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CERTAIN RESULTS ON A SUBCLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS ASSOCIATED WITH COEFFICIENT ESTIMATES AND QUASI-SUBORDINATION

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Abstract. In the present paper, we obtain the estimates on initial coefficients of normalized analytic function f in the open unit disk with f and its inverse $g = f^{-1}$ satisfying the conditions that $zf'(z)/f(z)$ and $zg'(z)/g(z)$ are both quasi-subordinate to a univalent function whose range is symmetric with respect to the real axis. Several related classes of functions are also considered, and connections to earlier known results are established.

1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disk $\mathbb{D} = \{z : z \in \mathcal{C}; |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The Koebe-one quarter theorem [4] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius 1/4. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z,\ z \in \mathbb{D}$ and

$$f\left(f^{-1}(w)\right) = w \quad (|w| < r_0(f),\ r_0(f) \geq 1/4).$$

In fact, the inverse function f^{-1} is given by (see, e.g. [1], [7], [14]])

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \tag{1.1}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let Σ denote the class of bi-univalent functions defined in $\mathbb{D}$.

Ma and Minda [9] introduced the following class

$$S^*(h) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec h(z) \right\} \tag{1.2}$$

where h is an analytic function with positive real part in $\mathbb{D}$, $h(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $h(0) = 1$ and $h'(0) > 0$. A function $f \in S^*(h)$ is called Ma-Minda starlike (with respect to h). $C(h)$ is the class of convex functions $f \in \mathcal{A}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec h(z). \tag{1.3}$$

The classes $S^*(h)$ and $C(h)$ include several well-known subclasses of starlike and convex functions as special cases.

In the year 1970, Robertson [13] introduced the concept of quasi-subordination. For two analytic functions f and g, the function f is quasi-subordinate to g, written as

$$f(z) \prec_q g(z)\ \ (z \in \mathbb{D}), \tag{1.4}$$

if there exist analytic functions φ and w, with $|\varphi(z)| \leq 1, w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$. Observe that if $\varphi(z) = 1$, then $f(z) = g(w(z))$, so that $f(z) \prec \varphi(z)$ in $\mathbb{D}$. Also notice that if $w(z) = z$, then $f(z) = \varphi(z)g(z)$ and it is said that f is majorized by g and written $f(z) \ll g(z)$ in $\mathbb{D}$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. (see, e.g. [5]-[7], [10], [12] for works related to quasi-subordination).

Lewin [8] investigated the bi-univalent function class Σ and showed that

$$|a_2| < 1.51$$

. Subsequently, Brannan et al. [2] conjuctured that

$$|a_2| < \sqrt{2}$$

. Netanyahu [11], on the other hand, showed that

$$\max_{f\in\Sigma} |a_2| = \frac{4}{3}.$$

The coefficient estimate problem for each of the following Taylor Maclaurin coefficients

$$|a_n| \quad (n \in \mathbb{N}\backslash\{1,2,3\} : \mathbb{N} = \{1,2,3,4,...\})$$

is presently still an open problem.

Brannan and Taha [3] obtained initial coefficient bounds for certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. Later, Srivastava et al. [14] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Recently, Ali et al. [1] obtained the coefficient bounds for bi-univalent Ma-Minda starlike and convex functions.

Throughout this paper it is assumed that h is analytic in $\mathbb{D}$ with $h(0) = 1$ and let

$$f(z) = z + a_2z^2 + a_3z^3 + ...,$$

$$\varphi(z) = A_0 + A_1z + A_2z^2 + ..., (|\varphi(z)| \le 1, z \in \mathbb{D}) \tag{1.5}$$

$$h(z) = 1 + B_1z + B_2z^2 + ..., \quad B_1 \in \mathbb{R}^+. \tag{1.6}$$

Motivated by earlier work on quasi-subordination we define the following classes:

Definition 1.1. A function $f \in \Sigma$ is said to be in the class $\mathcal{J}^q_\beta(h)$ $(\beta \ge 0)$ if the following quasi-subordination holds:

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\beta - 1 \prec_q (h(z) - 1) \qquad and \qquad \left[\frac{wg'(w)}{g(w)}\right]\left[\frac{g(w)}{w}\right]^\beta - 1 \prec_q (h(w) - 1). \tag{1.7}$$

Definition 1.2. A function $f \in \Sigma$ is said to be in the class $\mathcal{K}^q_{\gamma,\tau}(h)$ $\;(0 \le \gamma < 1, \tau \in \mathbb{C}/\{0\})$ if the following quasi-subordination holds:

$$\frac{1}{\tau}(f'(z) + \gamma zf''(z) - 1) \prec_q (h(z) - 1) \qquad and \qquad \frac{1}{\tau}(g'(w) + \gamma wg''(w) - 1) \prec_q (h(w) - 1). \tag{1.8}$$

Definition 1.3. A function $f \in \Sigma$ is said to be in the class $\mathcal{H}^q_\alpha(h)$ $(\alpha \ge 0)$ if the following quasi-subordination holds:

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)} - 1 \prec_q (h(z) - 1) \qquad and \qquad \frac{wg'(w)}{g(w)} + \alpha\frac{w^2g''(w)}{g(w)} - 1 \prec_q (h(w) - 1). \tag{1.9}$$

It is known that a function $f \in \mathcal{A}$ with $\mathrm{Re}f'(z) > 0$ in $\mathbb{D}$ is univalent. The above classes of functions defined in terms of the quasi-subordination are associated with the classes of functions with positive real part.

In the present paper, the coefficient bounds of $|a_2|$ and $|a_3|$ for functions in the classes $\mathcal{J}^q_\beta(h)$, $\mathcal{K}^q_{\gamma,\tau}(h)$ and $\mathcal{H}^q_\alpha(h)$ are obtained.

We now state and prove the main results of our present investigation:

2 Main Results

Theorem 2.1. *If $f \in \mathcal{J}^q_\beta(h)$ is given by*

$$f(z) = z + \sum_{n=2}^{\infty} a_nz^n \tag{2.1}$$

then

$$|a_2| \le \frac{|A_0| B_1\sqrt{2B_1}}{\sqrt{(\beta+1)\left|(\beta+2)A_0B_1^2 - 2(\beta+1)(B_2 - B_1)\right|}} \tag{2.2}$$

and

$$|a_3| \le \frac{|A_1| B_1}{(\beta+2)} + \frac{|A_0| B_1((\beta+3) + |\beta-1|) + 4|A_0||B_2 - B_1|}{2(\beta+1)(\beta+2)}. \tag{2.3}$$

Proof. Let $f \in \mathcal{J}_{\beta}^{q}(h)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(z)| < 1$ and a functions φ in $\mathbb{D}$ defined by (1.5) satisfying

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^{\beta}-1=\varphi(z)\left(h\left(u(z)\right)-1\right) \quad and \quad \left[\frac{wg'(w)}{g(w)}\right]\left[\frac{g(w)}{w}\right]^{\beta}-1=\varphi(w)\left(h\left(v(w)\right)-1\right). \tag{2.4}$$

Define the function p_1 and p_2 by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + ...$$

and

$$p_2(z) = \frac{1+v(z)}{1-v(z)} = 1 + b_1 z + b_2 z^2 + ...$$

Or equivalently

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + ...\right] \tag{2.5}$$

and

$$v(z) = \frac{p_2(z)-1}{p_2(z)+1} = \frac{1}{2}\left[b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + ...\right]. \tag{2.6}$$

Then p_1 and p_2 are analytic in $\mathbb{D}$ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \mathbb{D} \to \mathbb{D}$, the functions p_1 and p_2 have a positive real part in $\mathbb{D}$, and $|b_i| \le 2$ and $|c_i| \le 2$ $(i = 1, 2)$. In view of (2.4) – (2.6), clearly we have

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^{\beta} - 1 = \varphi(z)\left[h\left(\frac{p_1(z)-1}{p_1(z)+1}\right) - 1\right] \tag{2.7}$$

and

$$\left[\frac{wg'(w)}{g(w)}\right]\left[\frac{g(w)}{w}\right]^{\beta} - 1 = \varphi(w)\left[h\left(\frac{p_2(w)-1}{p_2(w)+1}\right) - 1\right]. \tag{2.8}$$

Using (2.5) and (2.6) together with (1.5) and (1.6), it is evident that

$$\phi(z)\left[h\left(\frac{p_1(z)-1}{p_1(z)+1}\right)-1\right] = \frac{1}{2}A_0B_1c_1z+\left\{\frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2}{4}c_1^2\right\}z^2+... \tag{2.9}$$

and

$$\phi(w)\left[h\left(\frac{p_2(w)-1}{p_2(w)+1}\right)-1\right] = \frac{1}{2}A_0B_1b_1w+\left\{\frac{1}{2}A_1B_1b_1 + \frac{1}{2}A_0B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{A_0B_2}{4}b_1^2\right\}w^2+... \tag{2.10}$$

Since $f \in \Sigma$ has the Maclaurin series given by (2.1), a computation shows that its inverse $g = f^{-1}$ has the expansion given by (1.1).
Since

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^{\beta} - 1 = (\beta+1)\,a_2 z + \left((\beta+2)\,a_3 + \frac{(\beta-1)(\beta+2)}{2}a_2^2\right) z^2 + ... \tag{2.11}$$

and

$$\left[\frac{wg'(w)}{g(w)}\right]\left[\frac{g(w)}{w}\right]^{\beta} - 1 = -(\beta+1)\,a_2 w + \left(-(\beta+2)\,a_3 + \frac{(\beta+3)(\beta+2)}{2}a_2^2\right) w^2 + ... \tag{2.12}$$

Now using (2.9) and (2.11) in (2.7) and comparing the coefficients of z and z^2, we get

$$(\beta+1)\,a_2 = \frac{1}{2}A_0B_1c_1 \tag{2.13}$$

$$(\beta+2)\,a_3 + \frac{(\beta-1)(\beta+2)}{2}a_2^2 = \frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2}{4}c_1^2. \tag{2.14}$$

Similarly using (2.10) and (2.12) in (2.8) and comparing the coefficients of w and w^2, we get

$$-(\beta+1)\,a_2 = \frac{1}{2}A_0B_1b_1 \tag{2.15}$$

$$-(\beta+2)\,a_3+\frac{(\beta+3)\,(\beta+2)}{2}a_2^2=\frac{1}{2}A_1B_1b_1+\frac{1}{2}A_0B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_0B_2}{4}b_1^2. \tag{2.16}$$

From (2.13) and (2.15), it follows that

$$c_1=-b_1 \tag{2.17}$$

and (2.12)-(2.16) and (2.17) yields

$$a_2^2=\frac{A_0^2B_1^3\,(b_2+c_2)}{2\,(\beta+1)\left[(\beta+2)\,A_0B_1^2-2\,(\beta+1)\,(B_2-B_1)\right]}.$$

Using well-known inequalities $|b_i|\le 2$ and $|c_i|\le 2$ $(i=1,2)$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.2).
Now further computations (2.13) to (2.17) leads to

$$a_3=\frac{A_1B_1c_1\,(\beta+1)+\frac{A_0B_1}{2}\left[(\beta+3)\,c_2-(\beta-1)\,b_2\right]+A_0b_1^2\,(B_2-B_1)}{2\,(\beta+1)\,(\beta+2)}.$$

Using the above result and in view of the inequalities $|c_i|\le 2$ and $|b_i|\le 2\,(i=1,2)$ for functions with positive real part yield the desired estimate in (2.3). □

Remark 2.2. For $\beta=1,\varphi(z)\equiv 1$, the inequality (2.2) reduce to the result ([1], p. 345, Theorem 2.1). Further for

$$\beta=1,\varphi(z)\equiv 1 \;\; and \;\; h(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}=1+2\gamma z+2\gamma^2z^2+... \quad (0<\gamma\le 1)$$

the inequality (2.2) reduces to the result in ([14], p.3, Theorem 1)
and for

$$h(z)=\frac{1+(1-2\gamma)z}{1-z}=1+2(1-\gamma)z+2(1-\gamma)z^2+...$$

the inequality (2.2) reduce to the result in ([14], p.4, Theorem 2).

For $\beta=0$, the above theorem reduces to

Corollary 2.3. *Let f given by (2.1) be in the class $J_0^q\,(h)\equiv J^q\,(h)$. Then*

$$|a_2|\le\frac{|A_0|\,B_1\sqrt{B_1}}{\sqrt{\left|A_0B_1^2-B_2+B_1\right|}},\qquad |a_3|\le\frac{|A_1|\,B_1}{2}+|A_0|\,(B_1+|B_2-B_1|)\,.$$

For $\beta=0,\varphi(z)\equiv 1$, above theorem reduces to the coefficient estimates for Ma-Minda bi-starlike functions.

Remark 2.4. For $\beta=0,\varphi(z)\equiv 1$ and

$$h(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}=1+2\gamma z+2\gamma^2z^2+... \quad (0<\gamma\le 1)$$

the inequalities (2.2)and (2.3) reduce to the result ([4], Theorem 2.1)
and for

$$h(z)=\frac{1+(1-2\gamma)z}{1-z}=1+2(1-\gamma)z+2(1-\gamma)z^2+...$$

the inequalities (2.2)and (2.3) reduce to the result ([3], Theorem 3.1).

Theorem 2.5. *If f given by* (2.1) *be in the class $\mathcal{K}_{\gamma,\tau}^q(h)$, then*

$$|a_2|\le\frac{|\tau|\,|A_0|\,B_1\sqrt{B_1}}{\sqrt{\left|3\tau A_0B_1^2\,(1+2\gamma)+4\,(1+\gamma)^2\,(B_1-B_2)\right|}} \tag{2.18}$$

and

$$|a_3|\le\frac{|\tau|}{1+2\gamma}\left[\frac{|A_1|}{3}+|A_0|\left(\frac{1}{3}+\frac{(1+2\gamma)\,|\tau|\,|A_0|\,B_1}{4\,(1+\gamma)^2}\right)\right]B_1. \tag{2.19}$$

Proof. Let $f \in \mathcal{K}^q_{\gamma,\tau}(h)$ and $g = f^{-1}$. Then there exists analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(z)| < 1$ and a function φ in $\mathbb{D}$ defined by (1.4) satisfying

$$\frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) = \phi(z)\left(h(u(z)) - 1\right)$$

and

$$\frac{1}{\tau}\left(g'(w) + \gamma w g''(w) - 1\right) = \varphi(w)\left(h(v(w)) - 1\right) \tag{2.20}$$

where $u(z)$ and $v(z)$ are defined by (2.5) and (2.6) respectively.
Under the same restrictions for $p_1(z), p_2(z), b_i$ and c_i as mentioned in the Theorem 2.1, obviously we have

$$\frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) = \phi(z)\left[h\left(\frac{p_1(z)-1}{p_1(z)+1}\right) - 1\right] \tag{2.21}$$

and

$$\frac{1}{\tau}\left(g'(w) + \gamma w g''(w) - 1\right) = \phi(w)\left[h\left(\frac{p_2(w)-1}{p_2(w)+1}\right) - 1\right] \tag{2.22}$$

where the right-hand sides of (2.21) and (2.22) are given by (2.9) and (2.10) respectively.
Since

$$\frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) = \frac{1}{\tau}\left[2(1+\gamma)a_2 z + 3(1+2\gamma)a_3 z^2 + ...\right] \tag{2.23}$$

and

$$\frac{1}{\tau}\left(g'(w) + \gamma w g''(w) - 1\right) = \frac{1}{\tau}\left[-2(1+\gamma)a_2 w + 3(1+2\gamma)\left(2a_2^2 - a_3\right)w^2 + ...\right]. \tag{2.24}$$

Now using (2.9) and (2.23) in (2.21) and comparing the coefficients of z and z^2, we get

$$2\left(\frac{1+\gamma}{\tau}\right)a_2 = \frac{1}{2}A_0 B_1 c_1 \tag{2.25}$$

$$3\left(\frac{1+2\gamma}{\tau}\right)a_3 = \frac{1}{2}A_1 B_1 c_1 + \frac{1}{2}A_0 B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0 B_2}{4}c_1^2. \tag{2.26}$$

Similarly (2.10) and (2.22) and (2.24) yields

$$-2\left(\frac{1+\gamma}{\tau}\right)a_2 = \frac{1}{2}A_0 B_1 b_1 \tag{2.27}$$

and

$$\left(6a_2^2 - 3a_3\right)\left(\frac{1+2\gamma}{\tau}\right) = \frac{1}{2}A_1 B_1 b_1 + \frac{1}{2}A_0 B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{A_0 B_2}{4}b_1^2. \tag{2.28}$$

From (2.25) and (2.27), we have

$$c_1 = -b_1. \tag{2.29}$$

From (2.26), (2.28) and (2.29), we get

$$a_2^2 = \frac{\tau^2 A_0^2 B_1^3 (b_2 + c_2)}{4\left[3\tau(1+2\gamma)B_1^2 A_0 - 4(1+\gamma)^2(B_2 - B_1)\right]}.$$

Using well-known inequalities $|b_i| \le 2$ and $|c_i| \le 2$ $(i = 1, 2)$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.18).

Now, further computation (2.21)-(2.25) leads to

$$a_3 = \frac{\tau}{1+2\gamma}\left[\frac{A_1 B_1 c_1}{6} + \frac{A_0 B_1}{12}(c_2 - b_2) + \frac{\tau}{16}\frac{(1+2\gamma)}{(1+\gamma)^2}A_0^2 B_1^2 c_1^2\right].$$

Using above result and in view of the inequalities $|c_i| \le 2$ and $|b_i| \le 2\,(i = 1, 2)$ for functions with positive real part yield the desired estimate in (2.19). $\square$

For $\tau = 1, \gamma = 0$, the above theorem reduces to

Corollary 2.6. *Let f given by (2.1) be in the class $\mathcal{K}^q_{0,1}(h) \equiv \mathcal{K}^q(h)$. Then*

$$|a_2| \leq \frac{|A_0|\, B_1\sqrt{B_1}}{\sqrt{|3A_0B_1^2 - 4B_2 + 4B_1|}} \quad and \quad |a_3| \leq \frac{|A_1|}{3} + |A_0|\left(\frac{1}{3} + \frac{|A_0|\, B_1}{4}\right) B_1.$$

Remark 2.7. For $\tau = 1, \gamma = 0$, $\varphi(z) \equiv 1$, the inequalities (2.18) and(2.19) reduce to the result ([1], p. 345, Theorem 2.1). Further for

$$\tau = 1, \gamma = 0, \varphi(z) \equiv 1 \;\; and \;\; h(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + ... \quad (0 < \gamma \leq 1)$$

the inequalities (2.18) and(2.19) reduces to the result in ([14], p.3, Theorem 1), and for

$$h(z) = \frac{1 + (1-2\gamma)z}{1-z} = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + ...$$

the inequalities (2.18) and(2.19) reduce to the result in ([14], p.4, Theorem 2).

Theorem 2.8. *If f given by (2.1) be in the class $\mathcal{H}^q_{\alpha}(h)$, then*

$$|a_2| \leq \frac{|A_0|\, B_1\sqrt{B_1}}{\sqrt{\left|A_0B_1^2\,(1+4\alpha) + (1+2\alpha)^2\,(B_1 - B_2)\right|}} \tag{2.30}$$

and

$$|a_3| \leq \frac{|A_1|\, B_1}{2\,(1+3\alpha)} + \frac{|A_0|\,(B_1 + |B_2 - B_1|)}{(1+4\alpha)}. \tag{2.31}$$

Proof. The inequalities in (2.30) and (2.31) can be easily proved by following the lines similar to those mentioned with Theorem 2.1 and Theorem 2.5, therefore we leave the details of the proof. □

Remark 2.9. For $\alpha = 0$, the above theorem reduces to the result obtained in corollary (2.3).

Remark 2.10. For $\alpha = 0, \varphi(z) \equiv 1$, above theorem reduces to the coefficient estimates for Ma-Minda bi-starlike functions.

Remark 2.11. For $\alpha = 0, \varphi(z) \equiv 1$ and

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + ... \quad (0 < \gamma \leq 1)$$

the inequalities (2.30)and (2.31) reduce to the result ([4], Theorem 2.1) and for

$$h(z) = \frac{1 + (1-2\gamma)z}{1-z} = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + ...$$

the inequalities (2.30)and (2.31) reduce to the result ([3], Theorem 3.1).

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