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ON ϕ-2-ABSORBING PRIMARY ELEMENTS IN MULTIPLICATIVE LATTICES

Ece Yetkin Celikel, Gulsen Ulucak and Emel A. Ugurlu

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Abstract In this paper, we introduce the concept of ϕ-2-absorbing primary elements in multiplicative lattices as a generalization of ϕ-2-absorbing elements. Let $\phi : L \to L \cup \{\varnothing\}$ be a function. We will say a proper element q of L to be a ϕ-2-absorbing primary element of L if whenever $a, b, c \in L$ with $abc \leq q$ and $abc \nleq \phi(q)$ implies either $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$. We give some basic properties of this new type of elements and establish some characterizations for ϕ-2-absorbing primary elements in some special lattices.

1 Introduction

Throughout this paper R denotes a commutative ring with identity and $L(R)$ denotes the lattice of all ideals of R. An element a of L is said to be compact if whenever $a \leq \bigvee_{\alpha \in I} a_\alpha$ implies $a \leq \bigvee_{\alpha \in I_0} a_\alpha$ for some finite subset I_0 of I. By a *multiplicative lattice*, we mean a complete lattice L with the least element 0_L and compact greatest element 1_L, on which there is defined a commutative, associative, completely join distributive product for which 1_L is a multiplicative identity. By a C*-lattice* we mean a (not necessarily modular) multiplicative lattice which is generated under joins by a multiplicatively closed subset C of compact elements. Throughout this paper L denotes a C-lattice and L_* denotes the set of all compact elements of L. We note that in a C-lattice, a finite product of compact elements is again compact.

The study of generalizations of prime and primary ideals are carried out in [1] - [18]. We generalize these concepts and study their properties in C-lattices. An element $a \in L$ is said to be *idempotent* if $a = a^2$. For any $a \in L$, $L/a = \{b \in L \mid a \leq b\}$ is a multiplicative lattice with the multiplication $c \circ d = cd \vee a$. An element $a \in L$ is said to be *proper* if $a < 1_L$. A proper element p of L is said to be prime if $ab \leq p$ implies either $a \leq p$ or $b \leq p$. If 0_L is prime, then L is said to be a *domain*. A proper element m of L is said to be *maximal* in L if $m < x \leq 1_L$ implies $x = 1_L$. It can be easily shown that maximal elements are prime. For $a, b \in L$, we denote $(a : b) = \vee\{x \in L \mid xb \leq a\}$. For $a \in L$, we define $\sqrt{a} = \wedge\{p \in L \mid p$ is prime and $a \leq p\}$. Recall that a is said to be a radical element of L if $\sqrt{a} = a$. Note that in a C-lattice L, $\sqrt{a} = \wedge\{p \in L \mid a \leq p$ is a minimal prime over $a\} = \vee\{x \in L_* \mid x^n \leq a$ for some $n \in \mathbb{Z}^+\}$. A proper element q is said to be *primary* if $ab \leq q$ implies either $a \leq q$ or $b \leq \sqrt{q}$ for every pair of elements $a, b \in L$. Recall from [12] that a proper element q of L is said to be a 2-absorbing element (resp. 2-absorbing primary) if $abc \leq q$ implies either $ab \leq q$ or $bc \leq q$ or $ac \leq q$ (resp. $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$) for any $a, b, c \in L$. Let $\phi : L \to L \cup \{\varnothing\}$ be a function. A proper element p of L is called as ϕ*-prime* (ϕ*-primary*) if $ab \leq p$ and $ab \nleq \phi(p)$ implies either $a \leq p$ or $b \leq p$ ($a \leq p$ or $b \leq \sqrt{p}$) for all $a, b \in L$. A proper element q of L is said to be a ϕ-2-absorbing element of L if whenever $a, b, c \in L$ with $abc \leq q$ and $abc \nleq \phi(q)$ implies either $ab \leq q$ or $ac \leq q$ or $bc \leq q$ as it is defined in [10].

A multiplicative lattice is called a *Noether lattice* if it is modular, principally generated (every element is a join of some principal elements) which satisfies the ascending chain condition. A Noether lattice L is local if it contains precisely one maximal prime element. If L is a *Noether lattice* and 0_L is prime, then L is said to be a *Noether domain*. In [19], J. F. Wells studied the restricted cancellation law in a Noether lattice. An element a in a Noether lattice L satisfies the restricted cancellation law if $ab = ac \neq 0_L$ implies $b = c$ for any $a, b, c \in L$.

2 ϕ-2-absorbing primary elements

Throughout this paper, ϕ denotes a function defined from L to $L \cup \{\varnothing\}$.

Definition 2.1. A proper element q is said to be ϕ-2-absorbing primary element of L if whenever $a, b, c \in L$ with $abc \leq q$ and $abc \nleq \phi(q)$ implies either $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$.

The special functions ϕ_α can be defined as following: Let q be a ϕ_α-2-absorbing primary element of L. Then we say

$\phi_\varnothing(q) = \varnothing \quad \Rightarrow \quad q$ is a 2-absorbing primary element,
$\phi_0(q) = 0 \quad \Rightarrow \quad q$ is a weakly 2-absorbing primary element,
$\phi_2(q) = q^2 \quad \Rightarrow \quad q$ is an almost 2-absorbing primary element,
...
$\phi_n(q) = q^n \quad \Rightarrow \quad q$ is an n-almost 2-absorbing primary element for $n > 2$,
$\phi_\omega(q) = \wedge_{n=1}^{\infty} q^n \quad \Rightarrow \quad q$ is a ω-2-absorbing primary element.

Observe that for an element $a \in L$ with $a \leq q$ but $a \nleq \phi(q)$ implies that $a \nleq q \wedge \phi(q)$. So without loss of generality, throughout we assume $\phi(q) \leq q$.

Remark 2.2. For any two functions $\psi_1, \psi_2 : L \to L \cup \{\varnothing\}$, we say $\psi_1 \leq \psi_2$ if $\psi_1(a) \leq \psi_2(a)$ for each $a \in L$. Thus clearly we have the following order: $\phi_\varnothing \leq \phi_0 \leq \phi_\omega \leq ... \leq \phi_{n+1} \leq \phi_n \leq ... \leq \phi_2 \leq \phi_1$.

Lemma 2.3. *Let q be a proper element of L and $\psi_1, \psi_2 : L \to L \cup \{\varnothing\}$ be two functions with $\psi_1 \leq \psi_2$. If q is a ψ_1-2-absorbing primary element of L, then q is a ψ_2-2-absorbing primary element of L.*

Proof. Let $a, b, c \in L$ such that $abc \leq q$ and $abc \nleq \psi_2(q)$. Hence we have $abc \nleq \psi_1(q)$. Since q is a ψ_1-2-absorbing primary element of L and $abc \leq q$, we are done. □

Theorem 2.4. *Let q be a proper element of L. Then the following statements are satisfied:*

(i) q is a 2-absorbing primary element of $L \Rightarrow q$ is a weakly 2-absorbing primary element of $L \Rightarrow q$ is a ω-2-absorbing primary element of $L \Rightarrow q$ is an $(n+1)$-almost 2-absorbing primary element of $L \Rightarrow q$ is an n-almost 2-absorbing primary element of L for all $n \geq 2 \Rightarrow q$ is an almost 2-absorbing primary element of L.

(ii) q is a ϕ-prime element of $L \Rightarrow q$ is a ϕ-2-absorbing element of $L \Rightarrow q$ is a ϕ-2-absorbing primary element of L.

(iii) If q is a ϕ-primary element of L, then q is a ϕ-2-absorbing primary element of L.

(iv) If a proper element q is an idempotent element of L, then q is a ω-2-absorbing primary element of L and q is an n-almost 2-absorbing primary element of L for all $n \geq 2$.

(v) Suppose that q is a radical element of L. Then q is a ϕ-2-absorbing primary element of L if and only if q is a ϕ-2-absorbing element of L.

(vi) q is an n-almost 2-absorbing primary element of L for all $n \geq 2$ if and only if q is a ω-2-absorbing primary element of L.

Proof. (i) From Remark 2.2 we get the order $\phi_\varnothing \leq \phi_0 \leq \phi_\omega \leq ... \leq \phi_{n+1} \leq \phi_n \leq \phi_2 \leq \phi_1$. Hence the resut follows from Lemma 2.3.

(ii) Suppose that q is a ϕ-prime element of L and $a, b, c \in L$ such that $abc \leq q$, $abc \nleq \phi(q)$. Assume that $ab \nleq q$. Hence we get $c \leq q$, that is $ac \leq q$. Thus q is a ϕ-2-absorbing element of L and also it is a ϕ-2-absorbing primary element of L as $q \leq \sqrt{q}$.

(iii) Suppose that q is a ϕ-primary element of L and $a, b, c \in L$ such that $abc \leq q$, $abc \nleq \phi(q)$ and $ab \nleq q$. Then we have $c \leq \sqrt{q}$. This follows $ac \leq \sqrt{q}$, so we are done.

(iv) If q is idempotent, then observe that $q = q^n$ for all $n \geqslant 1$. Hence $\phi_\omega(q) = \wedge_{n=1}^{\infty} q^n = q$. Thus q is a ω-2-absorbing element of L. Morever q is an n-almost 2-absorbing element for all $n \geq 2$ from (1).

(v) Suppose that $q = \sqrt{q}$ and q is a ϕ-2-absorbing primary element of L. Let $a, b, c \in L$ such that $abc \leq q$, $abc \nleq \phi(q)$. Hence we get either $ab \leq q$ or $ac \leq \sqrt{q} = q$ or $bc \leq \sqrt{q} = q$, we are done. The converse is clear from (2).

(vi) Choose $a, b, c \in L$ such that $abc \leq q$ but $abc \nleq \wedge_{n=1}^{\infty} q^n$. Thus $abc \leq q$ but $abc \nleq q^m$ for some $m \geq 2$. Since q is n-almost 2-absorbing primary for all $n \geq 2$, we obtain either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. The converse is seen easily by (1).

□

The converses of (i), (ii) and (iii) are not true in general as it is shown in the following example.

Example 2.5. Let $R = \mathbb{Z}_{24}$. Then $L := L(R) = \{(0), (1), (2), (3), (4), (6), (8), (12)\}$. Consider the proper element $q = (8)$ of L. Since $\sqrt{q} = (2)$ is a prime elemet of L, q is a 2-absorbing primary element by Theorem 2.7(1) in [12]. But it is not 2-absorbing since $(2)(2)(2) \leq (8)$ but $(2)(2) \nleq (8)$. Next we show that (8) is an almost 2-absorbing element. Indeed, since $\phi_3((8)) = (8)$, (8) is obviously a 3-almost 2-absorbing element, and since $\phi_3 \leq \phi_2$, (8) is an almost 2-absorbing element by Lemma 2.3. So (8) is also an example of almost 2-absorbing element which is not 2-absorbing.

Theorem 2.6. *Let q be a ϕ-2-absorbing primary element of L. If $\phi(q)$ is a 2-absorbing primary element of L, then q is a 2-absorbing primary element of L.*

Proof. Suppose that q is ϕ-2-absorbing primary and $\phi(q)$ is a 2-absorbing primary element of L. Let $a, b, c \in L$ such that $abc \leq q$ and $ab \nleq q$. If $abc \nleq \phi(q)$, then we are done as q is a ϕ-2-absorbing primary element of L. So suppose that $abc \leq \phi(q)$. Since $ab \nleq \phi(q)$, we get either $bc \leq \sqrt{\phi(q)}$ or $ac \leq \sqrt{\phi(q)}$. So we conclude $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$ as there is an order $\sqrt{\phi(q)} \leq \sqrt{q}$. This completes the proof. □

Before giving a condition for a ϕ-2-absorbing primary element to be a 2-absorbing primary, we introduce the concept of ϕ-triple primary zero of q as the following:

Definition 2.7. Let q be a ϕ-2-absorbing primary element of L and $a, b, c \in L$. If $abc \leq \phi(q)$ but $ab \nleq q, bc \nleq \sqrt{q}, ac \nleq \sqrt{q}$, then (a, b, c) is called a ϕ-triple primary zero of q.

Remark 2.8. If q is a ϕ-2-absorbing primary element of L which is not 2-absorbing primary, then there exists (a, b, c) a ϕ-triple primary zero of q for some $a, b, c \in L$.

Lemma 2.9. *Let q be a ϕ-2-absorbing primary element of L and suppose that (a, b, c) is a ϕ-triple primary zero of q for some $a, b, c \in L$. Then the followings hold:*

(i) $abq, bcq, acq \leq \phi(q)$.

(ii) $aq^2, bq^2, cq^2 \leq \phi(q)$.

(iii) $q^3 \leq \phi(q)$.

Proof. (i) Assume on contrary that $abq \nleq \phi(q)$. Then $ab(c \vee q) \nleq \phi(q)$. Since $ab \nleq q$ and q is ϕ-2-absorbing primary, we have $a(c \vee q) \leq \sqrt{q}$ or $b(c \vee q) \leq \sqrt{q}$. Hence we have either $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$, a contradiction. Thus $abq \leq \phi(q)$. Similarly it is easily shown that $bcq \leq \phi(q)$ and $acq \leq \phi(q)$.

(ii) Suppose that $aq^2 \nleq \phi(q)$. Then we get $a(b \vee q)(c \vee q) \nleq \phi(q)$ implies that either $a(b \vee q) \leq q$ or $a(c \vee q) \leq \sqrt{q}$ or $(b \vee q)(c \vee q) \leq \sqrt{q}$. So either $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$, which is a contradiction. Thus $aq^2 \leq \phi(q)$. One can easily show that $bq^2, cq^2 \leq \phi(q)$.

(iii) Suppose that $q^3 \nleq \phi(q)$. So we can write $(a \vee q)(b \vee q)(c \vee q) \leq q$ but $(a \vee q)(b \vee q)(c \vee q) \nleq \phi(q)$. As q is ϕ-2-absorbing primary, we get $(a \vee q)(b \vee q) \leq q$ or $(a \vee q)(c \vee q) \leq \sqrt{q}$ or $(b \vee q)(c \vee q) \leq \sqrt{q}$, which means $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$, a contradiction. Thus $q^3 \leq \phi(q)$.

□

Corollary 2.10. *Let q be a ϕ-2-absorbing primary element of L such that $\phi \leq \phi_4$. Then q is a ϕ_n-2-absorbing primary element of L for every $n \geq 2$. Morever q is a ϕ_ω-2-absorbing primary element of L.*

Proof. If q is a 2-absorbing primary element of L, then clearly it is ϕ_n-2-absorbing primary for all $n \geq 2$ and ϕ_ω-2-absorbing primary element of L by Theorem 2.4. So assume that q is not a 2-absorbing primary element of L. From Lemma 2.9 (3), we have $q^3 \leq \phi(q)$. Hence we get $q^3 \leq \phi(q) \leq q^4$ as $\phi \leq \phi_4$. It follows $q^3 = q^n = \phi(q)$ for every $n \geq 3$. Thus q is a ϕ_n-2-absorbing primary element of L for every $n \geq 2$. Since $\phi_\omega(q) = q^n = q^3 = \phi_3(q)$, q is a ϕ_ω-2-absorbing primary element of L. □

The following corollary gives a condition for a ϕ-2-absorbing primary element to be 2-absorbing primary.

Corollary 2.11. *Let q be a proper element of L.*

(i) If q is a ϕ-2-absorbing primary element of L such that $q^3 \nleq \phi(q)$, then q is a 2-absorbing primary element of L.

(ii) If q is a ϕ-2-absorbing primary element of L that is not a 2-absorbing primary element of L, then $\sqrt{q} = \sqrt{\phi(q)}$.

Proof. (i) We conclude directly this result by Remark 2.8 and Lemma 2.9 (iii).

(ii) Suppose that q is a ϕ-2-absorbing primary element of L which is not 2-absorbing primary. Hence we get $q^3 \leq \phi(q)$ by Lemma 2.9 (iii). So we have $q \leq \sqrt{\phi(q)}$, which means $\sqrt{q} \leq \sqrt{\phi(q)}$. On the other hand, since $\phi(q) \leq q$, we have $\sqrt{\phi(q)} \leq \sqrt{q}$. Thus $\sqrt{q} = \sqrt{\phi(q)}$. □

Theorem 2.12. *Let q be a proper element of L such that $\sqrt{\phi(q)}$ is a primary (prime) element of L. Then the followings are equivalent:*

(i) q is a ϕ-2-absorbing primary element of L.

(ii) q is a 2-absorbing primary element of L.

Proof. (i) ⇒(ii) Suppose that q is a ϕ-2-absorbing primary element of L that is not 2-absorbing primary. Then $\sqrt{q} = \sqrt{\phi(q)}$ by Corollary 2.11 (ii). Hence $\sqrt{q}$ is a primary *(prime)* element. Thus q is a 2-absorbing element of L by Theorem 2.7 in [12], a contradiction. Consequently, q is a 2-absorbing primary element of L.

(ii) ⇒ (i) It is clear. □

Theorem 2.13. *Let q be a proper element of L. If q is a ϕ-2-absorbing primary element of L such that $\sqrt{\phi(q)} = \phi(\sqrt{q})$, then $\sqrt{q}$ is a ϕ-2-absorbing element of L.*

Proof. Let $p = \sqrt{q}$. Suppose that $abc \leq p$ and $abc \nleq \phi(p)$ but $ab \nleq p$ for some $a, b, c \in L$. Then there is a positive integer n such that $(abc)^n \leq q$. Also, $(abc)^n \nleq \phi(q)$ for every positive integer n by hypothesis. Since q is a ϕ-2-absorbing primary element of L and $(ab)^n \nleq q$ for all positive integer n, then $b^n c^n \leq \sqrt{q}$ or $a^n c^n \leq \sqrt{q}$. Thus $bc \leq \sqrt{\sqrt{q}} = \sqrt{q} = p$ or $ac \leq \sqrt{\sqrt{q}} = p$. Therefore, p is a ϕ-2-absorbing element of L. □

Theorem 2.14. *Let L be a local Noether domain. If q is a ϕ_n-2-absorbing primary element of L for all $n \geqslant 2$, then q is a 2-absorbing primary element of L.*

Proof. Let $abc \leq q$ for some $a, b, c \in L$. If $abc \nleq \phi_n(q)$, then we have either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$ as q is a ϕ_n-2-absorbing primary element of L. So assume that $abc \leq \phi_n(q)$. Since $\wedge_{n=1}^{\infty} q^n = 0_L$, from Corollary 3.3 of [14], we have $abc \leq 0$. Since L is a domain, we get either $a \leq 0_L$ or $b \leq 0_L$ or $c \leq 0_L$. Thus $ab \leq q$ or $bc \leq q$ or $ac \leq q$, we are done. □

Recall that for any $a \in L$, $L/a = \{b \in L \mid a \leq b\}$ is a multiplicative lattice with the multiplication $c \circ d = cd \vee a$. Now we conclude the following properties of ϕ-2-absorbing primary elements in quotient lattices.

Theorem 2.15. *Let q be a proper element of L. Then the following statements are equivalent:*

(i) q is a ϕ-2-absorbing primary element of L.

(ii) $q \vee \phi(q)$ is a weakly 2-absorbing primary element of $L/\phi(q)$.

Proof. (i)⇒ (ii): If $\phi(q) = \varnothing$, then it is clear. Assume that $\phi(q) \neq \varnothing$. Let $\phi(q) \neq (a \vee \phi(q)) \circ (b \vee \phi(q)) \circ (c \vee \phi(q)) = abc \vee \phi(q) \leq q \vee \phi(q)$ for some $a, b, c \in L$. Observe that $q \vee \phi(q) = q$ as $\phi(q) \leq q$. Then $abc \leq q$, but $abc \nleq \phi(q)$. Thus either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. So $(a \vee \phi(q)) \circ (b \vee \phi(q)) \leq q$ or $(b \vee \phi(q)) \circ (c \vee \phi(q)) \leq \sqrt{q}$ or $(a \vee \phi(q)) \circ (c \vee \phi(q)) \leq \sqrt{q}$. Consequently, q is a weakly 2-absorbing element of $L/\phi(q)$.

(ii)⇒ (i): Let $abc \leq q$ and $abc \nleq \phi(q)$ for some $a, b, c \in L$. Then $\phi(q) \neq (a \vee \phi(q)) \circ (b \vee \phi(q)) \circ (c \vee \phi(q)) \leq q$. Thus we get $(a \vee \phi(q)) \circ (b \vee \phi(q)) \leq q$ or $(b \vee \phi(q)) \circ (c \vee \phi(q)) \leq \sqrt{q}$ or $(a \vee \phi(q)) \circ (c \vee \phi(q)) \leq \sqrt{q}$. So we obtain $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. □

Observe that q is a primary element of L if and only if q is a weakly primary element of $L/\phi(q)$.

Corollary 2.16. *A proper element q of L is ϕ_n-2-absorbing primary if and only if q is a weakly 2-absorbing primary element of L/q^n for all $n \geq 2$.*

Recall from [12] that if $abc \leq q$ but $ab \nleq q$, $ac \nleq \sqrt{q}$, $bc \nleq \sqrt{q}$ for some $a, b, c \in L$, then (a, b, c) is called a triple zero of q.

Proposition 2.17. *Let q be a ϕ-2-absorbing primary element of L and $a, b, c \in L$. Then (a, b, c) is a ϕ-triple primary zero of q if and ony if $(a \vee \phi(q), b \vee \phi(q), c \vee \phi(q))$ is a triple zero of q.*

Proof. Suppose that (a, b, c) is a ϕ-triple primary zero of q. Then $abc \leq \phi(q)$ but $ab \nleq q$, $ac \nleq \sqrt{q}$ and $bc \nleq \sqrt{q}$. Thus $ab \vee \phi(q) \nleq q$, $ac \vee \phi(q) \nleq \sqrt{q}$ and $bc \vee \phi(q) \nleq \sqrt{q}$. Since q is a weakly 2-absorbing primary element of $L/\phi(q)$ by Theorem 2.15, $(a \vee \phi(q), b \vee \phi(q), c \vee \phi(q))$ is a triple zero of q. Conversely, suppose that $(a \vee \phi(q), b \vee \phi(q), c \vee \phi(q))$ is a triple zero of q. Hence $abc \leq \phi(q)$ with $ab \vee \phi(q) \nleq q$, $ac \vee \phi(q) \nleq \sqrt{q}$ and $bc \vee \phi(q) \nleq \sqrt{q}$. So $ab \nleq q$, $ac \nleq \sqrt{q}$ and $bc \nleq \sqrt{q}$. Therefore (a, b, c) is a ϕ-triple primary zero of q. □

Theorem 2.18. *Let x, y be proper elements of L with $x \leq y$ and let $n \geq 2$. If y is a ϕ_n-2-absorbing primary element of L, then y is a ϕ_n-2-absorbing primary element of L/x.*

Proof. Let y be a ϕ_n-2-absorbing primary element of L. Suppose that $(a \vee x) \circ (b \vee x) \circ (c \vee x) = abc \vee x \leq y$ and $(a \vee x) \circ (b \vee x) \circ (c \vee x) = abc \vee x \nleq y^n$ for some $a, b, c \in L$. As $y \in L/x$, then $y^n = y \circ y \circ y \circ ... \circ y = y^n \vee x$. Since $x \leq y$ and $abc \vee x \nleq y^n = y^n \vee x$, then we have $abc \leq y$ and $abc \nleq y^n$. Hence $ab \leq y$ or $ac \leq y$ or $bc \leq y$. Since $x \leq y$, we conclude that either $(a \vee x) \circ (b \vee x) \leq y$ or $(a \vee x) \circ (c \vee x) \leq \sqrt{y}$ or $(b \vee x) \circ (c \vee x) \leq \sqrt{y}$. Thus y is a ϕ_n-2-absorbing primary element of L/x. □

Corollary 2.19. *Let x and y be proper elements of L with $x \leq y$. If y is a ϕ_ω- 2-absorbing primary element of L, then y is a ϕ_ω- 2-absorbing primary element of L/x.*

Proof. The proof is obtained easily similar to the proof of Theorem 2.18. □

Definition 2.20. Let x be a proper element of L/q such that $q \leq x$. Then x is called a ϕ_q- 2-absorbing primary element of L/q if whenever $a, b, c \in L/q$ with $abc \leq x$ and $abc \nleq \phi(x) \vee q$ implies $ab \leq x$ or $ac \leq \sqrt{x}$ or $bc \leq \sqrt{x}$.

Theorem 2.21. *Let p and q be two proper elements of L with $q \leq p$. If p is a ϕ-2-absorbing primary element of L, then p is a ϕ_q-2-absorbing primary element of L/q.*

Proof. Assume that $(a \vee q) \circ (b \vee q) \circ (c \vee q) \leq p$ and $abc \vee q = (a \vee q) \circ (b \vee q) \circ (c \vee q) \nleq \phi(p) \vee q$ for some $a, b, c \in L$. Then we get $abc \leq p$ and $abc \nleq \phi(p)$. As p is ϕ-2-absorbing primary element of L, we have either $ab \leq p$ or $ac \leq \sqrt{p}$ or $bc \leq \sqrt{p}$. So we obtain $(a \vee q) \circ (b \vee q) \leq p$ or $(a \vee q) \circ (c \vee q) \leq \sqrt{p}$ or $(b \vee q) \circ (c \vee q) \leq \sqrt{p}$. □

Theorem 2.22. *Let p and q be two proper elements of L with $q \leq \phi(p)$. Then the following statements are equivalent:*

(i) p is a ϕ-2-absorbing primary element of L.

(ii) p is a ϕ_q-2-absorbing primary element of L/q.

(iii) p is a ϕ_{q^n}-2-absorbing primary element of L/q^n.

Proof. (i)⇒ (ii): Suppose that p is a ϕ-2-absorbing primary element of L. Then p is a ϕ_q-2-absorbing primary element of L/q by Theorem 2.21.

(ii)⇒(iii): Let $n \geq 1$. Observe that $q^n \leq q \leq \phi(p)$. Now suppose that $(a \vee q^n) \circ (b \vee q^n) \circ (c \vee q^n) \leq p$ and $(a \vee q^n) \circ (b \vee q^n) \circ (c \vee q^n) \nleq \phi(p) \vee q^n$ for some $a, b, c \in L$. So $abc \nleq \phi(p)$. As $q \leq \phi(p)$ and $abc \nleq \phi(p)$, we get $abc \nleq q$. Thus $(a \vee q) \circ (b \vee q) \circ (c \vee q) \leq p$ and $(a \vee q) \circ (b \vee q) \circ (c \vee q) \nleq \phi(p) \vee q$. Since p is ϕ_q- 2-absorbing element of L/q, we obtain $ab \leq p$ or $ac \leq \sqrt{p}$ or $bc \leq \sqrt{p}$. Consequently, $ab \vee q^n \leq p$ or $ac \vee q^n \leq \sqrt{p}$ or $bc \vee q^n \leq \sqrt{p}$ (in L/q^n).

(iii)⇒(i): Let $a, b, c \in L$ with $abc \leq p$ and $abc \nleq \phi(p)$. Since $q^n \leq \phi(p)$, one can see $abc \nleq q^n$. As $q^n \leq \phi(p) \leq p$, we get $(a \vee q^n) \circ (b \vee q^n) \circ (c \vee q^n) = abc \vee q^n \leq p$ and $(a \vee q^n) \circ (b \vee q^n) \circ (c \vee q^n) \nleq \phi(p) \vee q^n$. As p is a ϕ_{q^n}-2-absorbing element of L/q^n, we conclude that $ab \leq p$ or $ac \leq \sqrt{p}$ or $bc \leq \sqrt{p}$. □

Corollary 2.23. *Let q be a proper element of L which is not a weakly 2-absorbing primary. Then the following statements are equivalent:*

(i) q is a ϕ-2-absorbing primary element of L.

(ii) q is a ϕ_{p^3}-2-absorbing primary element of L/q^3.

(iii) q is a ϕ_{p^n}-2-absorbing primary element of L/q^n for every $n \geq 3$.

Proof. Assume that q is not a weakly 2-absorbing primary element of L. So q is not a 2-absorbing primary element of L. Hence we get $q^3 \leq \phi(q)$ by Lemma 2.9 (iii). Thus the results are clear by Theorem 2.22. □

Definition 2.24. Let q be a proper element of L and $n \geq 2$. Then q is said to be n-potent 2-absorbing primary if whenever $a, b, c \in L$ with $abc \leq q^n$, then $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$.

Theorem 2.25. *Let q be an n-almost 2-absorbing primary element for some $n \geq 2$. If q is k-potent 2-absorbing primary for some $k \leq n$, then q is a 2-absorbing primary element of L.*

Proof. Assume that q is an n-almost 2-absorbing primary element. Let $a, b, c \in L$ such that $abc \leq q$. If $abc \nleq q^k$, then clearly we have $abc \nleq q^n$. Since q is an n-almost 2-absorbing primary element, we conclude either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. Now suppose that $abc \leq q^k$. Since q is k-potent 2-absorbing primary, we conclude that either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$, which completes the proof. □

Recall that $J(L) = \wedge\{m \in L \mid m \text{ is a maximal element of } L\}$.

Theorem 2.26. *Let L be a Noether domain. Then an element q of L with $q \leq J(L)$ is a 2-absorbing primary element of L if and only if q is a ϕ_n-2-absorbing primary element of L for all $n \geq 2$.*

Proof. Assume that q is ϕ_n-2-absorbing primary for all $n \geq 2$. Let $a, b, c \in L$ such that $abc \leq q$. If $abc \nleq q^k$ for some $k \geq 2$, we get either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. Now suppose that $abc \leq q^n$ for all $n \geq 2$. From Corollary 1.4 in [4], we conclude $abc \leq \wedge_{n=1}^{\infty} q^n = 0_L$ as L is a Noether domain. Hence we get either $a = 0_L$ or $b = 0_L$ or $c = 0_L$. Without loss generality assume that $a = 0_L$. Thus we get $ab = 0_L \leq q$. The converse is clear from Theorem 2.4. □

Theorem 2.27. *Let L be a Noether lattice and a non-zero non-nilpotent proper element q of L satisfies the restricted cancellation law. Then q is a ϕ-2-absorbing primary element of L for some $\phi \leq \phi_n$ and for all $n \geq 2$ if and only if q is a 2-absorbing primary element of L.*

Proof. Suppose that q is a 2-absorbing primary element of L. Thus q is a ϕ-2-absorbing primary element of L for all ϕ. Therefore q is ϕ-2-absorbing primary for some $\phi \leq \phi_n$ and for all $n \geq 2$.

Conversely, we assume that q is a ϕ-2-absorbing primary element of L for some $\phi \leq \phi_n$ and for all $n \geq 2$. Then q is a ϕ_n-2-absorbing primary element of L for all $n \geq 2$ by Lemma 2.3. Let $abc \leq q$ for some $a, b, c \in L$. So we have two cases:

Case 1: Let $abc \nleq q^n$ for some $n \geq 2$. Then we obtain $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$ by the hypothesis.
Case 2: Let $abc \leq q^n$ for all $n \geq 2$. Note that $a(b \vee q)(c \vee q) = abc \vee abq \vee acq \vee aq^2 \leq q$. If $a(b \vee q)(c \vee q) \nleq q^n$, then $a(b \vee q) \leq q$ or $(b \vee q)(c \vee q) \leq \sqrt{q}$ or $a(c \vee q) \leq \sqrt{q}$. We get that either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. If $a(b \vee q)(c \vee q) \leq q^n$, then $a(b \vee q)(c \vee q) = abc \vee abq \vee acq \vee aq^2 \leq q^n \leq q^2$. We conclude either $ab \leq q$ or $ac \leq q$ by [19, Lemma 1.11]. Consequently, q is a 2-absorbing primary element of L. ◻

Proposition 2.28. *Let q be a ϕ-2-absorbing primary element of L and $\phi(q) \leq \phi(p)$ for some radical element p of L with $p < q$. Then q is a 2-absorbing primary element of L.*

Proof. Assume on the contrary that q is not a 2-absorbing primary element. Hence $\sqrt{q} = \sqrt{\phi(q)}$ by Corollary 2.11. Since we have the order $\phi(q) \leq \phi(p) \leq p$ and p is a radical element, we conclude $\sqrt{q} = \sqrt{\phi(q)} \leq \sqrt{\phi(p)} \leq p$ which means $q \leq p$, a contradiction. Thus q is a 2-absorbing primary element of L. ◻

3 ϕ-2-absorbing Primary Elements of Cartesian Product of C-lattices

Let $L = L_1 \times L_2 \times ...L_n$ where $L_1, L_2, ..., L_n$ are multiplicative lattices $(n \geq 1)$ and let $\phi = \psi_1 \times \psi_2 \times ... \times \psi_n$ where $\psi_i : L_i \to L_i \cup \{\varnothing\}$ $(i = 1, ..., n)$ be a function. Let $a = (a_1, a_2, ..., a_n)$ be an element of L. Observe that if $\psi_i(a_i) = \varnothing$ for some $i = 1, ..., n$, then there is no element of $\phi(a)$ and vice versa. Thus $\phi(a) = \varnothing$ if and only if $\psi_i(a_i) = \varnothing$ for some $i = 1, ..., n$.

Lemma 3.1. *Let $L = L_1 \times L_2$ where L_1, L_2 are C-lattices. Then $\sqrt{(a_1, a_2)} = (\sqrt{a_1}, \sqrt{a_2})$ for any $(a_1, a_2) \in L_1 \times L_2$.*

Proof. Let $(x, y) \leq \sqrt{(a_1, a_2)}$ for some $(x, y) \in L_1 \times L_2$. Then $(x, y)^n = (x^n, y^n) \leq (a_1, a_2)$ for some positive integer n. Thus $x^n \leq a_1$ and $y^n \leq a_2$. So $x \leq \sqrt{a_1}$ and $y \leq \sqrt{a_2}$, that is, $(x, y) \leq (\sqrt{a_1}, \sqrt{a_2})$. Conversely, let $(x, y) \leq (\sqrt{a_1}, \sqrt{a_2})$. Then $x \leq \sqrt{a_1}$ and $y \leq \sqrt{a_2}$. There are two positive integers n, m such that $x^n \leq a_1$ and $y^m \leq a_2$. Then $(x^{nm}, y^{nm}) = (x, y)^{nm} \leq (a_1, a_2)$ and so $(x, y) \leq \sqrt{(a_1, a_2)}$. ◻

Theorem 3.2. *Let $L = L_1 \times L_2$ where L_1, L_2 are C-lattices and $\phi = \psi_1 \times \psi_2$, where $\psi_i : L_i \longrightarrow L_i \cup \{\varnothing\}$ $(i = 1, 2)$ is a function such that $\psi_2(1_{L_2}) \neq 1_{L_2}$. Let q_1 be a proper element of L_1. Then the following statements are equivalent:*

(i) $q = (q_1, 1_{L_2})$ is a ϕ-2-absorbing primary element of L.

(ii) q_1 is a 2-absorbing primary element of L_1.

(iii) $q = (q_1, 1_{L_2})$ is a 2-absorbing primary element of L.

Proof. Let $\psi_1(q_1) = \varnothing$ or $\psi_2(1_{L_2}) = \varnothing$. Then we obtain $\phi(q) = \varnothing$. So it is clear from Theorem 2.21 in [12]. Hence we suppose that $\psi_1(q_1) \neq \varnothing$ and $\psi_2(1_{L_2}) \neq \varnothing$.

(i) $\Rightarrow$ (ii): Let $q = (q_1, 1_{L_2})$ be a ϕ-2-absorbing primary element of L. First we show that q_1 is a ψ_1-2-absorbing primary element of L_1. Assume on the contrary that q_1 is not ψ_1-2-absorbing primary. Then there exist a, b, c in L_1 such that $abc \leq q_1$ and $abc \nleq \psi_1(q_1)$ but $ab \nleq q_1$ and $bc \nleq \sqrt{q_1}$ and $ac \nleq \sqrt{q_1}$. Hence $(abc, 1_{L_2}) = (a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \leq q$ and $(abc, 1_{L_2}) = (a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \nleq (\psi_1(q_1), \psi_2(1_{L_2})) = \phi(q)$. This implies either $(ab, 1_{L_2}) = (a, 1_{L_2})(b, 1_{L_2}) \leq q$ or $(bc, 1_{L_2}) = (b, 1_{L_2})(c, 1_{L_2}) \leq \sqrt{q}$ or $(ac, 1_{L_2}) = (a, 1_{L_2})(c, 1_{L_2}) \leq \sqrt{q}$. Thus either $ab \leq q_1$ or $bc \leq \sqrt{q_1}$ or $ac \leq \sqrt{q_1}$, a contradiciton. Hence q_1 is a ψ_1-2-absorbing primary element of L_1.

Next we prove that q_1 is a 2-absorbing primary element of L_1. If q_1 is not a 2-absorbing primary element of L_1, then there is a ψ_1-triple-zero (x, y, z) of q_1 for some $x, y, z \in L_1$. Since $\psi_2(1_{L_2}) \neq 1_{L_2}$, then we get $(xyz, 1_{L_2}) = (x, 1_{L_2})(y, 1_{L_2})(z, 1_{L_2}) \leq q$ and $(xyz, 1_{L_2}) = (x, 1_{L_2})(y, 1_{L_2})(z, 1_{L_2}) \nleq \phi(q)$. Then $(x, 1_{L_2})(y, 1_{L_2}) \leq q$ or $(y, 1_{L_2})(z, 1_{L_2}) \leq \sqrt{q}$ or $(x, 1_{L_2})(z, 1_{L_2}) \leq \sqrt{q}$. Thus we have $xy \leq q_1$ or $yz \leq \sqrt{q_1}$ or $xz \leq \sqrt{q_1}$, a contradiction. Therefore q_1 is a 2-absorbing primary element of L_1.

(ii)$\Rightarrow$ (iii) It is obvious by Theorem 2.21 in [12].

(iii) $\Rightarrow$ (i) It is clear from Theorem 2.4. ◻

Theorem 3.3. *Let $L = L_1 \times L_2$ where L_1, L_2 are C-lattices and $\phi = \psi_1 \times \psi_2$, where $\psi_i : L_i \to L_i \cup \{\varnothing\}$ $(i = 1, 2)$ is a function. Then the following statements hold:*

(i) If q_i is a proper element of L_i with $\psi_i(q_i) = q_i$ $(i = 1, 2)$, then $q = (q_1, q_2)$ is a ϕ-2-absorbing primary element of L.

(ii) If q_1 is ψ_1-2-absorbing primary element of L_1 and $\psi_2(1_{L_2}) = 1_{L_2}$, then $q = (q_1, 1_{L_2})$ is a ϕ-2-absorbing primary element of L.

(iii) If q_2 is a ψ_2-2-absorbing primary element of L_2 and $\psi_1(1_{L_1}) = 1_{L_1}$, then $q = (1_{L_1}, q_2)$ is a ϕ-2-absorbing primary element of L.

Proof. (i) Let $\psi_1(q_1) = q_1$ and $\psi_2(q_2) = q_2$. Then we know that there is no an element (a, b) such that $(a, b) \leq (q_1, q_2)$ and $(a, b) \nleq \phi(q_1, q_2) = (q_1, q_2)$. Thus the proof is completed.

(ii) Suppose that $\psi_1(q) = \varnothing$. Then $q = (q_1, 1_{L_2})$ is a ϕ-2-absorbing primary element of L by Theorem 3.2 ($2 \Rightarrow 1$). So assume that $\psi_1(q) \neq \varnothing$. Let $abc \leq q$ and $abc \nleq \phi(q)$ for some $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2) \in L$. Thus $a_1b_1c_1 \leq q_1$ and $a_1b_1c_1 \nleq \psi_1(q_1)$. Since q_1 is ψ_1-2-absorbing element of L_1, we get either $a_1b_1 \leq q_1$ or $b_1c_1 \leq \sqrt{q_1}$ or $a_1c_1 \leq \sqrt{q_1}$. Then we have either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. Therefore q is a ϕ-2-absorbing primary element of L.

(iii) It can be easily seen similar to (ii).

□

Theorem 3.4. *Let $L = L_1 \times L_2$, where L_1, L_2 are C-lattices q_1 and q_2 be elements of L_1, L_2, respectively. Let $\phi = \psi_1 \times \psi_2$, where $\psi_i : L_i \to L_i \cup \{\varnothing\}$ $(i = 1, 2)$ is a function with $\psi_i(q_i) \neq q_i$ $(i = 1, 2)$. If $q = (q_1, q_2)$ is a proper element of L, then the following statements are equivalent:*

(i) q is a ϕ-2-absorbing primary element of L.

(ii) $q_1 = 1_{L_1}$ and q_2 is a 2-absorbing primary element of L_2 or $q_2 = 1_{L_2}$ and q_1 is a 2-absorbing primary element of L_1 or q_1, q_2 are primary elements of L_1, L_2, respectively.

(iii) q is a 2-absorbing primary element of L.

Proof. (i)⇒ (ii): Suppose that q is a ϕ-2-absorbing primary element of L. From Theorem 2.15, $(q_1 \vee \psi_1(q_1), q_2 \vee \psi_2(q_2))$ is a weakly 2-absorbing element of $L_1/\psi_1(q_1) \times L_2/\psi_2(q_2)$. Hence we conclude either $q_1 \vee \psi_1(q_1) = 1_{L_1} \vee \psi_1(q_1)$ and $q_2 \vee \psi_2(q_2)$ is a 2-absorbing primary element of $L_2/\psi_2(q_2)$ or $q_2 \vee \psi_2(q_2) = 1_{L_2} \vee \psi_1(q_2)$ and $q_1 \vee \psi_1(q_1)$ is a 2-absorbing primary element of $L_1/\psi_1(q_1)$ or $q_1 \vee \psi_1(q_1)$ and $q_2 \vee \psi_2(q_2)$ are primary elements of $L_1/\psi_1(q_1)$ and $L_2/\psi_2(q_2)$, respectively by Theorem 2.22 in [12]. Therefore from Theorem 2.15, we get either $q_1 = 1_{L_1}$ and q_2 is a 2-absorbing primary element of L_2 or $q_2 = 1_{L_2}$ and q_1 is a 2-absorbing primary element of L_1 or q_1 and q_2 are primary elements of L_1 and L_2, respectively.

(ii)⇒ (iii): It is clear from Theorem 2.22 in [12].

(iii)⇒ (i): Suppose that q is a 2-absorbing primary element of L. Then q is a 2-absorbing primary element of L by Theorem 2.21 of [12], so we are done. □

Theorem 3.5. *Let $(L_1, 0_{L_1})$ and $(L_2, 0_{L_2})$ be quasi-local C-lattices which are not field and $L = L_1 \times L_2$. Then the followings are hold:*

(i) Every proper element of L is a 2-absorbing primary element of L.

(ii) Every proper element of L is a ϕ-2-absorbing primary element of L.

Proof. (i) Let $q = (q_1, q_2)$ be a proper element of L. Then $\sqrt{q_i} = \sqrt{0_{L_i}}$, $(i = 1, 2)$ as 0_{L_1} and 0_{L_2} are maximal elements of L_1 and L_2, respectively. Hence q_1 and q_2 are primary elements of L_1 and L_2, respectively. So $q = (q_1, q_2)$ is a 2-absorbing primary element of L by Theorem 3.4.

(ii) Since every 2-absorbing primary element is a ϕ-2-absorbing primary element of L, we are done from (i).

□

Theorem 3.6. *If $L = L_1 \times L_2$ where L_1, L_2 are C-lattices, then the following statements are equivalent:*

(i) Every proper element of L is a 2-absorbing primary element of L.

(ii) Every proper element of L_1 is a primary element of L_2 and every proper element of L_2 is a primary element of L_2.

Proof. $(i) \Rightarrow (ii)$ Let q_1 is a proper element of L_1 and $ab \leq q_1$ for some $a, b \in L_1$. Then $(q_1, 0_{L_2})$ is a 2-absorbing primary element of L from (i). Hence $(a, 1_{L_2})(b, 1_{L_2})(1_{L_1}, 0_{L_2}) \leq (q_1, 0_{L_2})$. Since $(ab, 1_{L_2}) \nleq \sqrt{(q_1, 0_{L_2})}$, we have either $(a, 0_{L_2}) = (a, 1_{L_2})(1_{L_1}, 0_{L_2}) \leq (q_1, 0_{L_2})$ or $(b, 0_{L_2}) = (b, 1_{L_2})(1_{L_1}, 0_{L_2}) \leq \sqrt{(q_1, 0_{L_2})}$. This means that $a \leq q_1$ or $b \leq \sqrt{q_1}$. Thus q_1 is a primary element of L. Similarly one can easily show that every proper element of L_2 is a primary element of L_2.

$(ii) \Rightarrow (i)$ It is clear from Theorem 3.4. □

Lemma 3.7. *Let $L = L_1 \times L_2 \times L_3$ where L_1, L_2, L_3 are C-lattices. Let $\phi = \psi_1 \times \psi_2 \times \psi_3$, where $\psi_i : L_i \longrightarrow L_i \cup \{\varnothing\}$ $(i = 1, 2, 3)$ is a function with $\psi_i(1_{L_i}) \neq 1_{L_i}$. If $q = (q_1, q_2, q_3)$ is a ϕ-2-absorbing primary element of L and $q \neq \phi(q)$, then q is a 2-absorbing primary element of L.*

Proof. The result is clear if $\phi(q) = \varnothing$. Suppose that $\phi(q) \neq \varnothing$ and $q \neq \phi(q)$. So $(a, b, c) \leq q$ but $(a, b, c) \nleq \phi(q)$ for some $(a, b, c) \in L$. Hence $(a, b, c) = (a, 1_{L_2}, 1_{L_3})(1_{L_1}, b, 1_{L_3})(1_{L_1}, 1_{L_2}, c) \leq q$ implies that either $(a, 1_{L_2}, 1_{L_3})(1_{L_1}, b, 1_{L_3}) \leq q$ or $(1_{L_1}, b, 1_{L_3})(1_{L_1}, 1_{L_2}, c) \leq \sqrt{q}$ or $(a, 1_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, c) \leq \sqrt{q}$. Without loss of generality assume that $(1_{L_1}, b, 1_{L_3})(1_{L_1}, 1_{L_2}, c) \leq q$. Then $q_1 = 1_{L_1}$ which means that $q^3 \nleq \phi(q)$. Thus q is a 2-absorbing primary element of L by Corollary 2.11. □

Theorem 3.8. *Let $L = L_1 \times L_2 \times L_3$ where L_1, L_2, L_3 are C-lattices. Let $\phi = \psi_1 \times \psi_2 \times \psi_3$, where $\psi_i : L_i \longrightarrow L_i \cup \{\varnothing\}$ $(i = 1, 2, 3)$ is a function with $\psi_i(1_{L_i}) \neq 1_{L_i}$. If $q \neq \phi(q)$, then the followings are equivalent:*

(i) q is a ϕ-2-absorbing primary element of L.

(ii) q is a 2-absorbing primary element of L.

(iii) q is in one of the following type:

I) $q = (1_{L_1}, q_2, q_3)$, where q_2 is a primary element of L_2 and q_3 is a primary element of L_3.

II) $q = (q_1, 1_{L_2}, q_3)$, where q_1 is a primary element of L_1 and q_3 is a primary element of L_3.

III) $q = (q_1, q_2, 1_{L_3})$, where q_1 is a primary element of L_1 and q_2 is a primary element of L_2.

IV) For some $i \in \{1, 2, 3\}$, q_i is a 2-absorbing primary element of L_i and $q_j = 1_{L_j}$ for every $j \in \{1, 2, 3\} \backslash \{i\}$.

Proof. (i)⇒ (ii): If $\phi(q) = \varnothing$ and q is a ϕ-2-absorbing primary element, then q is a 2-absorbing primary element of L. Assume that $\phi(q) \neq \varnothing$. Let $q = (q_1, q_2, q_3)$ be a ϕ-2-absorbing primary element of L, then q is a 2-absorbing primary element of L by Lemma 3.7.

(ii)⇒ (iii): Suppose that q is a 2-absorbing primary element of L. Since $q \neq \phi(q)$, there is a compact element $(a_1, a_2, a_3) \in L$ such that $(a_1, a_2, a_3) \leq q$ and $(a_1, a_2, a_3) \nleq \phi(q)$. Since $(a_1, a_2, a_3) = (a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, a_2, 1_{L_3})(1_{L_1}, 1_{L_2}, a_3)$ and q is ϕ-2-absorbing primary, we have $(a_1, a_2, 1_{L_3}) \leq q$ or $(1_{L_1}, a_2, a_3) \leq \sqrt{q}$ or $(a_1, 1_{L_2}, a_3) \leq \sqrt{q}$. It means that either $q_1 = 1_{L_1}$ or $q_2 = 1_{L_2}$ or $q_3 = 1_{L_3}$.

Case I. Suppose that $q = (1_{L_1}, q_2, q_3)$ where $q_2 \neq 1_{L_2}$ and $q_3 \neq 1_{L_3}$. We show that q_2 is a primary element of L_2. Let $xy \leq q_2$. Hence $(1_{L_1}, x, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3})(1_{L_1}, y, 1_{L_3}) \leq q$ and it implies that $(1_{L_1}, x, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \leq q$ or $(1_{L_1}, x, 1_{L_3})(1_{L_1}, y, 1_{L_3}) \leq \sqrt{q}$ or $(1_{L_1}, 1_{L_2}, 0_{L_3})(1_{L_1}, y, 1_{L_3}) \leq \sqrt{q}$. Since q_3 is proper, we get $(1_{L_1}, xy, 1_{L_3}) = (1_{L_1}, x, 1_{L_3})(1_{L_1}, y, 1_{L_3}) \nleq \sqrt{q}$. Thus $x \leq q_2$ or $y \leq \sqrt{q_2}$, which shows that q_2 is primary. By the similar argument one can easily show that q_3 is a primary element of L_3.

Case II. $q = (q_1, 1_{L_2}, q_3)$, where $q_{1,3} \neq 1_{L_{1,3}}$ and Case III. $q = (q_1, q_2, 1_{L_3})$, where $q_{1,2} \neq 1_{L_{1,2}}$ can be easily obtained similar to Case I.

Case IV. Let $i = 1$. Then $q = (q_1, 1_{L_2}, 1_{L_3})$ where q_1 is a proper element of L_1. Then $x_1, x_2, x_3 \in L_1$ for some $x_1x_2x_3 \leq q_1$. Hence $(x_1x_2x_3, 1_{L_2}, 0_{L_3}) = (x_1, 1_{L_2}, 0_{L_3})(x_2, 1_{L_2}, 0_{L_3})(x_3, 1_{L_2}, 0_{L_3}) \leq q$ and $(x_1x_2x_3, 1_{L_2}, 0_{L_3}) \not\leq \phi(q)$. Since q is ϕ-2-absorbing primary, we have either $(x_1x_2, 1_{L_2}, 0_{L_3}) \leq q$ or $(x_2x_3, 1_{L_2}, 0_{L_3}) \leq \sqrt{q}$ or $(x_1x_3, 1_{L_2}, 0_{L_3}) \leq \sqrt{q}$. So $x_1x_2 \leq q_1$ or $x_2x_3 \leq \sqrt{q_1}$ or $x_2x_3 \leq \sqrt{q_1}$.

(iii)$\Rightarrow$ (i): Suppose that q_2 and q_3 are primary elements of L_2 and L_3, respectively and $q = (1_{L_1}, q_2, q_3)$. Let (a_1, a_2, a_3), (b_1, b_2, b_3), $(c_1, c_2, c_3) \in L$ such that $(a_1, a_2, a_3)(b_1, b_2, b_3)(c_1, c_2, c_3) \leq q$ and $(a_1, a_2, a_3)(b_1, b_2, b_3)(c_1, c_2, c_3) \not\leq \phi(q)$. Assume that $(b_1, b_2, b_3)(c_1, c_2, c_3) \not\leq \sqrt{q}$ and $(a_1, a_2, a_3)(c_1, c_2, c_3) \not\leq \sqrt{q}$. Hence $b_2c_2 \not\leq \sqrt{q_2}$ or $b_3c_3 \not\leq \sqrt{q_3}$, and $a_2c_2 \not\leq \sqrt{q_2}$ or $a_3c_3 \not\leq \sqrt{q_3}$. If $b_2c_2 \not\leq \sqrt{q_2}$ and $a_2c_2 \not\leq \sqrt{q_2}$, then since q_2 is a primary element of L_2 and $a_2b_2c_2 \leq q_2$, this is a contradiction. Similarly the case of $b_3c_3 \not\leq \sqrt{q_3}$ and $a_3c_3 \not\leq \sqrt{q_3}$ gives again a contradiction. So without loss of generality assume that $b_2c_2 \not\leq \sqrt{q_2}$ and $a_3c_3 \not\leq \sqrt{q_3}$. Since q_2 and q_3 are primary, we have $a_2 \leq q_2$ and $b_3 \leq q_3$. Thus $(a_1, a_2, a_3)(b_1, b_2, b_3) \leq q$, which shows that q is a ϕ-2-absorbing primary element of L. Similar to this way, one can easily obtain that q is a ϕ-2-absorbing primary element of L if it is in type of $i)$ or $ii)$.

Last suppose that q is in type of $iv)$. Let $i = 1$. Then $q = (q_1, 1_{L_2}, 1_{L_3})$ where q_1 is a 2-absorbing primary element of L_1, then it can be seen that q is a 2-absorbing primary element of L. Therefore q is a ϕ-2-absorbing primary element of L by Theorem 2.4. □

Theorem 3.9. *Let $L = L_1 \times L_2 \times L_3$ where L_1, L_2, L_3 are C-lattices. Let $\phi = \psi_1 \times \psi_2 \times \psi_3$, where $\psi_i : L_i \longrightarrow L_i \cup \{\varnothing\}$ $(i = 1, 2, 3)$ is a function. If every element $a = (a_1, a_2, a_3)$ of L where $a_i \in L_i$ with $\sqrt{a_i}$ proper for all $i = 1, 2, 3$ is ϕ-2-absorbing primary, then $\psi(a) = \varnothing$ or $\psi(a) = a$.*

Proof. First observe that if $\psi(a_i) = \varnothing$ for some $i = 1, 2, 3$, then $\psi(a) = \varnothing$. So suppose that $\psi_i(a_i) \neq \varnothing$. Assume on the contrary that $\psi_1(a_1) \neq a_1$. From our hypothesis we can say that $a = (a_1, 0_{L_2}, 0_{L_3})$ is a ϕ-2-absorbing primary element. Hence $(a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 0_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \leq a$, but $(a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 0_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \not\leq \phi(a)$. So we get either $(a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 0_{L_2}, 1_{L_3}) \leq a$ or $(1_{L_1}, 0_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \leq \sqrt{a}$ or $(a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \leq \sqrt{a}$. It follows either $1_{L_3} \leq a_3$ or $1_{L_1} \leq \sqrt{a_1}$ or $1_{L_2} \leq \sqrt{a_2}$ by Lemma 3.1, which is a contradiction. Therefore $\psi_i(a_i) = a_i$ for every a_i of L_i, and thus $\psi(a) = a$. □

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Author information

Ece Yetkin Celikel, Gaziantep University, Department of Mathematics, 27310, Gaziantep, Turkey.
E-mail: `yetkinece@gmail.com`

Gulsen Ulucak, Gebze Technical University, Department of Mathematics, 141 41400 Kocaeli, Turkey.
E-mail: `gulsenulucak@gtu.edu.tr`

Emel A. Ugurlu, Marmara University, Department of Mathematics, Ziverbey, Goztepe, Istanbul, Turkey.
E-mail: `emel.aslankarayigit@marmara.edu.tr`

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