(b)\Phi_i(a)\Phi_i(x)\left(\Phi_i(ab) - \sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right) = \\
&= -\Phi_n(ab)\Phi_n(x)(a^b)^n - \Phi_n(ab)\Phi_n(x)(a_b)^n + \Phi_n(a)\Phi_n(b)(\Phi_n(x)(a_b)^n + \\
&\quad + \Phi_n(b)\Phi_n(a)\Phi_n(x)(a^b)^n - \sum_{i=1}^{n-1}\Phi_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^{n-1}\Phi_i(ab)\Phi_i(x)(a_b)^i + \\
&\quad + \sum_{i=1}^{n-1}\Phi_i(a)\Phi_i(b)\Phi_i(x)(a_b)^i + \sum_{i=1}^{n-1}\Phi_i(b)\Phi_i(a)\Phi_i(x)(a^b)^i = \\
&= -(a_b)^n\Phi_n(x)(a^b)^n - (a^b)^n\Phi_P n(x)(a_b)^n - \sum_{i=1}^{n-1}(a_b)^i\Phi_i(x)(a^b)^i - \sum_{i=1}^{n-1}(a^b)^i\Phi_i(x)(a_b)^i = \\
&= \left((a_b)^n\Phi_n(x)(a^b)^n + (a^b)^n\Phi_n(x)(a_b)^n\right) - \sum_{i=1}^{n-1}\left((a^b)^i\Phi_i(x)(a_b)^i + (a_b)^i\Phi_i(x)(a^b)^i\right).
\end{aligned}$$

By our hypothesis, we have $(a^b)^n\Phi_n(x)(a_b^n + (a_b)^n\Phi_n(x)(a^b)^n = 0$, for all $a, b, x \in R$, $n \in \mathbb{N}$. ■

As an immediate consequence of Lemma 2.1, Lemma 2.2 and Lemma 2.9, we get the following

Corollary 2.2. Let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a Jordan Higher Homomorphism of a ring R onto a 2-torsion-free semiprime ring R'. Then $(a^b)^n x'(c_d)^n = 0$ for all $a, b, c, d \in R$, $x' \in R'$ and $n \in \mathbb{N}$.

Thus, immediately from Corollary 2.2 we get the following

Remark 2.2. Let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a Jordan Higher Homomorphism of a ring R onto a 2-torsion-free prime ring R'. Then either Θ is a Higher Homomorphism or Θ is a Higher Anti-Homomorphism.

Proposition 2.1. Let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a Jordan Higher Homomorphism of a ring R onto a 2-torsion-free semiprime ring R' such that $[\Phi_i(a), \Phi_i(b)] = 0$, for all $a, b \in R$, $n \in\mathbb{N}$, $i < n$. Then there exist ideals U_n and V_n of R, for all $n \in\mathbb{N}$, such that:

(i). $\Phi_n(ux) = \sum\limits_{i=1}^{n} \Phi_i(u)\Phi_i(x)$, for all $u \in U_n$, $n \in\mathbb{N}$, $x \in R$ and

(ii). $\Phi_n(vx) = \sum\limits_{i=1}^{n} \Phi_i(x)\Phi_i(v)$, for all $v \in V_n$, $n \in\mathbb{N}$, $x \in R$.

Proof. The proposition is true for $n = 1$, by Lemma 2.3. Thus, we can assume that there exist ideals $U_m = \bigcap\limits_{i=1}^{m} \Phi_i^{-1}(ann(V'_{i-1}))$ and $V_m = \bigcap\limits_{i=1}^{m} \Phi_i^{-1}(ann(ann(V'_{i-1})))$ for all $m, n \in\mathbb{N}$, $m < n$, where V'_{i-1} is the ideal generated by the set $\{(a^b)^j \mid j = 1, \cdots, i < n,\ a, b \in R\}$ such that, for all $x \in R$, $\Phi_n(ux) = \sum\limits_{i=1}^{m} \Phi_i(u)\Phi_i(x)$, for all $u \in U_m$ and $\Phi_m(vx) = \sum\limits_{i=1}^{m} \Phi_i(v)\Phi_i(x)$, for all $v \in V_m$.

Let V'_{n-1} be the ideal generated by the set $\{(a^b)^j \mid i = 1, \cdots, n,\ a, b \in R\}$. Thus, by Lemma 2.9 and since Φ_i is onto, for all $i \in\mathbb{N}$ and hence by Lemma 2.1, we have $(a^b)^n(a_b)^n = (a_b)^n(a^b)^n = 0$, for all $a, b \in R$, $n \in\mathbb{N}$. We claim that $(a_b)^i(a^b)^j = 0$, for all $a, b \in R$, $i, j \in\mathbb{N}$ and $i \neq j$ to show that:

Case I: if $i < j$, then

$$\begin{aligned}(a_b)^i(a_b)^j &= \left(\Phi_i(ab) - \sum_{l=1}^{i} \Phi_l(b)\Phi_l(a)\right)\left(\Phi_j(ab) - \sum_{t=1}^{j} \Phi_t(a)\Phi_t(b)\right) = \\ &= -\Phi_i(ab)\sum_{t=1}^{j} \Phi_t(a)\Phi_t(b) + \sum_{l=1}^{i} \Phi_l(b)\Phi_l(a)\sum_{t=1}^{j} \Phi_t(a)\Phi_t(b)(a_b)^i(a_b)^j = \\ &= -(a_n)^i \sum_{t=1}^{j} \Phi_t(a)\Phi_t(b).\end{aligned} \tag{2.2}$$

On the other hand, since Θ is JHH, we find that

$$\begin{aligned}(a_b)^i(a_b)^j &= \left(\Phi_i(ab) - \sum_{l=1}^{i} \Phi_l(b)\Phi_l(a)\right)\left(-\Phi_j(ba) + \sum_{t=1}^{j} \Phi_t(b)\Phi_t(a)\right) = \\ &= \Phi_i(ab)\sum_{t=1}^{j} \Phi_t(b)\Phi_t(a) - \sum_{l=1}^{i} \Phi_l(b)\Phi_l(a)\sum_{t=1}^{j} \Phi_t(b)\Phi_t(a)(a_b)^i(a_b)^j = \\ &= -(a_n)^i \sum_{t=1}^{j} \Phi_t(b)\Phi_t(a).\end{aligned} \tag{2.3}$$

Now adding (2.2) and (2.3), we get

$$(a_b)^i(a_b)^j = (a_b)^i\left(-\sum_{t=1}^{j} \Phi_t(a)\Phi_t(b) + \sum_{t=1}^{j} \Phi_t(b)\Phi_t(a)\right) = (a_b)^i \sum_{t=1}^{j}[\Phi_t(b), \Phi_t(a)] = 0.$$

Since R' is 2-torsion-free, we obtain $(a_b)^i(a^b)^j = 0$ for all $a, b \in R$ and $i, j \in\mathbb{N}$, $i < j$.

Using the same arguments as above one can show that:

Case II: If $i > j$, then $(a_b)^i(a^b)^j = 0$, for all $a, b \in R$ and $i, j \in\mathbb{N}$, $i > j$. Therefore, we conclude that $\{(a_b)^i \mid i = 1, \cdots, n,\ a, b \in R\} \subseteq ann(V'_{n-1})$ and $\{(a^b)^i \mid i = 1, \cdots, n,\ a, b \in R\} \subseteq ann(ann(V'_{n-1}))$.

Now we set

$$\begin{aligned}U_n &= \bigcap_{i=1}^{n} \Phi_i^{-1}(ann(V'_{n-1})) = \Phi_n^{-1}(ann(V'_{n-1})) \bigcap \bigcap_{i=1}^{n-1} \Phi_i^{-1}(ann(V'_{i-1})) \text{ and} \\ V_n &= \bigcap_{i=1}^{n} \Phi_i^{-1}(ann(ann(V'_{n-1}))) = \Phi_n^{-1}(ann(ann(V'_{n-1}))) \bigcap \bigcap_{i=1}^{n-1} \Phi_i^{-1}(ann(ann(V'_{i-1}))).\end{aligned}$$

Given $u \in U_n$, for all $y \in R$, $x \in R'$, we have

$$(u^y)^n x'(u^y)^n = (u^y)^n x'\left((u^y)^n - (u_y)^n\right) = (u^y)^n x'\left(\sum_{i=1}^{n} \Phi_i(y)\Phi_i(u) - \sum_{i=1}^{n} \Phi_i(u)\Phi_i(y)\right) = 0.$$

Hence, by semiprimness of R' forces that $(u^y)^n = 0$. That is,

$$\Phi_n(uy) = \sum_{i=1}^{n} \Phi_i(u)\Phi_i(y), \text{ for all } u \in U_n,\ y \in R.$$

Now to prove that U_n is an ideal of R, it is enough to show that $\Phi_n^{-1}(ann(V'_{n-1}))$ is an ideal of R.

Let $u \in U_n$, $x \in R$. Since $\Phi_n(ux) = \sum_{i=1}^{n} \Phi_i(u)\Phi_i(x)$ and $\Phi_n(u) \in ann(V'_{n-1})$, it remains to show that $\Phi_m(u) \in ann(V'_{n-1})$, for all $u \in U_n$, $m < n$ and hence

$$\Phi_m(u)(a^b)^n = \Phi_m(u)\left(-\Phi_m(a)\Phi_m(b)\right). \tag{2.4}$$

On the other hand,

$$\Phi_m(u)(a^b)^n = \Phi_m(u)\left(\Phi_m(b)\Phi_m(a)\right). \tag{2.5}$$

Thus, by adding (2.4) and (2.5) and our hypothesis we obtain $\Phi_m(u)(a^b)^n = 0$. Similarly, we can prove that $\Phi_n(vy) = \sum_{i=1}^{n} \Phi_i(y)\Phi_i(v)$, for all $v \in V_n$, $y \in R$ and therefore V_n is an ideal of R. ∎

Lemma 2.10. Let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a Jordan Triple Higher Homomorphism of a ring R into a ring R'. Then $S_n(a,b,c)\Phi_n(x)T_n(a,b,c)+T_n(a,b,c)\Phi_n(x)S_n(a,b,c) = 0$, for all $a,b,c,x \in R$, $n \in \mathbb{N}$.

Proof. By Lemma 2.4, the statement is true for $n = 1$. Then, by induction, we assume that

$$S_m(a,b,c)\Phi_m(x)T_m(a,b,c) + T_m(a,b,c)\Phi_m(x)S_m(a,b,c) = 0,$$

for all $a,b,c,x \in R$, $m \in \mathbb{N}$ and $m < n$.

Let $W = \Phi_n(abcxcba + cbaxabc)$. Then, by the definition of JTHH, we have

$$\begin{aligned}
W &= \Phi_n(a(bcxcb)a + c(baxab)c) = \sum_{i=1}^{n} \Phi_i(a)\Phi_i(bcxcb)\Phi_i(a) + \sum_{i=1}^{n} \Phi_i(c)\Phi_i(baxab)\Phi_i(c) = \\
&= \sum_{i=1}^{n} \Phi_i(a)\left(\sum_{l=1}^{i} \Phi_l(b)\Phi_i(cxc)\Phi_l(b)\right)\Phi_i(a) + \sum_{i=1}^{n} \Phi_i(c)\left(\sum_{l=1}^{i} \Phi_l(b)\Phi_i(axa)\Phi_l(b)\right)\Phi_i(c) = \\
&= \sum_{i=1}^{n} \Phi_i(a)\Phi_i(b)\Phi_i(cxc)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^{n} \Phi_i(c)\Phi_i(b)\Phi_i(axa)\Phi_i(b)\Phi_i(c) = \\
&= \sum_{i=1}^{n} \Phi_i(a)\Phi_i(b)\Phi_i(c)\Phi_i(x)\Phi_i(c)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^{n} \Phi_i(c)\Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(a)\Phi_i(b)\Phi_i(c) = \\
&= \sum_{i=1}^{n}\left(\sum_{l=1}^{i} \Phi_l(a)\Phi_l(b)\Phi_l(c)\right)\Phi_i(x)\left(\sum_{l=1}^{i} \Phi_l(c)\Phi_l(b)\Phi_l(a)\right) + \\
&\quad + \sum_{i=1}^{n}\left(\sum_{l=1}^{i} \Phi_l(c)\Phi_l(b)\Phi_l(a)\right)\Phi_i(x)\left(\sum_{l=1}^{i} \Phi_l(a)\Phi_l(b)\Phi_l(c)\right).
\end{aligned}$$

On the other hand, by Corollary 2.1, we have

$$\begin{aligned}
W &= \Phi_n((abc)x(cba) + (cba)x(abc)) = \sum_{i=1}^{n} \Phi_i(abc)\Phi_i(x)\Phi_i(cba) + \sum_{i=1}^{n} \Phi_i(cba)\Phi_i(x)\Phi_i(abc) = \\
&= \sum_{i=1}^{n} \Phi_i(abc)\Phi_i(x)\left(\sum_{l=1}^{i} \Phi_l(a)\Phi_l(b)\Phi_l(c) + \Phi_l(c)\Phi_l(b)\Phi_l(a) - \Phi_i(abc)\right) + \\
&\quad + \sum_{i=1}^{n}\left(\sum_{l=1}^{i} \Phi_l(a)\Phi_l(b)\Phi_l(c) + \Phi_l(c)\Phi_l(b)\Phi_l(a) - \Phi_i(abc)\right)(\Phi_i(x)\Phi_i(abc) = \\
&= \sum_{i=1}^{n} \Phi_i(abc)\Phi_i(x)\sum_{l=1}^{i} \Phi_l(a)\Phi_l(b)\Phi_l(c) + \sum_{i=1}^{n} \Phi_i(abc)\Phi_i(x)\sum_{l=1}^{i} \Phi_l(c)\Phi_l(b)\Phi_l(a) - \\
&\quad - \sum_{i=1}^{n} \Phi_i(abc)\Phi_i(x)\Phi_i(abc) + \sum_{i=1}^{n}\sum_{l=1}^{i} \Phi_l(a)\Phi_l(b)\Phi_l(c)\Phi_i(x)\Phi_i(abc) + \\
&\quad + \sum_{i=1}^{n}\sum_{l=1}^{i} \Phi_l(c)\Phi_l(b)\Phi_l(a)\Phi_i(x)\Phi_i(abc) - \sum_{i=1}^{n} \Phi_i(abc)\Phi_i(x)\Phi_i(abc).
\end{aligned}$$

On comparing the right hand side of W and by our hypothesis, we get the required result. ∎

Theorem 2.1. Let $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be a Jordan Triple Higher Homomorphism of a ring R onto a 2-torsion-free prime ring R'. Then $\Phi_i = \pm\Psi_i$, for all $i \in \mathbb{N}$, where $\Psi_i \in \Psi = (\Psi_i)_{i\in\mathbb{N}}$, which Ψ is either a Higher Homomorphism or a Higher Anti-Homomorphism of R onto R'.

Proof. We proceed by induction on $n \in \mathbb{N}$. By Lemma 2.5, the statment of the Theorem is true for $n = 1$. Now suppose that $\Phi_m = \pm\Psi_m$ for $m < n$, $m, n \in \mathbb{N}$. Then we have two cases:

Case I: If $\Phi_m = \pm\Psi_m$, for all $m < n$, where $\Psi_i \in \Psi$ such that Ψ is a HH. Since Φ_i is onto for all $i \in \mathbb{N}$ and R' is prime, then by Lemma 2.10, either $S_n(a,b,c) = 0$ or $T_n(a,b,c) = 0$, for all $a, b, c \in R$, $n \in \mathbb{N}$. If $T_n(a,b,c) = 0$, then $\Phi_n(abc) = \sum\limits_{i=1}^{n} \Phi_i(c)\Phi_i(b)\Phi_i(a)$ and hence

$$\Phi_n(abxab) = \Phi_n((ab)x(ab)) = \sum_{i=1}^{n} \Phi_i(ab)\Phi_i(x)\Phi_i(ab). \tag{2.6}$$

On the other hand,

$$\Phi_n(abxab) = \Phi_n(a(bxa)b) = \sum_{i=1}^{n} \Phi_i(b)\Phi_i(bxa)\Phi_i(a) = \sum_{i=1}^{n} \Phi_i(b)\sum_{l=1}^{i} \Phi_l(a)\Phi_l(x)\Phi_l(b)\Phi_i(a).$$

That is,

$$\Phi_n(abxab) = \sum_{i=1}^{n}\sum_{l=1}^{i} \Phi_l(b)\Phi_l(a)\Phi_l(x)\sum_{l=1}^{i} \Phi_l(b)\Phi_l(a). \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$\begin{aligned}
0 &= \sum_{i=1}^{n}\left(\Phi_i(ab) - \sum_{l=1}^{i}\Phi_l(b)\Phi_l(a)\right)\Phi_i(x)\left(\Phi_i(ab) + \sum_{l=1}^{i}\Phi_l(b)\Phi_l(a)\right) + \\
&\quad + \sum_{i=1}^{n}\left(\Phi_i(ab) + \sum_{l=1}^{i}\Phi_l(b)\Phi_l(a)\right)\Phi_i(x)\left(\Phi_i(ab) - \sum_{l=1}^{i}\Phi_l(b)\Phi_l(a)\right) = \\
&= \left(\Phi_n(ab) - \sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)\right)\Phi_n(x)\left(\Phi_n(ab) + \sum_{l=1}^{n}\Phi_b\Phi_l(a)\right) + \\
&\quad + \left(\Phi_n(ab) + \sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)\right)\Phi_n(x)\left(\Phi_n(ab) - \sum_{l=1}^{n}\Phi_b\Phi_l(a)\right) + \\
&\quad + \sum_{i=1}^{n-1}\left(\Phi_i(ab) - \sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)\right)\Phi_i(x)\left(\Phi_i(ab) + \sum_{l=1}^{n}\Phi_b\Phi_l(a)\right) + \\
&\quad + \sum_{i=1}^{n-1}\left(\Phi_i(ab) + \sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)\right)\Phi_i(x)\left(\Phi_i(ab) - \sum_{l=1}^{n}\Phi_b\Phi_l(a)\right).
\end{aligned} \tag{2.8}$$

Since $\Phi_m = \pm\Psi_m$, for all $m < n$ and $\Psi_m \in \Psi$, where $\Psi_m \in \Psi$ is a HH, then the last two lines of equation (2.8) can be reduced to

$$\sum_{i=1}^{n-1}\left(\Phi_i(ab) \mp \Phi_i(ba)\right)\Phi_i(x)\left(\Phi_i(ab) \pm \Phi_i(ba)\right) + \sum_{i=1}^{n-1}\left(\Phi_i(ab) \pm \Phi_i(ba)\right)\Phi_i(x)\left(\Phi_i(ab) \mp \Phi_i(ba)\right) =$$

$$= 2\sum_{i=1}^{n-1} \phi_i(ab)\Phi_i(x)\Phi_i(ab) - \Phi_i(ba)\Phi_i(x)\Phi_i(ba). \tag{2.9}$$

Since $T_n(a,b,c) = 0$ and $\Phi_m = \pm\Psi_m$, for all $m < n$, where $\Psi_m \in \Psi$ such that $\Psi_m \in \Psi$ is a HH. Thus,

$$\begin{aligned}
\sum_{i=1}^{n-1}\Phi_i(ab)\Phi_i(x)\Phi_i(ab) &= \sum_{i=1}^{n-1}\Phi_i(a)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(x)\Phi_l(a)\Phi_l(b) = \sum_{i}^{n-1}\Phi_i(a)\Phi_i(axb)\Phi_i(b) = \\
&= \sum_{i}^{n-1}\Phi_i(a)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(x)\Phi_l(b)\Phi_i(b) = \sum_{i}^{n-1}\sum_{l=1}^{i}\Phi_l(a)\Phi_l(a)\Phi_l(x)\Phi_l(b)\Phi_i(b) = \\
&= \sum_{i=1}^{n-1}\Phi_i(xaa)\Phi_i(b)\Phi_i(b) = \sum_{i}^{n-1}\sum_{l=1}^{i}\Phi_l(x)\Phi_l(a)\Phi_l(a)\Phi_l(b)\Phi_i(b) = \\
&= \sum_{i=1}^{n-1}\Phi_i(x)\Phi_i(a)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\Phi_l(b) = \sum_{i=1}^{l}\Phi_i(x)\Phi_i(a)\Phi_i(bba) = \\
&= \sum_{i=1}^{n-1}\Phi_i(x)\Phi_i(a)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(b)\Phi_l(a) = \sum_{i=1}^{n-1}\sum_{l=1}^{i}\Phi_l(x)\Phi_l(a)\Phi_l(b)\Phi_l(b)\Phi_l(a) = \\
&= \sum_{i=1}^{n-1}\Phi_i(bax)\Phi_i(b)\Phi_i(a) = \sum_{i=1}^{n-1}\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a)\Phi_l(x)\Phi_l(b)\Phi_l(a) = \\
&= \sum_{i=1}^{n-1}\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a)\Phi_l(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) = \sum_{i=1}^{n-1}\Phi_i(ba)\Phi_i(x)\Phi_i(ba).
\end{aligned}$$

Then equation (2.9) is equal to zero and therefore equation (2.8) becomes

$$\left(\Phi_n(ab)+\sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)\right)\Phi_n(x)\left(\Phi_n(ab)-\sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)\right)+$$
$$+\left(\Phi_n(ab)-\sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)\right)\Phi_n(x)\left(\Phi_n(ab)+\sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)\right)=0.$$

Since Φ_i is onto, for all $i \in \mathbb{N}$ and R' is prime, then either $\Phi_n(ab)=\sum\limits_{l=1}^{n}\Phi_i(b)\Phi_i(a)$ or $\Phi_n(ab)=-\sum\limits_{l=1}^{n}\Phi_i(b)\Phi_i(a)$. Thus Θ is a HAH or a negative of a HAH and this is a contradiction. Hence, $S_n(a,b,c)=0$, for all $a,b,c\in R$, $n\in\mathbb{N}$ and this lead us to

$$\begin{aligned}\Phi_n(abxab) &= \sum_{i=1}^{n}\Phi_i(a)\Phi_i(bxa)\Phi_i(b)=\sum_{i=1}^{n}\Phi_i(a)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(x)\Phi_l(a)\Phi_l(b)=\\ &= \sum_{i=1}^{n}\Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(a)\Phi_i(b)=\\ &= \sum_{i=1}^{n}\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right).\end{aligned}\tag{2.10}$$

On the other hand,

$$\Phi_n(abxab)=\sum_{i=1}^{n}\Phi_i(ab)\Phi_i(x)\Phi_i(ab).\tag{2.11}$$

Now, comparing (2.10) and(2.11), we get

$$\begin{aligned}0 &= \sum_{i=1}^{n}\left(\Phi_i(ab)-\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)\Phi_i(x)\left(\Phi_i(ab)+\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)+\\ &\quad+\sum_{i=1}^{n}\left(\Phi_i(ab)+\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)\Phi_i(x)\left(\Phi_i(ab)-\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)=\\ &= \left(\Phi_n(ab)-\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)\Phi_n(x)\left(\Phi_n(ab)+\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)+\\ &\quad+\left(\Phi_n(ab)+\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)\Phi_n(x)\left(\Phi_n(ab)-\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)+\\ &\quad+\sum_{i=1}^{n-1}\left(\Phi_i(ab)-\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)\Phi_i(x)\left(\Phi_i(ab)+\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)+\\ &\quad+\sum_{i=1}^{n-1}\left(\Phi_i(ab)+\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right)\Phi_i(x)\left(\Phi_i(ab)-\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right).\end{aligned}\tag{2.12}$$

Since $\Phi_m=\pm\Psi_m$, for all $m<n$, where $\Psi_m=\Psi$ and Ψ is a HH, then the last term of equation (2.12) are equal to zero. Therefore equation (2.12) becomes

$$\begin{aligned}&\left(\Phi_n(ab)-\sum_{l=1}^{n}\Phi_l(a)\Phi_l(b)\right)\Phi_n(x)\left(\Phi_n(ab)+\sum_{l=1}^{n}\Phi_l(a)\Phi_l(b)\right)+\\ +&\left(\Phi_n(ab)+\sum_{l=1}^{n}\Phi_l(a)\Phi_l(b)\right)\Phi_n(x)\left(\Phi_n(ab)-\sum_{l=1}^{n}\Phi_l(a)\Phi_l(b)\right)=0.\end{aligned}$$

Since Φ_i is onto, for all $i\in\mathbb{N}$ and R' is prime, then by Lemma 2.2, we get either $\Phi_n(ab)=\sum\limits_{l=1}^{n}\Phi_l(a)\Phi_l(b)$ or $\Phi_n(ab)=-\sum\limits_{l=1}^{n}\Phi_l(a)\Phi_l(b)$, for all $a,b\in R$ and $n\in\mathbb{N}$. That is, Θ is a HH or a negative of a HH.

Using similar arguments as used Case I we can prove that:

Case II: If $\Phi_m=\pm\Psi_m$, for all $m<n$, where $\Psi_m\in\Psi$ and Ψ is a HAH, then $\Theta=(\Phi_i)_{i\in\mathbb{N}}$ is also a HAH. ■

3 Generalized Jordan Higher Homomorphism

Let $\mathbb{N}$ be the set of all natural numbers and let $F=(f_i)_{i\in\mathbb{N}}$ be a family of additive mappings of a ring R into ring R'. Then F is said to be:

- a *Generalized Higher Homomorphism* (GHH, for short), if there exists a higher homomorphism $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ such that $f_n(ab) = \sum_{i=1}^{n} f_i(a)\Phi_i(b)$, for all $a, b \in R$ and Θ is called the *relating higher homomorphism* (RHH, for short);
- a *Generalized Jordan Higher Homomorphism* (GJHH, for short), if there exists a Jordan higher homomorphism $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ such that $f_n(ab+ba) = \sum_{i=1}^{n} f_i(a)\Phi_i(b) + f_i(b)\Phi_i(a)$, for all $a, b \in R$ and Θ is called the *relating Jordan higher homomorphism* (RJHH, for short). If R' is 2-torsion-free, then the definition of GJHH is equivalent to the following condition: $f_n(a^2) = \sum_{i=1}^{n} f_i(a)\Phi_i(b)$, for all $a, b \in R$;
- a *Generalized Jordan Triple Homomorphism* (GJTHH, for short), if there exists a Jordan triple higher homomorphism $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ such that $f_n(aba) = \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(a)$, for all $a, b \in R$ and Θ is called the *relating Jordan triple higher homomorphism* (RJTHH, for short);
- a *Generalized Higher Anti-Homomorphism* (GAH, for short), if there exists a higher anti-homomorphism $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ such that $f_n(ab) = \sum_{i=1}^{n} f_i(b)\Phi_i(a)$, for all $a, b \in R$ and Θ is called the *relating higher anti-homomorphism* (RHAH, for short).

Remark 3.1. It is clear that every Generalized Higher Homomorphism is a Generalized Jordan Higher Homomorphism, but the converse need not be true in general. Following example shows that:

Example 3.1. Let S be any ring with involution $*$, set $R = S \oplus S$ and $a \in S$ such that $a \in Z(S)$ and $s_1 a s_2 = 0$, for all $s_1, s_2 \in S$. Let $F = (f_i)_{i \in \mathbb{N}}$ be a family of mappings of R into R defined by

$$f_n(s,t) = \begin{cases} (-(2-n_a s, (n-1)t^*), & n = 1,2, \quad \text{for all } (s,t) \in R; \\ 0, & n \geq 3. \end{cases}$$

Then there exists a JHH which is defined in Example 2.1. It becomes clear that F is a GJHH, but not a GHH.

Lemma 3.1. Let $F = (f_i)_{i \in \mathbb{N}}$ be a Generalized Jordan Higher Homomorphism of a ring R into a 2-torsion-free ring R'. Then:

(i) F is a Generalized Jordan Triple Higher Homomorphisms of R into R';

(ii) $f_n(abc+cba) = \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(c) + f_i(c)\Phi_i(b)\Phi_i(a)$, for all $a, b \in R$, $n \in \mathbb{N}$ and $\Phi_i \in \Theta = (\Phi_i)_{i \in \mathbb{N}}$, where Θ is the Related Jordan Higher Homomorphism.

Proof. (i). We have

$$f_n(ab+ba) = \sum_{i=1}^{n} f_i(a)\Phi_i(b) + f_i(b)\Phi_i(a). \tag{3.1}$$

Replacing a by $ab+ba$ in (3.1), we get

$$\begin{aligned} & f_n(a(ab+ba)+(ab+ba)a) = \sum_{i=1}^{n} f_i(a)\Phi_i(ab+ba) + f_i(ab+ba)\Phi_i(a) = \\ = \ & \sum_{i=1}^{n} f_i(a)\left(\sum_{l=1}^{i} \Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a)\right) + \sum_{i=1}^{n}\left(\sum_{l=1}^{i} f_l(a)\Phi_l(b) + f_l(a)\Phi_l(a)\right)\Phi_i(a) = \\ = \ & \sum_{i=1}^{n} f_i(a)\Phi_i(a)\Phi_i(b) + \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^{n} f_i(b)\Phi_i(a)\Phi_i(b). \end{aligned} \tag{3.2}$$

On the other hand,

$$f_n(a(ab+ba)+(ab+ba)a) = \sum_{i=1}^{n} f_i(a^2)\Phi_i(b) + f_i(b)\Phi_i(a^2) + 2f_n(aba). \tag{3.3}$$

Compare (3.2) and (3.3), to get $f_n(aba) = \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(a)$, for all $a, b \in R$, $n \in \mathbb{N}$. Therefore F is a GJTHH.

(ii). Replacing a by $a+b$ in (i)., we get

$$\begin{aligned} f_n((a+c)b(a+c)) &= \sum_{i=1}^{n} f_i(a+c)\Phi_i(b)\Phi_i(a+c) = \\ &= \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(a) + f_i(a)\Phi_i(b)\Phi_i(c) + f_i(c)\Phi_i(b)\Phi_i(a) + f_i(c)\Phi_i(b)\Phi_i(c). \end{aligned} \tag{3.4}$$

On the other hand,

$$f_n((a+c)b(a+c)) = f_n(aba) + f_n(abc+cba) + f_n(cba). \tag{3.5}$$

By comparing (3.4) and (3.5), we get $f_n(abc+cba) = \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(c) + f_i(c)\Phi_i(b)\Phi_i(a)$, for all $a, b, c \in R$, $n \in \mathbb{N}$. ■

Remark 3.2. Let $F = (f_i)_{i\in\mathbb{N}}$ be a Generalized Jordan Higher Homomorphism of a ring R into a ring R' and $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be the Related Jordan Higher Homomorphism. We shall write, for all $a, b \in R$, $n \in \mathbb{N}$:

- $(a^{b*})^n = f_n(ab) - \sum_{i=1}^{n} f_i(a)\Phi_i(b)$;
- $(a_b^*)^n = f_n(ab) - \sum_{i=1}^{n} f_i(b)\Phi_i(a)$.

Note that if $(a^{b*})^n = 0$ (resp. $(a_b^*)^n = 0$), then F is a Generalized Higher Homomorphism (resp. a Generalized Higher Anti-Homomorphism).

Lemma 3.2. Let $F = (f_i)_{i\in\mathbb{N}}$ be a Generalized Jordan Higher Homomorphism of a ring R into a ring R'. Then

$$(a^{b*})^n\Phi_n(x)(a_b)^n + (a_b^*)^n\Phi_n(x)(a^b)^n,$$

for all $a, b, x \in R, n \in \mathbb{N}$, where $\Phi_i \in \Theta = (\Phi_i)_{i\in\mathbb{N}}$ such that Θ is the Related Jordan Higher Homomorphism.

Proof. We proceed by induction on $n \in \mathbb{N}$. If $n = 1$, let $W = abxba + baxab$. Since F is a GJHH, then

$$\begin{aligned} f_1(W) &= f_1(a(bxb)a + b(axa)b) = f_1(a)\Phi_1(bxb)\Phi_1(a) + f_1(b)\Phi_1(axa)\Phi_1(b) = \\ &= f_1(a)\Phi_1(b)\Phi_1(x)\Phi_1(b)\Phi_1(a) + f_1(b)\Phi_1(a)\Phi_1(x)\Phi_1(a)\Phi_1(b). \end{aligned} \tag{3.6}$$

On the other hand, by Lemma 3.1 (ii)., we have

$$f_1(W) = f_1((ab)x(ba) + (ba)x(ab)) = f_1(ab)\Phi_1(x)\Phi_1(ba) + f_1(ba)\Phi_1(x)\Phi_1(x)\Phi_1(ab). \tag{3.7}$$

Since $f_1(ab+ba) = f_1(a)\Phi_1(a)\Phi_1(b)+f_1(b)\Phi_1(a)$ and $\Phi_1(ab+ba) = \Phi_1(a)\Phi_1(b)+\Phi_1(b)\Phi_1(a)$, then (3.7) can reduces to

$$\begin{aligned} f_1(W) &= f_1(ab)\Phi_1(x)\left(\Phi_1(a)\Phi_1(b) + \Phi_1(b)\Phi_1(a)\right) + \\ &\quad + \left(-f_1(ab) + f_1(a)\Phi_1(b) + f_1(b)\Phi_1(a)\right)\Phi_1(x)\Phi_1(ab) = \\ &= -f_1(ab)\Phi_1(x)\left(\Phi_1(ab) - \Phi_1(a)\Phi_1(b)\right) - f_1(ab)\Phi_1(x)\left(\Phi_1(ab) - \Phi_1(b)\Phi_1(a)\right) + \\ &\quad + f_1(a)\Phi_1(b)\Phi_1(x)\Phi_1(ab) + f_1(b)\Phi_1(a)\Phi_1(x)\Phi_1(ab). \end{aligned} \tag{3.8}$$

Comparing (3.6) and (3.8), we get

$$\begin{aligned} 0 &= -f_1(ab)(a^b) - f_1(ab)\Phi_1(x)(a_b) + f_1(a)\Phi_1(b)\Phi_1(x)\Phi_1(ab) + f_1(b)\Phi_1(a)\Phi_1(x)\Phi_1(ab) - \\ &\quad - f_1(a)\Phi_1(b)\Phi_1(x)\Phi_1(b)\Phi_1(a) - f_1(b)\Phi_1(a)\Phi_1(x)\Phi_1(a)\Phi_1(b) = \\ &= -f_1(ab)\Phi_1(x)(a^b) - f_1(ab)\Phi_1(x)(a_b) + f_1(a)\Phi_1(b)\Phi_1(x)(a_b) + f_1(b)\Phi_1(a)\Phi_1(x)(a^b) = \\ &= -\left(f_1(ab) - f_1(a)\Phi_1(b)\right) - \left(f_1(ab) - f_1(a)\Phi_1(b)\right)\Phi_1(x)\Phi_1(a). \end{aligned}$$

Thus we have $(a^{b*})\Phi_1(x)(a_b) + (a^*_b)\Phi_1(x)(a^b) = 0$, for all $a, b, x \in R$. Then we can assume that $(a^{b*})^m\Phi_m(x)(a_b)^m + (a^*_b)^m\Phi_m(x)(a^b)^m = 0$, for all $a, b, x \in R$, $m, n \in \mathbb{N}$, $m < n$. Since every GJHH of 2-torsion-free ring is a GJTHH, then

$$\begin{aligned}
f_n(W) &= f_n(a(bxb)a + b(axa)b) = \sum_{i=1}^{n} f_i(a)\Phi_i(bxb)\Phi_i(a) + f_i(b)\Phi_i(axa)\Phi_i(b) = \\
&= \sum_{i=1}^{n} f_i(a)\left(\sum_{l=1}^{i} f_l(b)\Phi_l(x)\Phi_l(b)\right)\Phi_i(a) + \sum_{i=1}^{n} f_i(b)\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(x)\Phi_l(a)\right)\Phi_i(b) = \\
&= \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^{n} f_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(a)\Phi_i(b) = \\
&= \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) + \sum_{i=1}^{n} f_i(b)\Phi_i(a)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) = \\
&= f_n(a)\Phi_n(b)\Phi_n(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) + \sum_{i=1}^{n-1} f_i(a)\Phi_i(b)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) + \\
&\quad + f_n(b)\Phi_n(a)\Phi_n(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) + \sum_{i=1}^{n-1} f_i(b)\Phi_i(a)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b).
\end{aligned} \tag{3.9}$$

On the other hand, by Lemma 3.1 (ii)., we have

$$f_n(W) = f_n((ab)x(ba) + (ba)x(ab)) = \sum_{i=1}^{n} f_i(ab)\Phi_i(x)\Phi_i(ba) + f_i(ba)\Phi_i(x)\Phi_i(ab).$$

Since F is a GJHH and Θ is the RJHH, then

$$\begin{aligned}
f_n(W) &= \sum_{i=1}^{n} f_i(ab)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a) - \Phi_i(ab)\right) + \\
&\quad + \sum_{i=1}^{n}\left(\sum_{l=1}^{i} f_l(a)\Phi_l(b) + f_l(b)\Phi_l(a) - f_i(ab)\right)\Phi_i(x)\Phi_i(ab) = \\
&= \sum_{i=1}^{n} f_i(ab)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) + \sum_{i=1}^{n} f_i(ab)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) - \\
&\quad - \sum_{i=1}^{n} f_i(ab)\Phi_i(x)\Phi_i(ab) + \sum_{i=1}^{n}\sum_{l=1}^{i} f_l(a)\Phi_l(b)\Phi_i(x)\Phi_i(ab) + \\
&\quad + \sum_{i=1}^{n}\sum_{l=1}^{i} f_l(b)\Phi_l(a)\Phi_i(x)\Phi_i(ab) - \sum_{i=1}^{n} f_i(ab)\Phi_i(x)\Phi_i(ab) = \\
&= -\sum_{i=1}^{n} f_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^{n} f_i(ab)\Phi_i(x)(a_b)^i + \\
&\quad + \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \sum_{i=1}^{n} f_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab) = \\
&= -f_n\Phi_n(x)(a^b)^n - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a^b)^i - f_n\Phi_n(x)(a_b)^n - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a_b)^i + \\
&\quad + f_n(a)f_n(b)\Phi_n(x)\Phi_n(ab) + \sum_{i=1}^{n-1} f_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \\
&\quad + f_n(b)f_n(a)\Phi_n(x)\Phi_n(ab) + \sum_{i=1}^{n-1} f_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab).
\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned}
0 &= -f_n(an)\Phi_n(x)(a^b)^n - f_n(ab)\Phi_n(x)(a_b)^n + f_n(a)\Phi_n(b)\Phi_n(x)\left(\Phi_n(ab) - \sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)\right) + \\
&\quad + f_n(b)\Phi_n(a)\Phi_n(x)\left(\Phi_n(ab) - \sum_{l=1}^{n}\Phi_l(a)\Phi_l(b)\right) - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a_b)^i + \\
&\quad + \sum_{i=1}^{n-1} f_i(a)\Phi_i(b)\Phi_i(x)\left(\Phi_i(ab) - \sum_{l=1}^{i}\Phi_l(b)\Phi_l(a)\right) + \sum_{i=1}^{n-1} f_i(b)\Phi_i(a)\Phi_i(x)\left(\Phi_i(ab) - \sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\right) = \\
&= -f_n(ab)\Phi_n(x)(a^b)^n - f_n(ab)\Phi_n(x)(a_b)^n + f_n(a)\Phi_n(b)\Phi_n(x)(a_b)^n + f_n(b)\Phi_n(a)\Phi_n(x)(a^b)^n - \\
&\quad - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a_b)^i + \sum_{i=1}^{n-1} f_i(a)\Phi_i(b)\Phi_i(x)(a_b)^i + \sum_{i=1}^{n-1} f_i(b)\Phi_i(a)\Phi_i(x)(a^b)^i = \\
&= -(a^*_b)^n\Phi_n(x)(a^b)^n - (a^{b*})^n\Phi_x(a_b)^n - \sum_{i=1}^{n-1}(a^{b*})^i\Phi_i(x)(a_b)^i - \sum_{i=1}^{n-1}(a^*_b)^i\Phi_i(x)(a_b).
\end{aligned}$$

Therefore, by our assumption, we find that $(a^{b*})^n\Phi_n(x)(a^b)^n + (a^*_b)^n\Phi_n(x)(a^b)^n$, for all $a, b, x \in R$, $n \in \mathbb{N}$. ■

Remark 3.3. Let $F = (f_i)_{i\in\mathbb{N}}$ be a Generalized Jordan Triple Higher Homomorphism of a ring R into a ring R' and $\Theta = (\Phi_i)_{i\in\mathbb{N}}$ be the Relating Jordan Triple Higher Homomorphism. Then, for all $a, b, c \in R$, $n \in \mathbb{N}$, we define:

- $S_n^*(a,b,c) = f_n(abc) - \sum\limits_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(c)$;

- $T_n^*(a,b,c) = f_n(abc) - \sum\limits_{i=1}^{n} f_i(c)\Phi_i(b)\Phi_i(a)$.

Lemma 3.3. Let $F = (f_i)_{i\in N}$ be a Generalized Jordan Triple Higher Homomorphisms of a ring R into a ring R' and $\Theta = (\Phi_i)_{i\in \mathbb{N}}$ be the Relating Jordan Triple Higher Homomorphism. Then

$$S_n^*(a,b,c)\Phi_n(x)T_n(a,b,c) + T_n^*(a,b,c)\Phi_n(x)S_n(a,b,c) = 0, \text{ for all } a,b,c,x \in R,\ n \in \mathbb{N}.$$

Proof. By Lemma 2.6, the statment of the Lemma is true for $n = 1$. Then, by induction, we assume that $S_m^*(a,b,c)\Phi_n(x)T_m(a,b,c) + T_m^*(a,b,c)\Phi_m(x)S_m(a,b,c) = 0$, for all $a,b,c,x \in R$, $m,n \in \mathbb{N}$, $m < n$. Let

$$\begin{aligned}
W &= f_n(abcxcba + cbaxabc) = \sum_{i=1}^{n} f_i(a)\Phi_i(bcxcb)\Phi_i(a) + \sum_{i=1}^{n} f_i(c)\Phi_i(baxab)\Phi_i(c) = \\
&= \sum_{i=1}^{n} f_i(a)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(cxc)\Phi_l(b)\Phi_i(a) + \sum_{i=1}^{n} f_i(c)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(axa)\Phi_l(b)\Phi_i(c) = \\
&= \sum_{i=1}^{n} f_i(a)\sum_{l=1}^{i}\Phi_l(b)\left(\sum_{t=1}^{l}\Phi_t(c)\Phi_t(x)\Phi_t(c)\right)\Phi_l(b)\Phi_i(a) = \\
&= \sum_{i=1}^{n} f_i(c)\sum_{l=1}^{i}\Phi_l(b)\left(\sum_{t=1}^{l}\Phi_t(a)\Phi_t(x)\Phi_t(a)\right)\Phi_l(b)\Phi_i(c) = \\
&= \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(c)\Phi_i(x)\Phi_i(c)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^{n} f_i(c)\Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(a)\Phi_i(b)\Phi_i(c) = \\
&= \sum_{i=1}^{n}\left(\sum_{l=1}^{i} f_l(a)\Phi_l(b)\Phi_l(c)\right)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(c)\Phi_l(b)\Phi_l(a)\right) + \\
&\quad + \sum_{i=1}^{n}\left(\sum_{l=1}^{i} f_l(c)\Phi_l(b)\Phi_l(a)\right)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\Phi_l(c)\right) = \\
&= \left(\sum_{l=1}^{n} f_l(a)\Phi_l(b)\Phi_l(c)\right)\Phi_n(x)\left(\sum_{l=1}^{n}\Phi_l(c)\Phi_l(b)\Phi_l(a)\right) + \\
&\quad + \sum_{i=1}^{n-1}\left(\sum_{l=1}^{i} f_l(a)\Phi_l(b)\Phi_l(c)\right)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(c)\Phi_l(b)\Phi_l(a)\right) + \\
&\quad + \left(\sum_{l=1}^{n} f_l(c)\Phi_l(b)\Phi_l(a)\right)\Phi_n(x)\left(\sum_{l=1}^{n}\Phi_l(a)\Phi_l(b)\Phi_l(c)\right) + \\
&\quad + \sum_{i=1}^{n-1}\left(\sum_{l=1}^{i} f_l(c)\Phi_l(b)\Phi_l(a)\right)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\Phi_l(c)\right).
\end{aligned}$$

On the other hand by Lemma 3.1 (ii)., we get

$$W = f_n((abc)x(cba) + (cba)x(abc)) = \sum_{i=1}^{n} f_i(abc)\Phi_i(x)\Phi_i(cba) + f_i(cba)\Phi_i(x)\Phi_i(abc).$$

Since

$$f_n(abc + cba) = \sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(c) + f_i(c)\Phi_i(b)\Phi_i(a) \text{ and}$$

$$\Phi_n(abc + cba) = \sum_{i=1}^{n} \Phi_i(a)\Phi_i(b)\Phi_i(c) + \Phi_i(c)\Phi_i(b)\Phi_i(a), \text{ for all } a,b,c \in R \text{ and } n \in \mathbb{N},$$

then

$$
\begin{aligned}
W \;=\;& \sum_{i=1}^{n} f_i(abc)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\Phi_l(c)\right)+\sum_{i=1}^{n} f_i(abc)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(c)\Phi_l(b)\Phi_l(a)\right)-\\
&-\sum_{i=1}^{n} f_i(abc)\Phi_i(x)\Phi_i(abc)+\sum_{i=1}^{n}\left(\sum_{l=1}^{i} f_l(a)\Phi_l(b)\Phi_l(c)\right)\Phi_i(x)\Phi_i(abc)+\\
&+\sum_{i=1}^{n}\left(\sum_{l=1}^{i} f_l(c)\Phi_l(b)\Phi_l(a)\right)\Phi_i(x)\Phi_i(abc)-\sum_{i=1}^{n} f_i(abc)\Phi_i(x)\Phi_i(abc)=\\
=\;& f_n(abc)\Phi_n(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\Phi_l(c)+\sum_{i=1}^{n-1} f_i(abc)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\Phi_l(c)+\\
&+f_n(abc)\Phi_n(x)\sum_{l=1}^{i}\Phi_l(c)\Phi_l(b)\Phi_l(a)+\sum_{i=1}^{n-1} f_i(abc)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(c)\Phi_l(b)\Phi_l(a)-\\
&-f_n(abc)\Phi_n(x)\Phi_n(abc)-\sum_{i=1}^{n-1} f_i(abc)\Phi_i(x)\Phi_i(abc)+\sum_{l=1}^{n} f_l(a)\Phi_l(b)\Phi_l(c)\Phi_l(x)\Phi_n(abc)+\\
&+\sum_{i=1}^{n-1}\sum_{l=1}^{i} f_l(a)\Phi_l(b)\Phi_l(c)\Phi_i(x)\Phi_i(abc)+\sum_{l=1}^{n} f_l(c)\Phi_l(b)\Phi_l(a)\Phi_n(x)\Phi_n(abc)+\\
&+\sum_{i=1}^{n-1}\sum_{l=1}^{i} f_l(c)\Phi_l(b)\Phi_l(a)\Phi_i(x)\Phi_i(abc)-f_n(abc)\Phi_n(x)\Phi_n(abc)-\sum_{i=1}^{n-1} f_i(abc)\Phi_i(x)\Phi_i(abc).
\end{aligned}
$$

Now, comparing the right hand side of W, we obtain

$$
\begin{aligned}
0 \;=\;& \left(f_n(abc)-\sum_{l=1}^{n} f_l(c)\Phi_l(b)\Phi_l(a)\right)\Phi_n(x)\sum_{l=1}^{n}\Phi_l(a)\Phi_l(b)\Phi_l(c)+\\
&+\left(f_n(abc)-\sum_{l=1}^{n} f_l(a)\Phi_l(b)\Phi_l(c)\right)\Phi_n(x)\sum_{l=1}^{n}\Phi_l(c)\Phi_l(b)\Phi_l(a)-\\
&-\left(f_n(abc)-\sum_{l=1}^{n} f_l(a)\Phi_l(b)\Phi_l(c)\right)\Phi_n(x)\Phi_n(abc)-\\
&-\left(f_n(abc)-\sum_{l=1}^{n} f_l(c)\Phi_l(b)\Phi_l(a)\right)\Phi_n(x)\Phi_n(abc)+\\
&+\sum_{i=1}^{n-1}\left(f_i(abc)-\sum_{l=1}^{i} f_l(c)\Phi_l(b)\Phi_l(a)\right)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b)\Phi_l(c)\right)+\\
&+\sum_{i=1}^{n-1}\left(f_i(abc)-\sum_{l=1}^{i} f_l(a)\Phi_l(b)\Phi_l(c)\right)\Phi_i(x)\left(\sum_{l=1}^{i}\Phi_l(c)\Phi_l(b)\Phi_l(a)\right)-\\
&-\sum_{i=1}^{n-1}\left(f_i(abc)-\sum_{l=1}^{i} f_l(a)\Phi_l(b)\Phi_l(c)\right)\Phi_i(x)\Phi_i(abc)-\\
&-\sum_{i=1}^{n-1}\left(f_i(abc)-\sum_{l=1}^{i} f_l(c)\Phi_l(b)\Phi_l(a)\right)\Phi_i(x)\Phi_i(abc)=\\
=\;& S_n^*(a,b,c)\Phi_n(x)T_n(a,b,c)+T_n^*(a,b,c)\Phi_n(x)S_n(a,b,c)+\\
&+\sum_{i=1}^{n-1} S_i^*(a,b,c)\Phi_i(x)T_i(a,b,c)+T_i^*(a,b,c)\Phi_i(x)S_i(a,b,c),\ \text{for all } a,b,c,x\in R \text{ and } n\in\mathbb{N}.
\end{aligned}
$$

Then, by our assumption, we get $S_n^*(a,b,c)\Phi_n(x)T_n(a,b,c)+T_n^*(a,b,c)\Phi_n(x)S_n(a,b,c)=0$, for all $a,b,c,x\in R$, $n\in\mathbb{N}$. ■

Theorem 3.1. Let $F=(f_i)_{i\in\mathbb{N}}$ be a Generalized Jordan Triple Higher Homomorphism of a ring R into a 2-torsion-free prime ring R' and $\Theta=(\Phi_i)_{i\in\mathbb{N}}$ be the Relating Jordan Triple Higher Homomorphism. Then $f_i=\pm\Psi_i$, for all $i\in\mathbb{N}$, where $\Psi_i\in\Psi=(\Psi_i)_{i\in\mathbb{N}}$ and Ψ is a Generalized Higher Homomorphism or a Generalized Higher Anti-Homomorphism.

Proof. We proceed by induction on $n\in\mathbb{N}$. By Lemma 2.7, the statment of the Theorem is true for $n=1$. Then we can assume that $f_m=\pm\Psi_m$, for all $m,n\in\mathbb{N}$, $m<n$. Now, we have two cases:

Case I: If $f_m=\pm\Psi_m$, for all $m,n\in\mathbb{N}$, $m<n$ and $\Psi_m\in\Psi$ such that Ψ is a GHH. By Theorem 2.1, either

(i). $\Phi_i=\pm\delta_i$, for all $i\in\mathbb{N}$ and $\delta_i\in\delta=(\delta_i)_{i\in\mathbb{N}}$ such that δ is a HH, or

(ii). $\Phi_i=\pm\delta_i$, for all $i\in\mathbb{N}$ and $\delta_i\in\delta=(\delta_i)_{i\in\mathbb{N}}$ such that δ is a HAH.

If we have (i)., then $\Phi_n(abc)=\sum_{i=1}^{n}\Phi_i(a)\Phi_i(b)\Phi_i(c)$ and this means $S_n(a,b,c)=0$. Since R' is prime and Θ is onto, then by Lemma 3.3, we have either $T_n(a,b,c)=0$ or $S_n^*(a,b,c)=0$, for all $a,b,c\in R$, $n\in\mathbb{N}$. If $T_n(a,b,c)=0$, for all $a,b,c\in R$, $n\in\mathbb{N}$, then as in the proof of Theorem 2.1, we have $\Phi_i=\pm\delta_i$, for all $i\in\mathbb{N}$ and δ is a HAH and this will be a contradiction with assumption (i). Therefore $S_n^*(a,b,c)=0$, for all $a,b,c\in R$, $n\in\mathbb{N}$, that is, $f_n(abc)=\sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(c)$.

Let $W = f_n(abxab)$. Since F is a GJTHH and $\Phi_i \pm \delta_i$, for all $i \in \mathbb{N}$ and $\delta_i \in \delta = (\delta_i)_{i\in\mathbb{N}}$ such that δ is a HH,

$$\begin{aligned} W &= \sum_{i=1}^{n} f_i(ab)\Phi_i(x)\Phi_i(ab) = \pm\sum_{i=1}^{n} f_i(abc)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) = \\ &= \pm f_n(ab)\Phi_n(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) \pm \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b). \end{aligned}$$

On the other hand, since $S_n^*(a,b,c) = 0$, for all $a,b,c \in R$, $n \in \mathbb{N}$, then

$$\begin{aligned} W &= f_n(a(bxa)b) = \sum_{i=1}^{n} f_i(a)\Phi_i(bxa)\Phi_i(b) = \pm\sum_{i=1}^{n} f_i(a)\left(\sum_{l=1}^{i}\Phi_l(b)\Phi_l(xa)\right)\Phi_i(b) = \\ &= \pm\sum_{i=1}^{n} f_i(a)\Phi_i(b)\Phi_i(xa)\Phi_i(b) = \pm\sum_{i=1}^{n} f_i(a)\Phi_i(b)\left(\sum_{l=1}^{i}\Phi_l(x)\Phi_l(a)\right)\Phi_i(b) = \\ &= \pm\sum_{i=1}^{n}\sum_{l=1}^{i} f_l(a)\Phi_l(b)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) = \\ &= \pm\sum_{l=1}^{n} f_l(a)\Phi_l(b)\Phi_l(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b) \pm \sum_{i=1}^{n-1}\sum_{l=1}^{i} f_l(a)\Phi_l(b)\Phi_l(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b). \end{aligned}$$

By comparing the right hand side of W, we get

$$0 = \left(f_n(ab) \pm \sum_{l=1}^{n} f_l(a)\Phi_l(b)\right)\Phi_n(x)\sum_{l=1}^{n}\Phi_l(a)\Phi_l(b) + \sum_{i=1}^{n-1}\left(f_i(ab) \pm \sum_{l=1}^{i} f_l(a)\Phi_l(b)\right)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(a)\Phi_l(b).$$

Note that the assumption of Case I reduces the last equation to

$$\left(f_n(ab) \pm \sum_{l=1}^{n} f_l(a)\Phi_l(b)\right)\Phi_n(x)\sum_{l=1}^{n}\Phi_l(a)\Phi_l(b) = 0,$$

and this implies that $\left(f_n(ab) \pm \sum_{l=1}^{n} f_l(a)\Phi_l(b)\right)R' = 0$, for all $a,b \in R$, $n \in \mathbb{N}$. Since R' is prime, then $f_n(ab) = \pm\sum_{l=1}^{n} f_l(a)\Phi_l(b)$ and this means $f_i = \pm\Psi_i$, for all $i \in \mathbb{N}$, $\Psi_i \in \Psi = (\Psi_i)_{i\in\mathbb{N}}$, where Ψ is a GHH.

Now, if have case (ii)., then $\Phi_n(abc) = \sum_{i=1}^{n}\Phi_i(c)\Phi_i(b)\Phi_i(a)$, for all $a,b,c \in R$, $n \in \mathbb{N}$. Since R' is prime and Θ is onto, then by Lemma 3.3, we have either $S_n(a,b,c) = 0$ or $T_n^*(a,b,c) = 0$, for all $a,b,c \in R$, $n \in \mathbb{N}$. If $S_n(a,b,c) = 0$, then as in the proof of Theorem 2.1, we get $\Phi_i = \pm\Psi_i$, $i \in \mathbb{N}$, $\Psi_i \in \Psi = (\Psi_i)_{i\in\mathbb{N}}$, where Ψ is a HH and this is a contradiction. Thus, $T_n^*(a,b,c) = 0$ and this gives us $f_n(abc) = \sum_{i=1}^{n} f_i(c)\Phi_i(b)\Phi_i(a)$, for all $a,b,c \in R$, $n \in \mathbb{N}$.

Let $W = f_n(abxab)$. Then

$$\begin{aligned} W &= \sum_{i=1}^{n} f_i(ab)\Phi_i(x)\Phi_i(ab) = \pm\sum_{i=1}^{n} f_i(ab)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) = \\ &= f_n(ab)\Phi_n(x)\sum_{l=1}^{n}\Phi_l(b)\Phi_l(a) \pm \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a). \end{aligned}$$

On the other hand,

$$\begin{aligned} W &= f_n(a(bxa)b) = \sum_{i=1}^{n} f_i(b)\Phi_i(bxa)\Phi_i(a) = \pm\sum_{i=1}^{n} f_i(b)\left(\sum_{l=1}^{i}\Phi_l(xa)\Phi_l(b)\right)\Phi_i(a) = \\ &= \pm\sum_{i=1}^{n} f_i(b)\Phi_i(xa)\Phi_i(b)\Phi_i(a) = \pm\sum_{i=1}^{n} f_i(b)\left(\sum_{l=1}^{i}\Phi_l(a)\Phi_l(x)\right)\Phi_i(b)\Phi_i(a) = \\ &= \sum_{i=1}^{n}\sum_{l=1}^{i} f_l(b)\Phi_l(a)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a) = \sum_{l=1}^{n} f_l ab)\Phi_l(a)\Phi_n(x)\sum_{l=1}^{n}\Phi_l(b)\Phi_l(a) = \\ &= \sum_{i=1}^{n-1}\sum_{l=1}^{i} f_l(b)\Phi_l(a)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a). \end{aligned}$$

Comparing both right hand sides of W, we get

$$\begin{aligned} 0 &= \left(f_n(ab) \pm \sum_{l=1}^{n} f_l(b)\Phi_l(a)\right)\Phi_n(x)\sum_{l=1}^{n}\Phi_l(b)\Phi_l(a) + \\ &\quad + \sum_{i=1}^{n-1}\left(f_i(ab) \pm \sum_{l=1}^{i} f_l(b)\Phi_l(a)\right)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a). \end{aligned} \tag{3.11}$$

By the assumption of cases (i). and (ii). and since $T_n^*(a,b,c)=0$, then the last line of equation (3.11) becomes zero. Hence,

$$\left(f_n(ab)\pm\sum_{l=1}^{n} f_l(b)\Phi_l(a)\right)\Phi_n(x)\sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)=0.$$

Since R' is prime, then $f_n(ab)=\pm\sum_{l=1}^{n} f_l(b)\Phi_l(a)$, but this is a contradiction (this means we have only case (i).).

Case II: If $f_m=\pm\Psi_m$, for all $m,n\in\mathbb{N}$, $m<n$ and $\Psi_m\in\Psi=(\Psi_i)_{i\in\mathbb{N}}$ such that Ψ is a GHAH, then by Theorem 2.1 we have either

(i') $\Phi_i=\pm\delta_i$, for all $i\in\mathbb{N}$, $\delta_i\in\delta=(\delta_i)_{i\in\mathbb{N}}$ such that δ is a HH, or

(ii') $\Phi_i=\pm\delta_i$, for all $i\in\mathbb{N}$, $\delta_i\in\delta=(\delta_i)_{i\in\mathbb{N}}$ such that δ is a HAH.

If we have case (i')., then $\Phi_n(abc)=\sum_{i=1}^{n}\Phi_i(a)\Phi_i(b)\Phi_i(c)$, for all $a,b,c\in R$, $n\in\mathbb{N}$. Since Θ is onto and R' is prime, then by Lemma 3.3 we have either $T_n(a,b,c)=0$ or $S_n^*(a,b,c)=0$, for all $a,b,c\in R$, $n\in\mathbb{N}$. If $T_n(a,b,c)=0$, then as in the proof of Case I, we get $\Phi_i=\pm\delta_i$, for all $i\in\mathbb{N}$ and $\delta_i\in\delta=(\delta_i)_{i\in\mathbb{N}}$ such that δ is a HAH and this is a contradiction. Hence $S_n^*(a,b,c)=0$. Again, as in the proof of case I above, we arrive at equation (3.11) and hence by assumption of case II and (i') and since $S_n^*(a,b,c)=0$. Thus, equation (3.11) becomes $\left(f_n(ab)\pm\sum_{l=1}^{n} f_l(a)\Phi_l(b)\right)\Phi_n(x)\sum_{l=1}^{n}\Phi_l(a)\Phi_l(b)=0$, for all $a,b,x\in R$, $n\in\mathbb{N}$, but this is a contradiction with assumption of case II.

Hence, we have case (ii')., which give us $\Phi_n(abc)=\sum_{i=1}^{n}\Phi_i(c)\Phi_i(b)\Phi_i(a)$. Thus, by Lemma 3.3, we have either $S_n(a,b,c)=0$ or $T_n^*(a,b,c)=0$, for all $a,b,c\in R$, $n\in\mathbb{N}$. As in the proof of Case I (ii)., if $S_n(a,b,c)=0$, then $\Phi_i=\pm\Psi_i$, for all $i\in\mathbb{N}$, $\Psi_i\in\Psi=(\Psi_i)_{i\in\mathbb{N}}$ such that Ψ is a HH, but this is a contradiction with assumption (ii'). Therefore, $T_n^*(a,b,c)=0$. As in the proof of case I $(ii.)$, we have

$$0=\left(f_n(ab)\pm\sum_{l=1}^{n} f_l(b)\Phi_l(a)\right)\Phi_n(x)\sum_{l=1}^{n}\Phi_l(b)\Phi_l(a)+\sum_{i=1}^{n-1}\left(f_i(ab)\pm\sum_{l=1}^{i} f_l(b)\Phi_l(a)\right)\Phi_i(x)\sum_{l=1}^{i}\Phi_l(b)\Phi_l(a),$$

for all $a,b,x\in R$, $n\in\mathbb{N}$. By the assumption case II, the last term of the equation above becomes zero. Since R' is prime, then $f_n(ab)=\pm\sum_{l=1}^{n} f_l(b)\Phi_l(a)$, for all $a,b,c\in R$, $n\in\mathbb{N}$ and this means $f_i=\pm\Psi_i$ for all $i\in\mathbb{N}$, $\Psi_i\in\Psi=(\Psi_i)_{i\in\mathbb{N}}$ such that Ψ is a HAH. ■

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