

On Generalized Jordan Higher Homomorphisms

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Dedicated to Patrick Smith and John Clark on their 70th. birthday.

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Abstract. In this paper we extend to the higher homomorphism a well known result proved by Herstein concerning homomorphism in prime rings. We prove results which imply that every Jordan triple higher homomorphism (*resp.* *generalized Jordan triple higher homomorphism*) of a ring R onto a 2-torsion-free prime rings R' is of the form $\Phi_i = \pm\Psi_i$ for all $i \in \mathbb{N}$, where $\Psi_i \in \Psi = (\Psi_i)_{i \in \mathbb{N}}$ which Ψ is either higher homomorphism or higher anti-homomorphism (*resp.* Ψ is either *generalized higher homomorphism* or *generalized higher anti-homomorphism*) of R onto R' .

1 Introduction

The study of additive mappings from one ring R into another ring R' which preserve squares was initiated by Ancochea [1], [2] in connection with problem arising in projective geometry. Hua [9] and Kaplansky [12] took the subject up where Ancochea had left off and pushed his results further. Jacobson and Rickart [10] then proceeded to carry out an extensive study of such functions.

An additive mapping Θ of a ring R into a 2-torsion-free ring R' is called a *Jordan homomorphism* if $\Theta(ab + ba) = \Theta(a)\Theta(b) + \Theta(b)\Theta(a)$ for all $a, b \in R$. A well known result due to Herstein [6] states that every Jordan homomorphism of a ring R onto a prime ring R' of characteristic different from 2 and 3 is either a homomorphism or an anti-homomorphism. Later Smilley [14] provided a brief proof of this result and also same time removed the requirement that the characteristic be different from 3. Suppose R' contains ideals U' and V' with null intersection. Let $\Phi: R \rightarrow U'$ be a homomorphism and $\Psi: R \rightarrow V'$ be an anti-homomorphism. A mapping $\Theta = \Phi + \Psi$ is a so-called *direct sum* of mappings Φ and Ψ . Obviously, Θ is a Jordan homomorphism. According to this construction we see that Herstein's result [6] does not hold in semiprime rings. Moreover, Baxter and Martindale [3] showed by an example that a Jordan homomorphism Θ of a ring R onto a semiprime ring R' is not necessarily a direct sum of a homomorphism and an anti-homomorphism. Although they proved that there always exists an essential ideal E of R such that the restriction of Θ to E is a direct sum of a homomorphism $\Phi: E \rightarrow R'$ and an anti-homomorphism $\Psi: E \rightarrow R'$ [3, Theorem 2.7]. In 1989, Brešar [4, Theorem 2.3] extended this result by showing that E can be chosen so that it is a sum of ideals U and V of R such that Φ vanishes on V and Ψ vanishes on U . Even more, for each $x \in R$ we have $\Theta(ux) = \Theta(u)\Theta(x)$ for all $u \in U$ and $\Theta(vx) = \Theta(v)\Theta(x)$ for all $v \in V$.

An additive mapping Θ of ring R into a ring R' which satisfies $\Theta(aba) = \Theta(a)\Theta(b)\Theta(a)$ for all $a, b \in R$ will be called a *Jordan homomorphism*. An easy computation shows that every Jordan homomorphism is also a Jordan triple homomorphism (see for example [8, Lemma 3.1]). In [7] Herstein proved that a Jordan triple homomorphism Θ of a ring R into a prime ring R' of characteristic different from 2 and 3 is of the form $\Theta = \pm\Phi$, where Φ is a homomorphism or an anti-homomorphism of R onto R' . In his paper Brešar [4, Theorem 3.3] generalized this result by removing the requirement that the characteristic be different from 3. In the same paper, Brešar [4] obtained a more general result where Θ is a Jordan triple homomorphism of a ring R onto a 2-torsion-free semiprime ring R' .

An additive mapping $F: R \rightarrow R'$ is said to be a *generalized Jordan homomorphism* if there exists a Jordan homomorphism $\Theta: R \rightarrow R'$ such that $F(ab + ba) = F(a)\Theta(b) + F(b)\Theta(a)$ for all $a, b \in R$ and Θ is called the *relating Jordan homomorphism*. An additive mapping $F: R \rightarrow R'$ is said to be a *generalized Jordan triple homomorphism* if there exists a Jordan triple

homomorphism $\Theta: R \rightarrow R'$ such that $F(aba) = F(a)\Theta(b)\Theta(a)$, for all $a, b \in R$ and Θ is called the relating Jordan triple homomorphism.

In the present paper our objective is to study the concept of higher homomorphism, Jordan higher homomorphism, Jordan triple higher homomorphism and generalized Jordan triple higher homomorphism.

The rings R and R' used in the whole paper will be all associative ones with center $Z(R)$, and for each $x, y \in R$, the symbol $[x, y]$ will represent the usual commutator $xy - yx$.

2 Preliminary results

Our goal is to study the case of higher homomorphisms. Let \mathbb{N} be the set of all natural numbers and let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a family of additive mappings of a ring R into a ring R' . Then Θ is called:

- (a). a *Higher Homomorphism* (HH, for short) if for each $n \in \mathbb{N}$, $\Phi_n(ab) = \sum_{i=1}^n \Phi_i(a)\Phi_i(b)$, for all $a, b \in R$;
- (b). a *Jordan Higher Homomorphism* (JHH, for short) if for each $n \in \mathbb{N}$, $\Phi_n(ab + ba) = \sum_{i=1}^n \Phi_i(a)\Phi_i(b) + \Phi_i(b)\Phi_i(a)$, for all $a, b \in R$. When R' is 2-torsion-free we define a JHH by merely insisting that $\Phi_n(a^2) = \sum_{i=1}^n (\Phi_i(a))^2$, for all $a \in R$, $n \in \mathbb{N}$;
- (c). a *Jordan Triple Higher Homomorphism* (JTHH, for short) if for each $n \in \mathbb{N}$, $\Phi_n(aba) = \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\phi_i(a)$, for all $a, b \in R$;
- (d). a *Higher Anti-Homomorphism* (HAH, for short) if for each $n \in \mathbb{N}$, $\Phi_n(ab) = \sum_{i=1}^n \Phi_i(b)\Phi_i(a)$, for all $a, b \in R$.

Remark 2.1. Every Higher Homomorphism is a Jordan Higher Homomorphism, but the converse need not be true in general, following example shows:

Example 2.1. Let S be any ring with involution $*$, set $R = S \oplus S$ and $a \in S$ such that $a \in Z(S)$ and $s_1as_2 = 0$ for all $s_1, s_2 \in S$. Let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a family of mappings of R into itself defined by

$$\Phi_n(s, t) = \begin{cases} ((2 - n_a s, (n - 1)t^*), & n = 1, 2, \quad \text{for all } (s, t) \in R; \\ 0, & n \geq 3. \end{cases}$$

Then it is easy to see that Θ is JHH. Since

$$\begin{aligned} (0, (t_1, t_2))^* &= \Phi_2(s_1s_2, t_1t_2) = \Phi_2((s_1, t_1), (s_2, t_2)) \neq \\ &\neq \Phi_1(s_1, t_1)\Phi_1(s_2, t_2) + \Phi_2(s_1, t_1)\Phi_2(s_2, t_2) = \\ &= 0 + (0, t_1^*)(0, t_2^*) = (0, t_1^*t_2^*), \end{aligned}$$

therefore Θ is not HH.

To facilitate our discussion, we shall begin with the following known results:

Lemma 2.1. [4, Lemma 1.1] Let R be a semiprime ring. If $a, b \in R$ are such that $axb + bxa = 0$, for all $x \in R$, then $axb = bxa = 0$, for all $x \in R$. If R is semiprime, then $axb = 0$, for all $x \in R$, implies $bxa = ab = ba = 0$, too.

Lemma 2.2. [4, Lemma 1.2] Let G_1, G_2, \dots, G_n be additive groups, R a 2-torsion-free semiprime ring. Suppose that mappings $S: G_1 \times G_2 \times \dots \times G_n \rightarrow R$ and $T: G_1 \times G_2 \times \dots \times G_n \rightarrow R$ are additive in each argument. If $S(a_1, a_2, \dots, a_n)xT(a_1, a_2, \dots, a_n) = 0$, for all $x \in R$, $a_i \in G_i$, $i = 1, 2, \dots, n$, then $S(a_1, a_2, \dots, a_n)xT(b_1, b_2, \dots, b_n) = 0$, for all $x \in R$, $a_i, b_i \in G_i$, $i = 1, 2, \dots, n$.

Lemma 2.3. [4, Theorem 2.3] Let Θ be a Jordan homomorphism of a ring R onto a 2-torsion-free semiprime ring R' . Then there exist ideals U and V of R such that, for all $x \in R$, $\Theta(ux) =$

$\Theta(u)\Theta(x)$, for all $u \in U$ and $\Theta(vx) = \Theta(v)\Theta(x)$, for all $v \in V$. The ideal $U + V$ is an essential ideal of R , $U \cap V = \ker(\Theta)$, $\Theta(U)$ and $\Theta(V)$ are ideals of R' , $\Theta(U) \cap \Theta(V) = 0$, and $\Theta(U) \oplus \Theta(V)$ is an essential ideal of R' . If $U = \ker(\Theta)$, then Θ is an anti-homomorphism, and $V = \ker(\Theta)$ if and only if Θ is a homomorphism.

Lemma 2.4. [4, Lemma 3.1] Let Θ be a Jordan triple homomorphism of a ring R onto a ring R' . Then $S(a, b, c)\Theta(x)T(a, b, c) + T(a, b, c)\Theta(x)S(a, b, c) = 0$, for all $a, b, c, x \in R$, where $S(a, b, c) = \Theta(abc) - \Theta(a)\Theta(b)\Theta(c)$ and $T(a, b, c) = \Theta(abc) - \Theta(c)\Theta(b)\Theta(a)$.

Lemma 2.5. [4, Theorem 3.3] Let Θ be a Jordan triple homomorphism of a ring R onto a prime ring R' of characteristic different from 2. Then $\Theta = \pm\Phi$, where Φ is a homomorphism or anti-homomorphism of R onto R' .

Lemma 2.6. [13] Let F be a generalized Jordan triple homomorphism of a ring R onto ring R' and let $\Theta: R \rightarrow R'$ be the relating Jordan triple homomorphism. Then $S^*(a, b, c)\Theta(x)T(a, b, c) + T^*(a, b, c)\Theta(x)S(a, b, c) = 0$, for all $a, b, c, x \in R$, where $S^*(a, b, c) = F(abc) - F(a)\Theta(b)\Theta(c)$ and $T^*(a, b, c) = F(abc) - F(c)\Theta(b)\Theta(a)$.

Lemma 2.7. [13] Let F be a generalized Jordan triple homomorphism of a ring R onto a 2-torsion-free prime ring R' . Then $F = \pm\Psi$, where Ψ is either a generalized homomorphism or a generalized anti-homomorphism of R onto R' .

We are going to continue our discussion with the following results which are essential for developing the proof of our main theorem.

Lemma 2.8. Let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a Jordan Higher Homomorphism of R into R' which is a 2-torsion-free ring. Then Θ is a Jordan Triple Higher Homomorphism.

Proof. We have

$$\Phi_n(ab + ba) = \sum_{i=1}^n \Phi_i(a)\Phi_i(b) + \Phi_i(b)\Phi_i(a), \text{ for all } a, b \in R. \tag{2.1}$$

Replacing b by $ab + ba$ in (2.1) and using (2.1), we get

$$\begin{aligned} \Phi_n(a(ab + ba) + (ab + ba)a) &= \sum_{i=1}^n \Phi_i(a) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a) \right) + \\ &+ \sum_{i=1}^n \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a) \right) \Phi_i(a) = \\ &= \sum_{i=1}^n \sum_{l=1}^i \Phi_i(a)\Phi_l(a)\Phi_l(b) + \sum_{i=1}^n \sum_{l=1}^i \Phi_i(a)\Phi_l(b)\Phi_l(a) + \\ &+ \sum_{i=1}^n \sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_i(a) + \sum_{i=1}^n \sum_{l=1}^i \Phi_l(b)\Phi_l(a)\Phi_i(a), \end{aligned}$$

and hence

$$\Phi_n(a(ab + ba) + (ab + ba)a) = \sum_{i=1}^n (\Phi_i(a))^2 \Phi_i(b) + 2 \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^n \Phi_i(b)(\Phi_i(a))^2.$$

On the other hand,

$$\Phi_n(a(ab + ba) + (ab + ba)a) = \Phi_n(a^2b + ba^2) + 2\Phi_n(aba) = \sum_{i=1}^n (\Phi_i(a^2)\Phi_i(b) + \Phi_i(b)\Phi_i(a^2)) + 2\Phi_n(aba).$$

On comparing the last two equations, we get the required result. ■

If we linearize $\Phi_n(aba) = \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(a)$ on a , we get the following

Corollary 2.1. Let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a Jordan Triple Higher Homomorphism of a ring R into ring R' which is a 2-torsion-free. Then $\phi_n(abc + cba) = \sum_{i=1}^n (\Phi_i(a)\Phi_i(b)\Phi_i(c) + \Phi_i(c)\Phi_i(b)\Phi_i(a))$, for all $a, b, c \in R, n \in \mathbb{N}$.

Let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a Jordan Triple Higher Homomorphism (JTHH) of a ring R into ring R' for every fixed $n \in \mathbb{N}$ and for each $a, b, c \in R$. For the purpose of this section, we shall write:

- (i). $(a^b)^n = \Phi_n(ab) - \sum_{i=1}^n \Phi_i(a)\Phi_i(b)$ and $(a_b)^n = \Phi_n(ab) - \sum_{i=1}^n \Phi_i(b)\Phi_i(a)$;
- (ii). $S_n(a, b, c) = \Phi_n(abc) - \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(c)$ and $T_n(a, b, c) = \Phi_n(abc) - \sum_{i=1}^n \Phi_i(c)\Phi_i(b)\Phi_i(a)$.

Lemma 2.9. Let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a Jordan Triple Higher Homomorphism of R into a ring R' . Then $(a^b)^n \Phi(x)(a_b)^n + (a_b)^n \Phi(x)(a^b)^n$, for all $x, a, b \in R, n \in \mathbb{N}$.

Proof. Since $(a^b)\Phi(x)(a_b) + (a_b)\Phi(x)(a^b)$, for all $x, a, b \in R$, by induction we can assume that $(a^b)^m \Phi_m(x)(a_b)^m + (a_b)^m \Phi_m(x)(a^b)^m$, for all $x, a, b \in R, m, n \in \mathbb{N}, m < n$.

Let $W = abxba + baxab$. Then

$$\begin{aligned} \Phi_n(W) &= \Phi_n(a(bxb)a + b(axa)b) = \sum_{i=1}^n \Phi_i(a)\Phi_i(bxb)\Phi_i(a) + \sum_{i=1}^n \Phi_i(b)\Phi_i(axa)\Phi_i(b) = \\ &= \sum_{i=1}^n \Phi_i(a) \left(\sum_{l=1}^i \Phi_l(b)\Phi_l(x)\Phi_l(b) \right) \Phi_i(a) + \sum_{i=1}^n \Phi_i(b) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(x)\Phi_l(a) \right) \Phi_i(b) = \\ &= \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^n \Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(a)\Phi_i(b) = \\ &= \left(\sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(x) \right) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) + \left(\sum_{i=1}^n \Phi_i(b)\Phi_i(a)\Phi_i(x) \right) \sum_{l=1}^i \Phi_l(a)\Phi_l(b) = \\ &= \Phi_n(a)\Phi_n(b)\Phi_n(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) + \left(\sum_{i=1}^{n-1} \Phi_i(a)\Phi_i(b)\Phi_i(x) \right) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) + \\ &\quad + \Phi_n(b)\Phi_n(a)\Phi_n(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b) + \left(\sum_{i=1}^{n-1} \Phi_i(b)\Phi_i(a)\Phi_i(x) \right) \sum_{l=1}^i \Phi_l(a)\Phi_l(b). \end{aligned}$$

On the other hand, $\Phi_n(W) = \Phi_n((ab)x(ba) + (ba)x(ab))$. Thus, by Corollary 2.1 and the definition of HH, we get

$$\begin{aligned} \Phi_n(W) &= \sum_{i=1}^n \Phi_i(ab)\Phi_i(x) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a) - \Phi_i(ab) \right) + \\ &\quad + \sum_{i=1}^n \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a) - \Phi_i(ab) \right) \Phi_i(x)\Phi_i(ab) = \\ &= \sum_{i=1}^n \Phi_i(ab)\Phi_i(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b) + \sum_{i=1}^n \Phi_i(ab)\Phi_i(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) - \sum_{i=1}^n \Phi_i(ab)\Phi_i(x)\Phi_i(ab) + \\ &\quad + \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \sum_{i=1}^n \Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab) - \sum_{i=1}^n \Phi_i(ab)\Phi_i(x)\Phi_i(ab) = \\ &= - \sum_{i=1}^n \Phi_i(ab)\Phi_i(x) \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) - \sum_{i=1}^n \Phi_i(ab)\Phi_i(x) \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) + \\ &\quad + \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \sum_{i=1}^n \Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab) = \\ &= - \sum_{i=1}^n \Phi_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^n \Phi_i(ab)\Phi_i(x)(a_b)^i + \\ &\quad + \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \sum_{i=1}^n \Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab) = \\ &= -\Phi_n(ab)\Phi_n(x)(a^b)^n - \sum_{i=1}^{n-1} \Phi_i(ab)\Phi_i(x)(a^b)^i - \Phi_n(ab)\Phi_n(x)(a_b)^n - \sum_{i=1}^{n-1} \Phi_i(ab)\Phi_i(x)(a_b)^i + \\ &\quad + \Phi_n(a)\Phi_n(b)\Phi_n(x)\Phi_n(ab) + \sum_{i=1}^{n-1} \Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \Phi_n(b)\Phi_n(a)\Phi_n(x)\Phi_n(ab) + \\ &\quad + \sum_{i=1}^{n-1} \Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab). \end{aligned}$$

On comparing the right hand side of $\Phi_n(W)$, we find that

$$\begin{aligned} 0 &= -\Phi_n(ab)\Phi_n(x)\Phi_n(a^b)^n - \Phi_n(ab)\Phi_n(x)(a_b)^n + \Phi_n(a)\Phi_n(b)\Phi_n(x) \left(\Phi_n(ab) - \sum_{l=1}^n \Phi_l(b)\Phi_l(a) \right) + \\ &\quad + \Phi_n(b)\Phi_n(a)\Phi_n(x) \left(\Phi_n(ab) - \sum_{l=1}^n \Phi_l(a)\Phi_l(b) \right) - \sum_{i=1}^{n-1} \Phi_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^{n-1} \Phi_i(ab)\Phi_i(x)(a_b)^i + \\ &\quad + \sum_{i=1}^{n-1} \Phi_i(a)\Phi_i(b)\Phi_i(x) \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) + \sum_{i=1}^{n-1} \Phi_i(b)\Phi_i(a)\Phi_i(x) \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) = \\ &= -\Phi_n(ab)\Phi_n(x)(a^b)^n - \Phi_n(ab)\Phi_n(x)(a_b)^n + \Phi_n(a)\Phi_n(b)\Phi_n(x)(a_b)^n + \\ &\quad + \Phi_n(b)\Phi_n(a)\Phi_n(x)(a^b)^n - \sum_{i=1}^{n-1} \Phi_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^{n-1} \Phi_i(ab)\Phi_i(x)(a_b)^i + \\ &\quad + \sum_{i=1}^{n-1} \Phi_i(a)\Phi_i(b)\Phi_i(x)(a_b)^i + \sum_{i=1}^{n-1} \Phi_i(b)\Phi_i(a)\Phi_i(x)(a^b)^i = \\ &= -(a_b)^n \Phi_n(x)(a^b)^n - (a^b)^n \Phi_n(x)(a_b)^n - \sum_{i=1}^{n-1} (a_b)^i \Phi_i(x)(a^b)^i - \sum_{i=1}^{n-1} (a^b)^i \Phi_i(x)(a_b)^i = \\ &= ((a_b)^n \Phi_n(x)(a^b)^n + (a^b)^n \Phi_n(x)(a_b)^n) - \sum_{i=1}^{n-1} ((a^b)^i \Phi_i(x)(a_b)^i + (a_b)^i \Phi_i(x)(a^b)^i). \end{aligned}$$

By our hypothesis, we have $(a^b)^n \Phi_n(x)(a_b)^n + (a_b)^n \Phi_n(x)(a^b)^n = 0$, for all $a, b, x \in R, n \in \mathbb{N}$. ■

As an immediate consequence of Lemma 2.1, Lemma 2.2 and Lemma 2.9, we get the following

Corollary 2.2. Let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a Jordan Higher Homomorphism of a ring R onto a 2-torsion-free semiprime ring R' . Then $(a^b)^n x'(ca)^n = 0$ for all $a, b, c, d \in R, x' \in R'$ and $n \in \mathbb{N}$.

Thus, immediately from Corollary 2.2 we get the following

Remark 2.2. Let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a Jordan Higher Homomorphism of a ring R onto a 2-torsion-free prime ring R' . Then either Θ is a Higher Homomorphism or Θ is a Higher Anti-Homomorphism.

Proposition 2.1. Let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a Jordan Higher Homomorphism of a ring R onto a 2-torsion-free semiprime ring R' such that $[\Phi_i(a), \Phi_i(b)] = 0$, for all $a, b \in R, n \in \mathbb{N}, i < n$. Then there exist ideals U_n and V_n of R , for all $n \in \mathbb{N}$, such that:

- (i). $\Phi_n(ux) = \sum_{i=1}^n \Phi_i(u)\Phi_i(x)$, for all $u \in U_n, n \in \mathbb{N}, x \in R$ and
- (ii). $\Phi_n(vx) = \sum_{i=1}^n \Phi_i(x)\Phi_i(v)$, for all $v \in V_n, n \in \mathbb{N}, x \in R$.

Proof. The proposition is true for $n = 1$, by Lemma 2.3. Thus, we can assume that there exist ideals $U_m = \bigcap_{i=1}^m \Phi_i^{-1}(ann(V'_{i-1}))$ and $V_m = \bigcap_{i=1}^m \Phi_i^{-1}(ann(ann(V'_{i-1})))$ for all $m, n \in \mathbb{N}, m < n$, where V'_{i-1} is the ideal generated by the set $\{(a^b)^j \mid j = 1, \dots, i < n, a, b \in R\}$ such that, for all $x \in R, \Phi_n(ux) = \sum_{i=1}^m \Phi_i(u)\Phi_i(x)$, for all $u \in U_m$ and $\Phi_m(vx) = \sum_{i=1}^m \Phi_i(v)\Phi_i(x)$, for all $v \in V_m$.

Let V'_{n-1} be the ideal generated by the set $\{(a^b)^j \mid i = 1, \dots, n, a, b \in R\}$. Thus, by Lemma 2.9 and since Φ_i is onto, for all $i \in \mathbb{N}$ and hence by Lemma 2.1, we have $(a^b)^n(a_b)^n = (a_b)^n(a^b)^n = 0$, for all $a, b \in R, n \in \mathbb{N}$. We claim that $(a_b)^i(a^b)^j = 0$, for all $a, b \in R, i, j \in \mathbb{N}$ and $i \neq j$ to show that:

Case I: if $i < j$, then

$$\begin{aligned} (a_b)^i(a_b)^j &= \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) \left(\Phi_j(ab) - \sum_{t=1}^j \Phi_t(a)\Phi_t(b) \right) = \\ &= -\Phi_i(ab) \sum_{t=1}^j \Phi_t(a)\Phi_t(b) + \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \sum_{t=1}^j \Phi_t(a)\Phi_t(b)(a_b)^i(a_b)^j = \quad (2.2) \\ &= -(a_n)^i \sum_{t=1}^j \Phi_t(a)\Phi_t(b). \end{aligned}$$

On the other hand, since Θ is JHH, we find that

$$\begin{aligned} (a_b)^i(a_b)^j &= \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) \left(-\Phi_j(ba) + \sum_{t=1}^j \Phi_t(b)\Phi_t(a) \right) = \\ &= \Phi_i(ab) \sum_{t=1}^j \Phi_t(b)\Phi_t(a) - \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \sum_{t=1}^j \Phi_t(b)\Phi_t(a)(a_b)^i(a_b)^j = \quad (2.3) \\ &= -(a_n)^i \sum_{t=1}^j \Phi_t(b)\Phi_t(a). \end{aligned}$$

Now adding (2.2) and (2.3), we get

$$(a_b)^i(a_b)^j = (a_b)^i \left(-\sum_{t=1}^j \Phi_t(a)\Phi_t(b) + \sum_{t=1}^j \Phi_t(b)\Phi_t(a) \right) = (a_b)^i \sum_{t=1}^j [\Phi_t(b), \Phi_t(a)] = 0.$$

Since R' is 2-torsion-free, we obtain $(a_b)^i(a^b)^j = 0$ for all $a, b \in R$ and $i, j \in \mathbb{N}, i < j$.

Using the same arguments as above one can show that:

Case II: If $i > j$, then $(a_b)^i(a^b)^j = 0$, for all $a, b \in R$ and $i, j \in \mathbb{N}, i > j$. Therefore, we conclude that $\{(a_b)^i \mid i = 1, \dots, n, a, b \in R\} \subseteq ann(V'_{n-1})$ and $\{(a^b)^i \mid i = 1, \dots, n, a, b \in R\} \subseteq ann(ann(V'_{n-1}))$.

Now we set

$$\begin{aligned} U_n &= \bigcap_{i=1}^n \Phi_i^{-1}(ann(V'_{n-1})) = \Phi_n^{-1}(ann(V'_{n-1})) \cap \bigcap_{i=1}^{n-1} \Phi_i^{-1}(ann(V'_{i-1})) \text{ and} \\ V_n &= \bigcap_{i=1}^n \Phi_i^{-1}(ann(ann(V'_{n-1}))) = \Phi_n^{-1}(ann(ann(V'_{n-1}))) \cap \bigcap_{i=1}^{n-1} \Phi_i^{-1}(ann(ann(V'_{i-1}))). \end{aligned}$$

Given $u \in U_n$, for all $y \in R, x \in R'$, we have

$$(u^y)^n x' (u^y)^n = (u^y)^n x' ((u^y)^n - (u_y)^n) = (u^y)^n x' \left(\sum_{i=1}^n \Phi_i(y)\Phi_i(u) - \sum_{i=1}^n \Phi_i(u)\Phi_i(y) \right) = 0.$$

Hence, by semiprimness of R' forces that $(u^y)^n = 0$. That is,

$$\Phi_n(uy) = \sum_{i=1}^n \Phi_i(u)\Phi_i(y), \text{ for all } u \in U_n, y \in R.$$

Now to prove that U_n is an ideal of R , it is enough to show that $\Phi_n^{-1}(ann(V'_{n-1}))$ is an ideal of R .

Let $u \in U_n, x \in R$. Since $\Phi_n(ux) = \sum_{i=1}^n \Phi_i(u)\Phi_i(x)$ and $\Phi_n(u) \in ann(V'_{n-1})$, it remains to show that $\Phi_m(u) \in ann(V'_{n-1})$, for all $u \in U_n, m < n$ and hence

$$\Phi_m(u)(a^b)^n = \Phi_m(u) (-\Phi_m(a)\Phi_m(b)). \tag{2.4}$$

On the other hand,

$$\Phi_m(u)(a^b)^n = \Phi_m(u) (\Phi_m(b)\Phi_m(a)). \tag{2.5}$$

Thus, by adding (2.4) and (2.5) and our hypothesis we obtain $\Phi_m(u)(a^b)^n = 0$. Similarly, we can prove that $\Phi_n(vy) = \sum_{i=1}^n \Phi_i(y)\Phi_i(v)$, for all $v \in V_n, y \in R$ and therefore V_n is an ideal of R . ■

Lemma 2.10. Let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a Jordan Triple Higher Homomorphism of a ring R into a ring R' . Then $S_n(a, b, c)\Phi_n(x)T_n(a, b, c) + T_n(a, b, c)\Phi_n(x)S_n(a, b, c) = 0$, for all $a, b, c, x \in R, n \in \mathbb{N}$.

Proof. By Lemma 2.4, the statement is true for $n = 1$. Then, by induction, we assume that

$$S_m(a, b, c)\Phi_m(x)T_m(a, b, c) + T_m(a, b, c)\Phi_m(x)S_m(a, b, c) = 0,$$

for all $a, b, c, x \in R, m \in \mathbb{N}$ and $m < n$.

Let $W = \Phi_n(abcxcba + cbaabc)$. Then, by the definition of JTHH, we have

$$\begin{aligned} W &= \Phi_n(a(bcxcba) + c(baabc)) = \sum_{i=1}^n \Phi_i(a)\Phi_i(bcxcba)\Phi_i(a) + \sum_{i=1}^n \Phi_i(c)\Phi_i(baabc)\Phi_i(c) = \\ &= \sum_{i=1}^n \Phi_i(a) \left(\sum_{l=1}^i \Phi_l(b)\Phi_l(cxc)\Phi_l(b) \right) \Phi_i(a) + \sum_{i=1}^n \Phi_i(c) \left(\sum_{l=1}^i \Phi_l(b)\Phi_l(axa)\Phi_l(b) \right) \Phi_i(c) = \\ &= \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(cxc)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^n \Phi_i(c)\Phi_i(b)\Phi_i(axa)\Phi_i(b)\Phi_i(c) = \\ &= \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(c)\Phi_i(x)\Phi_i(c)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^n \Phi_i(c)\Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(a)\Phi_i(b)\Phi_i(c) = \\ &= \sum_{i=1}^n \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) \right) \Phi_i(x) \left(\sum_{l=1}^i \Phi_l(c)\Phi_l(b)\Phi_l(a) \right) + \\ &\quad + \sum_{i=1}^n \left(\sum_{l=1}^i \Phi_l(c)\Phi_l(b)\Phi_l(a) \right) \Phi_i(x) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) \right). \end{aligned}$$

On the other hand, by Corollary 2.1, we have

$$\begin{aligned} W &= \Phi_n((abc)x(cba) + (cba)x(abc)) = \sum_{i=1}^n \Phi_i(abc)\Phi_i(x)\Phi_i(cba) + \sum_{i=1}^n \Phi_i(cba)\Phi_i(x)\Phi_i(abc) = \\ &= \sum_{i=1}^n \Phi_i(abc)\Phi_i(x) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) + \Phi_l(c)\Phi_l(b)\Phi_l(a) - \Phi_i(abc) \right) + \\ &\quad + \sum_{i=1}^n \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) + \Phi_l(c)\Phi_l(b)\Phi_l(a) - \Phi_i(abc) \right) (\Phi_i(x)\Phi_i(abc) = \\ &= \sum_{i=1}^n \Phi_i(abc)\Phi_i(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) + \sum_{i=1}^n \Phi_i(abc)\Phi_i(x) \sum_{l=1}^i \Phi_l(c)\Phi_l(b)\Phi_l(a) - \\ &\quad - \sum_{i=1}^n \Phi_i(abc)\Phi_i(x)\Phi_i(abc) + \sum_{i=1}^n \sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c)\Phi_i(x)\Phi_i(abc) + \\ &\quad + \sum_{i=1}^n \sum_{l=1}^i \Phi_l(c)\Phi_l(b)\Phi_l(a)\Phi_i(x)\Phi_i(abc) - \sum_{i=1}^n \Phi_i(abc)\Phi_i(x)\Phi_i(abc). \end{aligned}$$

On comparing the right hand side of W and by our hypothesis, we get the required result. ■

Theorem 2.1. Let $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be a Jordan Triple Higher Homomorphism of a ring R onto a 2-torsion-free prime ring R' . Then $\Phi_i = \pm\Psi_i$, for all $i \in \mathbb{N}$, where $\Psi_i \in \Psi = (\Psi_i)_{i \in \mathbb{N}}$, which Ψ is either a Higher Homomorphism or a Higher Anti-Homomorphism of R onto R' .

Proof. We proceed by induction on $n \in \mathbb{N}$. By Lemma 2.5, the statement of the Theorem is true for $n = 1$. Now suppose that $\Phi_m = \pm\Psi_m$ for $m < n, m, n \in \mathbb{N}$. Then we have two cases:

Case I: If $\Phi_m = \pm\Psi_m$, for all $m < n$, where $\Psi_i \in \Psi$ such that Ψ is a HH. Since Φ_i is onto for all $i \in \mathbb{N}$ and R' is prime, then by Lemma 2.10, either $S_n(a, b, c) = 0$ or $T_n(a, b, c) = 0$, for all $a, b, c \in R, n \in \mathbb{N}$. If $T_n(a, b, c) = 0$, then $\Phi_n(abc) = \sum_{i=1}^n \Phi_i(c)\Phi_i(b)\Phi_i(a)$ and hence

$$\Phi_n(abxab) = \Phi_n((ab)x(ab)) = \sum_{i=1}^n \Phi_i(ab)\Phi_i(x)\Phi_i(ab). \tag{2.6}$$

On the other hand,

$$\Phi_n(abxab) = \Phi_n(a(bxa)b) = \sum_{i=1}^n \Phi_i(b)\Phi_i(bxa)\Phi_i(a) = \sum_{i=1}^n \Phi_i(b) \sum_{l=1}^i \Phi_l(a)\Phi_l(x)\Phi_l(b)\Phi_i(a).$$

That is,

$$\Phi_n(abxab) = \sum_{i=1}^n \sum_{l=1}^i \Phi_l(b)\Phi_l(a)\Phi_l(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a). \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$\begin{aligned} 0 &= \sum_{i=1}^n \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) \Phi_i(x) \left(\Phi_i(ab) + \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) + \\ &+ \sum_{i=1}^n \left(\Phi_i(ab) + \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) \Phi_i(x) \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) = \\ &= \left(\Phi_n(ab) - \sum_{l=1}^n \Phi_l(b)\Phi_l(a) \right) \Phi_n(x) \left(\Phi_n(ab) + \sum_{l=1}^n \Phi_b\Phi_l(a) \right) + \\ &+ \left(\Phi_n(ab) + \sum_{l=1}^n \Phi_l(b)\Phi_l(a) \right) \Phi_n(x) \left(\Phi_n(ab) - \sum_{l=1}^n \Phi_b\Phi_l(a) \right) + \\ &+ \sum_{i=1}^{n-1} \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) \Phi_i(x) \left(\Phi_i(ab) + \sum_{l=1}^i \Phi_b\Phi_l(a) \right) + \\ &+ \sum_{i=1}^{n-1} \left(\Phi_i(ab) + \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) \Phi_i(x) \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_b\Phi_l(a) \right). \end{aligned} \tag{2.8}$$

Since $\Phi_m = \pm\Psi_m$, for all $m < n$ and $\Psi_m \in \Psi$, where $\Psi_m \in \Psi$ is a HH, then the last two lines of equation (2.8) can be reduced to

$$\begin{aligned} &\sum_{i=1}^{n-1} (\Phi_i(ab) \mp \Phi_i(ba)) \Phi_i(x) (\Phi_i(ab) \pm \Phi_i(ba)) + \sum_{i=1}^{n-1} (\Phi_i(ab) \pm \Phi_i(ba)) \Phi_i(x) (\Phi_i(ab) \mp \Phi_i(ba)) = \\ &= 2 \sum_{i=1}^{n-1} \phi_i(ab)\Phi_i(x)\Phi_i(ab) - \Phi_i(ba)\Phi_i(x)\Phi_i(ba). \end{aligned} \tag{2.9}$$

Since $T_n(a, b, c) = 0$ and $\Phi_m = \pm\Psi_m$, for all $m < n$, where $\Psi_m \in \Psi$ such that $\Psi_m \in \Psi$ is a HH. Thus,

$$\begin{aligned} \sum_{i=1}^{n-1} \Phi_i(ab)\Phi_i(x)\Phi_i(ab) &= \sum_{i=1}^{n-1} \Phi_i(a) \sum_{l=1}^i \Phi_l(b)\Phi_l(x)\Phi_l(a)\Phi_l(b) = \sum_{i=1}^{n-1} \Phi_i(a)\Phi_i(a) \sum_{l=1}^i \Phi_l(b)\Phi_l(x)\Phi_l(b) = \\ &= \sum_{i=1}^{n-1} \Phi_i(a) \sum_{l=1}^i \Phi_l(a)\Phi_l(x)\Phi_l(b)\Phi_i(b) = \sum_{i=1}^{n-1} \sum_{l=1}^i \Phi_l(a)\Phi_l(a)\Phi_l(x)\Phi_l(b)\Phi_i(b) = \\ &= \sum_{i=1}^{n-1} \Phi_i(xaa)\Phi_i(b)\Phi_i(b) = \sum_{i=1}^{n-1} \sum_{l=1}^i \Phi_l(x)\Phi_l(a)\Phi_l(a)\Phi_l(b)\Phi_i(b) = \\ &= \sum_{i=1}^{n-1} \Phi_i(x)\Phi_i(a) \sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(b) = \sum_{i=1}^l \Phi_i(x)\Phi_i(a)\Phi_i(bba) = \\ &= \sum_{i=1}^{n-1} \Phi_i(x)\Phi_i(a) \sum_{l=1}^i \Phi_l(b)\Phi_l(b)\Phi_l(a) = \sum_{i=1}^{n-1} \sum_{l=1}^i \Phi_l(x)\Phi_l(a)\Phi_l(b)\Phi_l(b)\Phi_l(a) = \\ &= \sum_{i=1}^{n-1} \Phi_i(bax)\Phi_i(b)\Phi_i(a) = \sum_{i=1}^{n-1} \sum_{l=1}^i \Phi_l(b)\Phi_l(a)\Phi_l(x)\Phi_l(b)\Phi_l(a) = \\ &= \sum_{i=1}^{n-1} \sum_{l=1}^i \Phi_l(b)\Phi_l(a)\Phi_l(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) = \sum_{i=1}^{n-1} \Phi_i(ba)\Phi_i(x)\Phi_i(ba). \end{aligned}$$

Then equation (2.9) is equal to zero and therefore equation (2.8) becomes

$$\begin{aligned} & \left(\Phi_n(ab) + \sum_{l=1}^n \Phi_l(b)\Phi_l(a) \right) \Phi_n(x) \left(\Phi_n(ab) - \sum_{l=1}^n \Phi_l(b)\Phi_l(a) \right) + \\ & + \left(\Phi_n(ab) - \sum_{l=1}^n \Phi_l(b)\Phi_l(a) \right) \Phi_n(x) \left(\Phi_n(ab) + \sum_{l=1}^n \Phi_l(b)\Phi_l(a) \right) = 0. \end{aligned}$$

Since Φ_i is onto, for all $i \in \mathbb{N}$ and R' is prime, then either $\Phi_n(ab) = \sum_{l=1}^n \Phi_l(b)\Phi_l(a)$ or $\Phi_n(ab) = -\sum_{l=1}^n \Phi_l(b)\Phi_l(a)$. Thus Θ is a HAH or a negative of a HAH and this is a contradiction. Hence, $S_n(a, b, c) = 0$, for all $a, b, c \in R, n \in \mathbb{N}$ and this lead us to

$$\begin{aligned} \Phi_n(abxab) &= \sum_{i=1}^n \Phi_i(a)\Phi_i(bxa)\Phi_i(b) = \sum_{i=1}^n \Phi_i(a) \sum_{l=1}^i \Phi_l(b)\Phi_l(x)\Phi_l(a)\Phi_l(b) = \\ &= \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(a)\Phi_i(b) = \\ &= \sum_{i=1}^n \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) \Phi_i(x) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right). \end{aligned} \tag{2.10}$$

On the other hand,

$$\Phi_n(abxab) = \sum_{i=1}^n \Phi_i(ab)\Phi_i(x)\Phi_i(ab). \tag{2.11}$$

Now, comparing (2.10) and(2.11), we get

$$\begin{aligned} 0 &= \sum_{i=1}^n \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) \Phi_i(x) \left(\Phi_i(ab) + \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) + \\ &+ \sum_{i=1}^n \left(\Phi_i(ab) + \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) \Phi_i(x) \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) = \\ &= \left(\Phi_n(ab) - \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) \Phi_n(x) \left(\Phi_n(ab) + \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) + \\ &+ \left(\Phi_n(ab) + \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) \Phi_n(x) \left(\Phi_n(ab) - \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) + \\ &+ \sum_{i=1}^{n-1} \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) \Phi_i(x) \left(\Phi_i(ab) + \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) + \\ &+ \sum_{i=1}^{n-1} \left(\Phi_i(ab) + \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) \Phi_i(x) \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right). \end{aligned} \tag{2.12}$$

Since $\Phi_m = \pm\Psi_m$, for all $m < n$, where $\Psi_m = \Psi$ and Ψ is a HH, then the last term of equation (2.12) are equal to zero. Therefore equation (2.12) becomes

$$\begin{aligned} & \left(\Phi_n(ab) - \sum_{l=1}^n \Phi_l(a)\Phi_l(b) \right) \Phi_n(x) \left(\Phi_n(ab) + \sum_{l=1}^n \Phi_l(a)\Phi_l(b) \right) + \\ & + \left(\Phi_n(ab) + \sum_{l=1}^n \Phi_l(a)\Phi_l(b) \right) \Phi_n(x) \left(\Phi_n(ab) - \sum_{l=1}^n \Phi_l(a)\Phi_l(b) \right) = 0. \end{aligned}$$

Since Φ_i is onto, for all $i \in \mathbb{N}$ and R' is prime, then by Lemma 2.2, we get either $\Phi_n(ab) = \sum_{l=1}^n \Phi_l(a)\Phi_l(b)$ or $\Phi_n(ab) = -\sum_{l=1}^n \Phi_l(a)\Phi_l(b)$, for all $a, b \in R$ and $n \in \mathbb{N}$. That is, Θ is a HH or a negative of a HH.

Using similar arguments as used Case I we can prove that:

Case II: If $\Phi_m = \pm\Psi_m$, for all $m < n$, where $\Psi_m \in \Psi$ and Ψ is a HAH, then $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ is also a HAH. ■

3 Generalized Jordan Higher Homomorphism

Let \mathbb{N} be the set of all natural numbers and let $F = (f_i)_{i \in \mathbb{N}}$ be a family of additive mappings of a ring R into ring R' . Then F is said to be:

- a *Generalized Higher Homomorphism* (GHH, for short), if there exists a higher homomorphism $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ such that $f_n(ab) = \sum_{i=1}^n f_i(a)\Phi_i(b)$, for all $a, b \in R$ and Θ is called the *relating higher homomorphism* (RHH, for short);
- a *Generalized Jordan Higher Homomorphism* (GJHH, for short), if there exists a Jordan higher homomorphism $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ such that $f_n(ab + ba) = \sum_{i=1}^n f_i(a)\Phi_i(b) + f_i(b)\Phi_i(a)$, for all $a, b \in R$ and Θ is called the *relating Jordan higher homomorphism* (RJHH, for short). If R' is 2-torsion-free, then the definition of GJHH is equivalent to the following condition: $f_n(a^2) = \sum_{i=1}^n f_i(a)\Phi_i(b)$, for all $a, b \in R$;
- a *Generalized Jordan Triple Homomorphism* (GJTTH, for short), if there exists a Jordan triple higher homomorphism $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ such that $f_n(aba) = \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(a)$, for all $a, b \in R$ and Θ is called the *relating Jordan triple higher homomorphism* (RJTTH, for short);
- a *Generalized Higher Anti-Homomorphism* (GAH, for short), if there exists a higher anti-homomorphism $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ such that $f_n(ab) = \sum_{i=1}^n f_i(b)\Phi_i(a)$, for all $a, b \in R$ and Θ is called the *relating higher anti-homomorphism* (RAAH, for short).

Remark 3.1. It is clear that every Generalized Higher Homomorphism is a Generalized Jordan Higher Homomorphism, but the converse need not be true in general. Following example shows that:

Example 3.1. Let S be any ring with involution $*$, set $R = S \oplus S$ and $a \in S$ such that $a \in Z(S)$ and $s_1 a s_2 = 0$, for all $s_1, s_2 \in S$. Let $F = (f_i)_{i \in \mathbb{N}}$ be a family of mappings of R into R defined by

$$f_n(s, t) = \begin{cases} -(2 - n_a s, (n - 1)t^*), & n = 1, 2, \quad \text{for all } (s, t) \in R; \\ 0, & n \geq 3. \end{cases}$$

Then there exists a JHH which is defined in Example 2.1. It becomes clear that F is a GJHH, but not a GHH.

Lemma 3.1. Let $F = (f_i)_{i \in \mathbb{N}}$ be a Generalized Jordan Higher Homomorphism of a ring R into a 2-torsion-free ring R' . Then:

- (i) F is a Generalized Jordan Triple Higher Homomorphisms of R into R' ;
- (ii) $f_n(abc + cba) = \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(c) + f_i(c)\Phi_i(b)\Phi_i(a)$, for all $a, b \in R, n \in \mathbb{N}$ and $\Phi_i \in \Theta = (\Phi_i)_{i \in \mathbb{N}}$, where Θ is the Related Jordan Higher Homomorphism.

Proof. (i). We have

$$f_n(ab + ba) = \sum_{i=1}^n f_i(a)\Phi_i(b) + f_i(b)\Phi_i(a). \tag{3.1}$$

Replacing a by $ab + ba$ in (3.1), we get

$$\begin{aligned} f_n(a(ab + ba) + (ab + ba)a) &= \sum_{i=1}^n f_i(a)\Phi_i(ab + ba) + f_i(ab + ba)\Phi_i(a) = \\ &= \sum_{i=1}^n f_i(a) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a) \right) + \sum_{i=1}^n \left(\sum_{l=1}^i f_l(a)\Phi_l(b) + f_l(a)\Phi_l(a) \right) \Phi_i(a) = \tag{3.2} \\ &= \sum_{i=1}^n f_i(a)\Phi_i(a)\Phi_i(b) + \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^n f_i(b)\Phi_i(a)\Phi_i(b). \end{aligned}$$

On the other hand,

$$f_n(a(ab + ba) + (ab + ba)a) = \sum_{i=1}^n f_i(a^2)\Phi_i(b) + f_i(b)\Phi_i(a^2) + 2f_n(aba). \tag{3.3}$$

Compare (3.2) and (3.3), to get $f_n(aba) = \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(a)$, for all $a, b \in R, n \in \mathbb{N}$. Therefore F is a GJTHH.

(ii). Replacing a by $a + b$ in (i)., we get

$$\begin{aligned} f_n((a + c)b(a + c)) &= \sum_{i=1}^n f_i(a + c)\Phi_i(b)\Phi_i(a + c) = \\ &= \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(a) + f_i(a)\Phi_i(b)\Phi_i(c) + f_i(c)\Phi_i(b)\Phi_i(a) + f_i(c)\Phi_i(b)\Phi_i(c). \end{aligned} \tag{3.4}$$

On the other hand,

$$f_n((a + c)b(a + c)) = f_n(aba) + f_n(abc + cba) + f_n(cba). \tag{3.5}$$

By comparing (3.4) and (3.5), we get $f_n(abc + cba) = \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(c) + f_i(c)\Phi_i(b)\Phi_i(a)$, for all $a, b, c \in R, n \in \mathbb{N}$. ■

Remark 3.2. Let $F = (f_i)_{i \in \mathbb{N}}$ be a Generalized Jordan Higher Homomorphism of a ring R into a ring R' and $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be the Related Jordan Higher Homomorphism. We shall write, for all $a, b \in R, n \in \mathbb{N}$:

- $(a^{b*})^n = f_n(ab) - \sum_{i=1}^n f_i(a)\Phi_i(b)$;
- $(a_b^*)^n = f_n(ab) - \sum_{i=1}^n f_i(b)\Phi_i(a)$.

Note that if $(a^{b*})^n = 0$ (resp. $(a_b^*)^n = 0$), then F is a Generalized Higher Homomorphism (resp. a Generalized Higher Anti-Homomorphism).

Lemma 3.2. Let $F = (f_i)_{i \in \mathbb{N}}$ be a Generalized Jordan Higher Homomorphism of a ring R into a ring R' . Then

$$(a^{b*})^n \Phi_n(x)(a_b)^n + (a_b^*)^n \Phi_n(x)(a^b)^n,$$

for all $a, b, x \in R, n \in \mathbb{N}$, where $\Phi_i \in \Theta = (\Phi_i)_{i \in \mathbb{N}}$ such that Θ is the Related Jordan Higher Homomorphism.

Proof. We proceed by induction on $n \in \mathbb{N}$. If $n = 1$, let $W = abxba + baxab$. Since F is a GJHH, then

$$\begin{aligned} f_1(W) &= f_1(a(bxb)a + b(axa)b) = f_1(a)\Phi_1(bxb)\Phi_1(a) + f_1(b)\Phi_1(axa)\Phi_1(b) = \\ &= f_1(a)\Phi_1(b)\Phi_1(x)\Phi_1(b)\Phi_1(a) + f_1(b)\Phi_1(a)\Phi_1(x)\Phi_1(a)\Phi_1(b). \end{aligned} \tag{3.6}$$

On the other hand, by Lemma 3.1 (ii)., we have

$$f_1(W) = f_1((ab)x(ba) + (ba)x(ab)) = f_1(ab)\Phi_1(x)\Phi_1(ba) + f_1(ba)\Phi_1(x)\Phi_1(ab). \tag{3.7}$$

Since $f_1(ab+ba) = f_1(a)\Phi_1(a)\Phi_1(b) + f_1(b)\Phi_1(a)$ and $\Phi_1(ab+ba) = \Phi_1(a)\Phi_1(b) + \Phi_1(b)\Phi_1(a)$, then (3.7) can reduce to

$$\begin{aligned} f_1(W) &= f_1(ab)\Phi_1(x) (\Phi_1(a)\Phi_1(b) + \Phi_1(b)\Phi_1(a)) + \\ &\quad + (-f_1(ab) + f_1(a)\Phi_1(b) + f_1(b)\Phi_1(a)) \Phi_1(x)\Phi_1(ab) = \\ &= -f_1(ab)\Phi_1(x) (\Phi_1(ab) - \Phi_1(a)\Phi_1(b)) - f_1(ab)\Phi_1(x) (\Phi_1(ab) - \Phi_1(b)\Phi_1(a)) + \\ &\quad + f_1(a)\Phi_1(b)\Phi_1(x)\Phi_1(ab) + f_1(b)\Phi_1(a)\Phi_1(x)\Phi_1(ab). \end{aligned} \tag{3.8}$$

Comparing (3.6) and (3.8), we get

$$\begin{aligned} 0 &= -f_1(ab)(a^b) - f_1(ab)\Phi_1(x)(a_b) + f_1(a)\Phi_1(b)\Phi_1(x)\Phi_1(ab) + f_1(b)\Phi_1(a)\Phi_1(x)\Phi_1(ab) - \\ &\quad - f_1(a)\Phi_1(b)\Phi_1(x)\Phi_1(b)\Phi_1(a) - f_1(b)\Phi_1(a)\Phi_1(x)\Phi_1(a)\Phi_1(b) = \\ &= -f_1(ab)\Phi_1(x)(a^b) - f_1(ab)\Phi_1(x)(a_b) + f_1(a)\Phi_1(b)\Phi_1(x)(a_b) + f_1(b)\Phi_1(a)\Phi_1(x)(a^b) = \\ &= -(f_1(ab) - f_1(a)\Phi_1(b)) - (f_1(ab) - f_1(a)\Phi_1(b)) \Phi_1(x)\Phi_1(a). \end{aligned}$$

Thus we have $(a^{b*})\Phi_1(x)(a_b) + (a_b^*)\Phi_1(x)(a^b) = 0$, for all $a, b, x \in R$. Then we can assume that $(a^{b*})^m\Phi_m(x)(a_b)^m + (a_b^*)^m\Phi_m(x)(a^b)^m = 0$, for all $a, b, x \in R, m, n \in \mathbb{N}, m < n$. Since every GJHH of 2-torsion-free ring is a GJTHH, then

$$\begin{aligned}
 f_n(W) &= f_n(a(bxb)a + b(axa)b) = \sum_{i=1}^n f_i(a)\Phi_i(bxb)\Phi_i(a) + f_i(b)\Phi_i(axa)\Phi_i(b) = \\
 &= \sum_{i=1}^n f_i(a) \left(\sum_{l=1}^i f_l(b)\Phi_l(x)\Phi_l(b) \right) \Phi_i(a) + \sum_{i=1}^n f_i(b) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(x)\Phi_l(a) \right) \Phi_i(b) = \\
 &= \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^n f_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(a)\Phi_i(b) = \\
 &= \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) + \sum_{i=1}^n f_i(b)\Phi_i(a)\Phi_i(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b) = \\
 &= f_n(a)\Phi_n(b)\Phi_n(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) + \sum_{i=1}^{n-1} f_i(a)\Phi_i(b)\Phi_i(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) + \\
 &\quad + f_n(b)\Phi_n(a)\Phi_n(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b) + \sum_{i=1}^{n-1} f_i(b)\Phi_i(a)\Phi_i(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b).
 \end{aligned} \tag{3.9}$$

On the other hand, by Lemma 3.1 (ii)., we have

$$f_n(W) = f_n((ab)x(ba) + (ba)x(ab)) = \sum_{i=1}^n f_i(ab)\Phi_i(x)\Phi_i(ba) + f_i(ba)\Phi_i(x)\Phi_i(ab).$$

Since F is a GJHH and Θ is the RJHH, then

$$\begin{aligned}
 f_n(W) &= \sum_{i=1}^n f_i(ab)\Phi_i(x) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b) + \Phi_l(b)\Phi_l(a) - \Phi_i(ab) \right) + \\
 &\quad + \sum_{i=1}^n \left(\sum_{l=1}^i f_l(a)\Phi_l(b) + f_l(b)\Phi_l(a) - f_i(ab) \right) \Phi_i(x)\Phi_i(ab) = \\
 &= \sum_{i=1}^n f_i(ab)\Phi_i(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b) + \sum_{i=1}^n f_i(ab)\Phi_i(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) - \\
 &\quad - \sum_{i=1}^n f_i(ab)\Phi_i(x)\Phi_i(ab) + \sum_{i=1}^n \sum_{l=1}^i f_l(a)\Phi_l(b)\Phi_i(x)\Phi_i(ab) + \\
 &\quad + \sum_{i=1}^n \sum_{l=1}^i f_l(b)\Phi_l(a)\Phi_i(x)\Phi_i(ab) - \sum_{i=1}^n f_i(ab)\Phi_i(x)\Phi_i(ab) = \\
 &= - \sum_{i=1}^n f_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^n f_i(ab)\Phi_i(x)(a_b)^i + \\
 &\quad + \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \sum_{i=1}^n f_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab) = \\
 &= -f_n\Phi_n(x)(a^b)^n - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a^b)^i - f_n\Phi_n(x)(a_b)^n - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a_b)^i + \\
 &\quad + f_n(a)f_n(b)\Phi_n(x)\Phi_n(ab) + \sum_{i=1}^{n-1} f_i(a)\Phi_i(b)\Phi_i(x)\Phi_i(ab) + \\
 &\quad + f_n(b)f_n(a)\Phi_n(x)\Phi_n(ab) + \sum_{i=1}^{n-1} f_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(ab).
 \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned}
 0 &= -f_n(an)\Phi_n(x)(a^b)^n - f_n(ab)\Phi_n(x)(a_b)^n + f_n(a)\Phi_n(b)\Phi_n(x) \left(\Phi_n(ab) - \sum_{l=1}^n \Phi_l(b)\Phi_l(a) \right) + \\
 &\quad + f_n(b)\Phi_n(a)\Phi_n(x) \left(\Phi_n(ab) - \sum_{l=1}^n \Phi_l(a)\Phi_l(b) \right) - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a_b)^i + \\
 &\quad + \sum_{i=1}^{n-1} f_i(a)\Phi_i(b)\Phi_i(x) \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \right) + \sum_{i=1}^{n-1} f_i(b)\Phi_i(a)\Phi_i(x) \left(\Phi_i(ab) - \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \right) = \\
 &= -f_n(ab)\Phi_n(x)(a^b)^n - f_n(ab)\Phi_n(x)(a_b)^n + f_n(a)\Phi_n(b)\Phi_n(x)(a_b)^n + f_n(b)\Phi_n(a)\Phi_n(x)(a^b)^n - \\
 &\quad - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a^b)^i - \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x)(a_b)^i + \sum_{i=1}^{n-1} f_i(a)\Phi_i(b)\Phi_i(x)(a_b)^i + \sum_{i=1}^{n-1} f_i(b)\Phi_i(a)\Phi_i(x)(a^b)^i = \\
 &= -(a_b^*)^n\Phi_n(x)(a^b)^n - (a^{b*})^n\Phi_n(x)(a_b)^n - \sum_{i=1}^{n-1} (a^{b*})^i\Phi_i(x)(a_b)^i - \sum_{i=1}^{n-1} (a_b^*)^i\Phi_i(x)(a^b)^i.
 \end{aligned}$$

Therefore, by our assumption, we find that $(a^{b*})^n\Phi_n(x)(a^b)^n + (a_b^*)^n\Phi_n(x)(a_b)^n$, for all $a, b, x \in R, n \in \mathbb{N}$. ■

Remark 3.3. Let $F = (f_i)_{i \in \mathbb{N}}$ be a Generalized Jordan Triple Higher Homomorphism of a ring R into a ring R' and $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be the Relating Jordan Triple Higher Homomorphism. Then, for all $a, b, c \in R, n \in \mathbb{N}$, we define:

- $S_n^*(a, b, c) = f_n(abc) - \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(c);$

- $T_n^*(a, b, c) = f_n(abc) - \sum_{i=1}^n f_i(c)\Phi_i(b)\Phi_i(a).$

Lemma 3.3. Let $F = (f_i)_{i \in \mathbb{N}}$ be a Generalized Jordan Triple Higher Homomorphisms of a ring R into a ring R' and $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be the Relating Jordan Triple Higher Homomorphism. Then

$$S_n^*(a, b, c)\Phi_n(x)T_n(a, b, c) + T_n^*(a, b, c)\Phi_n(x)S_n(a, b, c) = 0, \text{ for all } a, b, c, x \in R, n \in \mathbb{N}.$$

Proof. By Lemma 2.6, the statement of the Lemma is true for $n = 1$. Then, by induction, we assume that $S_m^*(a, b, c)\Phi_m(x)T_m(a, b, c) + T_m^*(a, b, c)\Phi_m(x)S_m(a, b, c) = 0$, for all $a, b, c, x \in R, m, n \in \mathbb{N}, m < n$. Let

$$\begin{aligned} W &= f_n(abcxcba + cbaabc) = \sum_{i=1}^n f_i(a)\Phi_i(bcxcba)\Phi_i(a) + \sum_{i=1}^n f_i(c)\Phi_i(baxab)\Phi_i(c) = \\ &= \sum_{i=1}^n f_i(a) \sum_{l=1}^i \Phi_l(b)\Phi_l(cxc)\Phi_l(b)\Phi_i(a) + \sum_{i=1}^n f_i(c) \sum_{l=1}^i \Phi_l(b)\Phi_l(axa)\Phi_l(b)\Phi_i(c) = \\ &= \sum_{i=1}^n f_i(a) \sum_{l=1}^i \Phi_l(b) \left(\sum_{t=1}^l \Phi_t(c)\Phi_t(x)\Phi_t(c) \right) \Phi_l(b)\Phi_i(a) = \\ &= \sum_{i=1}^n f_i(c) \sum_{l=1}^i \Phi_l(b) \left(\sum_{t=1}^l \Phi_t(a)\Phi_t(x)\Phi_t(a) \right) \Phi_l(b)\Phi_i(c) = \\ &= \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(c)\Phi_i(x)\Phi_i(c)\Phi_i(b)\Phi_i(a) + \sum_{i=1}^n f_i(c)\Phi_i(b)\Phi_i(a)\Phi_i(x)\Phi_i(a)\Phi_i(b)\Phi_i(c) = \\ &= \sum_{i=1}^n \left(\sum_{l=1}^i f_l(a)\Phi_l(b)\Phi_l(c) \right) \Phi_i(x) \left(\sum_{l=1}^i \Phi_l(c)\Phi_l(b)\Phi_l(a) \right) + \\ &\quad + \sum_{i=1}^n \left(\sum_{l=1}^i f_l(c)\Phi_l(b)\Phi_l(a) \right) \Phi_i(x) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) \right) = \\ &= \left(\sum_{l=1}^n f_l(a)\Phi_l(b)\Phi_l(c) \right) \Phi_n(x) \left(\sum_{l=1}^n \Phi_l(c)\Phi_l(b)\Phi_l(a) \right) + \\ &\quad + \sum_{i=1}^{n-1} \left(\sum_{l=1}^i f_l(a)\Phi_l(b)\Phi_l(c) \right) \Phi_i(x) \left(\sum_{l=1}^i \Phi_l(c)\Phi_l(b)\Phi_l(a) \right) + \\ &\quad + \left(\sum_{l=1}^n f_l(c)\Phi_l(b)\Phi_l(a) \right) \Phi_n(x) \left(\sum_{l=1}^n \Phi_l(a)\Phi_l(b)\Phi_l(c) \right) + \\ &\quad + \sum_{i=1}^{n-1} \left(\sum_{l=1}^i f_l(c)\Phi_l(b)\Phi_l(a) \right) \Phi_i(x) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) \right). \end{aligned}$$

On the other hand by Lemma 3.1 (ii), we get

$$W = f_n((abc)x(cba) + (cba)x(abc)) = \sum_{i=1}^n f_i(abc)\Phi_i(x)\Phi_i(cba) + f_i(cba)\Phi_i(x)\Phi_i(abc).$$

Since

$$f_n(abc + cba) = \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(c) + f_i(c)\Phi_i(b)\Phi_i(a) \text{ and}$$

$$\Phi_n(abc + cba) = \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(c) + \Phi_i(c)\Phi_i(b)\Phi_i(a), \text{ for all } a, b, c \in R \text{ and } n \in \mathbb{N},$$

then

$$\begin{aligned}
 W &= \sum_{i=1}^n f_i(abc)\Phi_i(x) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) \right) + \sum_{i=1}^n f_i(abc)\Phi_i(x) \left(\sum_{l=1}^i \Phi_l(c)\Phi_l(b)\Phi_l(a) \right) - \\
 &\quad - \sum_{i=1}^n f_i(abc)\Phi_i(x)\Phi_i(abc) + \sum_{i=1}^n \left(\sum_{l=1}^i f_l(a)\Phi_l(b)\Phi_l(c) \right) \Phi_i(x)\Phi_i(abc) + \\
 &\quad + \sum_{i=1}^n \left(\sum_{l=1}^i f_l(c)\Phi_l(b)\Phi_l(a) \right) \Phi_i(x)\Phi_i(abc) - \sum_{i=1}^n f_i(abc)\Phi_i(x)\Phi_i(abc) = \\
 &= f_n(abc)\Phi_n(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) + \sum_{i=1}^{n-1} f_i(abc)\Phi_i(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) + \\
 &\quad + f_n(abc)\Phi_n(x) \sum_{l=1}^i \Phi_l(c)\Phi_l(b)\Phi_l(a) + \sum_{i=1}^{n-1} f_i(abc)\Phi_i(x) \sum_{l=1}^i \Phi_l(c)\Phi_l(b)\Phi_l(a) - \\
 &\quad - f_n(abc)\Phi_n(x)\Phi_n(abc) - \sum_{i=1}^{n-1} f_i(abc)\Phi_i(x)\Phi_i(abc) + \sum_{l=1}^n f_l(a)\Phi_l(b)\Phi_l(c)\Phi_l(x)\Phi_n(abc) + \\
 &\quad + \sum_{i=1}^{n-1} \sum_{l=1}^i f_l(a)\Phi_l(b)\Phi_l(c)\Phi_i(x)\Phi_i(abc) + \sum_{l=1}^n f_l(c)\Phi_l(b)\Phi_l(a)\Phi_n(x)\Phi_n(abc) + \\
 &\quad + \sum_{i=1}^{n-1} \sum_{l=1}^i f_l(c)\Phi_l(b)\Phi_l(a)\Phi_i(x)\Phi_i(abc) - f_n(abc)\Phi_n(x)\Phi_n(abc) - \sum_{i=1}^{n-1} f_i(abc)\Phi_i(x)\Phi_i(abc).
 \end{aligned}$$

Now, comparing the right hand side of W , we obtain

$$\begin{aligned}
 0 &= \left(f_n(abc) - \sum_{l=1}^n f_l(c)\Phi_l(b)\Phi_l(a) \right) \Phi_n(x) \sum_{l=1}^n \Phi_l(a)\Phi_l(b)\Phi_l(c) + \\
 &\quad + \left(f_n(abc) - \sum_{l=1}^n f_l(a)\Phi_l(b)\Phi_l(c) \right) \Phi_n(x) \sum_{l=1}^n \Phi_l(c)\Phi_l(b)\Phi_l(a) - \\
 &\quad - \left(f_n(abc) - \sum_{l=1}^n f_l(a)\Phi_l(b)\Phi_l(c) \right) \Phi_n(x)\Phi_n(abc) - \\
 &\quad - \left(f_n(abc) - \sum_{l=1}^n f_l(c)\Phi_l(b)\Phi_l(a) \right) \Phi_n(x)\Phi_n(abc) + \\
 &\quad + \sum_{i=1}^{n-1} \left(f_i(abc) - \sum_{l=1}^i f_l(c)\Phi_l(b)\Phi_l(a) \right) \Phi_i(x) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(b)\Phi_l(c) \right) + \\
 &\quad + \sum_{i=1}^{n-1} \left(f_i(abc) - \sum_{l=1}^i f_l(a)\Phi_l(b)\Phi_l(c) \right) \Phi_i(x) \left(\sum_{l=1}^i \Phi_l(c)\Phi_l(b)\Phi_l(a) \right) - \\
 &\quad - \sum_{i=1}^{n-1} \left(f_i(abc) - \sum_{l=1}^i f_l(a)\Phi_l(b)\Phi_l(c) \right) \Phi_i(x)\Phi_i(abc) - \\
 &\quad - \sum_{i=1}^{n-1} \left(f_i(abc) - \sum_{l=1}^i f_l(c)\Phi_l(b)\Phi_l(a) \right) \Phi_i(x)\Phi_i(abc) = \\
 &= S_n^*(a, b, c)\Phi_n(x)T_n(a, b, c) + T_n^*(a, b, c)\Phi_n(x)S_n(a, b, c) + \\
 &\quad + \sum_{i=1}^{n-1} S_i^*(a, b, c)\Phi_i(x)T_i(a, b, c) + T_i^*(a, b, c)\Phi_i(x)S_i(a, b, c), \text{ for all } a, b, c, x \in R \text{ and } n \in \mathbb{N}.
 \end{aligned}$$

Then, by our assumption, we get $S_n^*(a, b, c)\Phi_n(x)T_n(a, b, c) + T_n^*(a, b, c)\Phi_n(x)S_n(a, b, c) = 0$, for all $a, b, c, x \in R, n \in \mathbb{N}$. ■

Theorem 3.1. Let $F = (f_i)_{i \in \mathbb{N}}$ be a Generalized Jordan Triple Higher Homomorphism of a ring R into a 2-torsion-free prime ring R' and $\Theta = (\Phi_i)_{i \in \mathbb{N}}$ be the Relating Jordan Triple Higher Homomorphism. Then $f_i = \pm \Psi_i$, for all $i \in \mathbb{N}$, where $\Psi_i \in \Psi = (\Psi_i)_{i \in \mathbb{N}}$ and Ψ is a Generalized Higher Homomorphism or a Generalized Higher Anti-Homomorphism.

Proof. We proceed by induction on $n \in \mathbb{N}$. By Lemma 2.7, the statment of the Theorem is true for $n = 1$. Then we can assume that $f_m = \pm \Psi_m$, for all $m, n \in \mathbb{N}, m < n$. Now, we have two cases:

Case I: If $f_m = \pm \Psi_m$, for all $m, n \in \mathbb{N}, m < n$ and $\Psi_m \in \Psi$ such that Ψ is a GHH. By Theorem 2.1, either

- (i). $\Phi_i = \pm \delta_i$, for all $i \in \mathbb{N}$ and $\delta_i \in \delta = (\delta_i)_{i \in \mathbb{N}}$ such that δ is a HH, or
- (ii). $\Phi_i = \pm \delta_i$, for all $i \in \mathbb{N}$ and $\delta_i \in \delta = (\delta_i)_{i \in \mathbb{N}}$ such that δ is a HAH.

If we have (i)., then $\Phi_n(abc) = \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(c)$ and this means $S_n(a, b, c) = 0$. Since R' is prime and Θ is onto, then by Lemma 3.3, we have either $T_n(a, b, c) = 0$ or $S_n^*(a, b, c) = 0$, for all $a, b, c \in R, n \in \mathbb{N}$. If $T_n(a, b, c) = 0$, for all $a, b, c \in R, n \in \mathbb{N}$, then as in the proof of Theorem 2.1, we have $\Phi_i = \pm \delta_i$, for all $i \in \mathbb{N}$ and δ is a HAH and this will be a contradiction with assumption (i). Therefore $S_n^*(a, b, c) = 0$, for all $a, b, c \in R, n \in \mathbb{N}$, that is, $f_n(abc) = \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(c)$.

Let $W = f_n(abxab)$. Since F is a GJTHH and $\Phi_i \pm \delta_i$, for all $i \in \mathbb{N}$ and $\delta_i \in \delta = (\delta_i)_{i \in \mathbb{N}}$ such that δ is a HH,

$$\begin{aligned} W &= \sum_{i=1}^n f_i(ab)\Phi_i(x)\Phi_i(ab) = \pm \sum_{i=1}^n f_i(abc)\Phi_i(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b) = \\ &= \pm f_n(ab)\Phi_n(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \pm \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b). \end{aligned}$$

On the other hand, since $S_n^*(a, b, c) = 0$, for all $a, b, c \in R, n \in \mathbb{N}$, then

$$\begin{aligned} W &= f_n(a(bxa)b) = \sum_{i=1}^n f_i(a)\Phi_i(bxa)\Phi_i(b) = \pm \sum_{i=1}^n f_i(a) \left(\sum_{l=1}^i \Phi_l(b)\Phi_l(xa) \right) \Phi_i(b) = \\ &= \pm \sum_{i=1}^n f_i(a)\Phi_i(b)\Phi_i(xa)\Phi_i(b) = \pm \sum_{i=1}^n f_i(a)\Phi_i(b) \left(\sum_{l=1}^i \Phi_l(x)\Phi_l(a) \right) \Phi_i(b) = \\ &= \pm \sum_{i=1}^n \sum_{l=1}^i f_l(a)\Phi_l(b)\Phi_l(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b) = \\ &= \pm \sum_{l=1}^n f_l(a)\Phi_l(b)\Phi_l(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b) \pm \sum_{i=1}^{n-1} \sum_{l=1}^i f_l(a)\Phi_l(b)\Phi_l(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b). \end{aligned}$$

By comparing the right hand side of W , we get

$$0 = \left(f_n(ab) \pm \sum_{l=1}^n f_l(a)\Phi_l(b) \right) \Phi_n(x) \sum_{l=1}^n \Phi_l(a)\Phi_l(b) + \sum_{i=1}^{n-1} \left(f_i(ab) \pm \sum_{l=1}^i f_l(a)\Phi_l(b) \right) \Phi_i(x) \sum_{l=1}^i \Phi_l(a)\Phi_l(b).$$

Note that the assumption of Case I reduces the last equation to

$$\left(f_n(ab) \pm \sum_{l=1}^n f_l(a)\Phi_l(b) \right) \Phi_n(x) \sum_{l=1}^n \Phi_l(a)\Phi_l(b) = 0,$$

and this implies that $\left(f_n(ab) \pm \sum_{l=1}^n f_l(a)\Phi_l(b) \right) R' = 0$, for all $a, b \in R, n \in \mathbb{N}$. Since R' is prime, then $f_n(ab) = \pm \sum_{l=1}^n f_l(a)\Phi_l(b)$ and this means $f_i = \pm \Psi_i$, for all $i \in \mathbb{N}, \Psi_i \in \Psi = (\Psi_i)_{i \in \mathbb{N}}$, where Ψ is a GHH.

Now, if have case (ii)., then $\Phi_n(abc) = \sum_{i=1}^n \Phi_i(c)\Phi_i(b)\Phi_i(a)$, for all $a, b, c \in R, n \in \mathbb{N}$. Since R' is prime and Θ is onto, then by Lemma 3.3, we have either $S_n(a, b, c) = 0$ or $T_n^*(a, b, c) = 0$, for all $a, b, c \in R, n \in \mathbb{N}$. If $S_n(a, b, c) = 0$, then as in the proof of Theorem 2.1, we get $\Phi_i = \pm \Psi_i, i \in \mathbb{N}, \Psi_i \in \Psi = (\Psi_i)_{i \in \mathbb{N}}$, where Ψ is a HH and this is a contradiction. Thus, $T_n^*(a, b, c) = 0$ and this gives us $f_n(abc) = \sum_{i=1}^n f_i(c)\Phi_i(b)\Phi_i(a)$, for all $a, b, c \in R, n \in \mathbb{N}$.

Let $W = f_n(abxab)$. Then

$$\begin{aligned} W &= \sum_{i=1}^n f_i(ab)\Phi_i(x)\Phi_i(ab) = \pm \sum_{i=1}^n f_i(ab)\Phi_i(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) = \\ &= f_n(ab)\Phi_n(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) \pm \sum_{i=1}^{n-1} f_i(ab)\Phi_i(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a). \end{aligned}$$

On the other hand,

$$\begin{aligned} W &= f_n(a(bxa)b) = \sum_{i=1}^n f_i(b)\Phi_i(bxa)\Phi_i(a) = \pm \sum_{i=1}^n f_i(b) \left(\sum_{l=1}^i \Phi_l(xa)\Phi_l(b) \right) \Phi_i(a) = \\ &= \pm \sum_{i=1}^n f_i(b)\Phi_i(xa)\Phi_i(b)\Phi_i(a) = \pm \sum_{i=1}^n f_i(b) \left(\sum_{l=1}^i \Phi_l(a)\Phi_l(x) \right) \Phi_i(b)\Phi_i(a) = \\ &= \sum_{i=1}^n \sum_{l=1}^i f_l(b)\Phi_l(a)\Phi_l(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a) = \sum_{l=1}^n f_l(ab)\Phi_l(a)\Phi_n(x) \sum_{l=1}^n \Phi_l(b)\Phi_l(a) = \\ &= \sum_{i=1}^{n-1} \sum_{l=1}^i f_l(b)\Phi_l(a)\Phi_l(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a). \end{aligned}$$

Comparing both right hand sides of W , we get

$$\begin{aligned} 0 &= \left(f_n(ab) \pm \sum_{l=1}^n f_l(b)\Phi_l(a) \right) \Phi_n(x) \sum_{l=1}^n \Phi_l(b)\Phi_l(a) + \\ &\quad + \sum_{i=1}^{n-1} \left(f_i(ab) \pm \sum_{l=1}^i f_l(b)\Phi_l(a) \right) \Phi_i(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a). \end{aligned} \tag{3.11}$$

By the assumption of cases (i). and (ii). and since $T_n^*(a, b, c) = 0$, then the last line of equation (3.11) becomes zero. Hence,

$$\left(f_n(ab) \pm \sum_{l=1}^n f_l(b)\Phi_l(a) \right) \Phi_n(x) \sum_{l=1}^n \Phi_l(b)\Phi_l(a) = 0.$$

Since R' is prime, then $f_n(ab) = \pm \sum_{l=1}^n f_l(b)\Phi_l(a)$, but this is a contradiction (this means we have only case (i).).

Case II: If $f_m = \pm\Psi_m$, for all $m, n \in \mathbb{N}$, $m < n$ and $\Psi_m \in \Psi = (\Psi_i)_{i \in \mathbb{N}}$ such that Ψ is a GHAH, then by Theorem 2.1 we have either

(i') $\Phi_i = \pm\delta_i$, for all $i \in \mathbb{N}$, $\delta_i \in \delta = (\delta_i)_{i \in \mathbb{N}}$ such that δ is a HH, or

(ii') $\Phi_i = \pm\delta_i$, for all $i \in \mathbb{N}$, $\delta_i \in \delta = (\delta_i)_{i \in \mathbb{N}}$ such that δ is a HAH.

If we have case (i')., then $\Phi_n(abc) = \sum_{i=1}^n \Phi_i(a)\Phi_i(b)\Phi_i(c)$, for all $a, b, c \in R$, $n \in \mathbb{N}$. Since Θ is onto and R' is prime, then by Lemma 3.3 we have either $T_n(a, b, c) = 0$ or $S_n^*(a, b, c) = 0$, for all $a, b, c \in R$, $n \in \mathbb{N}$. If $T_n(a, b, c) = 0$, then as in the proof of Case I, we get $\Phi_i = \pm\delta_i$, for all $i \in \mathbb{N}$ and $\delta_i \in \delta = (\delta_i)_{i \in \mathbb{N}}$ such that δ is a HAH and this is a contradiction. Hence $S_n^*(a, b, c) = 0$. Again, as in the proof of case I above, we arrive at equation (3.11) and hence by assumption of case II and (i') and since $S_n^*(a, b, c) = 0$. Thus, equation (3.11) becomes $\left(f_n(ab) \pm \sum_{l=1}^n f_l(a)\Phi_l(b) \right) \Phi_n(x) \sum_{l=1}^n \Phi_l(a)\Phi_l(b) = 0$, for all $a, b, x \in R$, $n \in \mathbb{N}$, but this is a contradiction with assumption of case II.

Hence, we have case (ii')., which give us $\Phi_n(abc) = \sum_{i=1}^n \Phi_i(c)\Phi_i(b)\Phi_i(a)$. Thus, by Lemma 3.3, we have either $S_n(a, b, c) = 0$ or $T_n^*(a, b, c) = 0$, for all $a, b, c \in R$, $n \in \mathbb{N}$. As in the proof of Case I (ii)., if $S_n(a, b, c) = 0$, then $\Phi_i = \pm\Psi_i$, for all $i \in \mathbb{N}$, $\Psi_i \in \Psi = (\Psi_i)_{i \in \mathbb{N}}$ such that Ψ is a HH, but this is a contradiction with assumption (ii'). Therefore, $T_n^*(a, b, c) = 0$. As in the proof of case I (ii.), we have

$$0 = \left(f_n(ab) \pm \sum_{l=1}^n f_l(b)\Phi_l(a) \right) \Phi_n(x) \sum_{l=1}^n \Phi_l(b)\Phi_l(a) + \sum_{i=1}^{n-1} \left(f_i(ab) \pm \sum_{l=1}^i f_l(b)\Phi_l(a) \right) \Phi_i(x) \sum_{l=1}^i \Phi_l(b)\Phi_l(a),$$

for all $a, b, x \in R$, $n \in \mathbb{N}$. By the assumption case II, the last term of the equation above becomes zero. Since R' is prime, then $f_n(ab) = \pm \sum_{l=1}^n f_l(b)\Phi_l(a)$, for all $a, b, c \in R$, $n \in \mathbb{N}$ and this means $f_i = \pm\Psi_i$ for all $i \in \mathbb{N}$, $\Psi_i \in \Psi = (\Psi_i)_{i \in \mathbb{N}}$ such that Ψ is a HAH. ■

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