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A Bolza Optimal Control Problem of a Clamped Thermoelastic Plate subject to Point Control

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Abstract. We consider a finite horizon optimal control problem of a clamped Thermoelastic plate system via a point control mechanism with the objective of minimizing a functional capturing the energy of the system and its final state. In our approach, we invoke the available theory on the linear-quadratic and Bolza type optimal control problems of infinite dimensional systems in the literature [1, 16, 9, 18, 22] and in particular the theory of singular estimate control systems and their generalizations [25, 29], developed especially to address a certain class of coupled parabolic-hyperbolic PDEs with point and boundary control. The main concern of this paper, is the formulation of this particular thermoelastic system into the abstract framework of the theory and most importantly the verification of the singular estimate assumption. Once this is achieved, existence and regularity of optimal solutions are deduced along with the feedback characterization of the optimal control via a self-adjoint operator solving a Riccati equation.

1 Introduction

In this paper, we consider a well established clamped thermoelastic plate system subject to interior point control with the objective of minimizing a generic Bolza type energy functional. The well-posedness and regularity properties of the system in the clamped and hinged cases were treated in several earlier works [21, 26, 27, 28]. Other works have considered similar types of Bolza control models of this system when subject to thermal boundary control [11, 10]. A comprehensive optimal control synthesis for some hyperbolic-parabolic systems with point and boundary control, has been made possible by advances in the theory of optimal control of the linear quadratic problem for infinite dimensional systems subject to unbounded controls. Such synthesis involves a complete characterization of the optimal solutions and a feedback description of the optimal control via special operators which solve Riccati equations. While earlier works on infinite dimensional systems have been concerned with systems driven by analytic semigroups and unbounded controls which are meant to address point or boundary control of parabolic partial differential equations, [1, 16, 9, 14, 18], more recent works have been concerned with a theoretical framework for hybrid systems involving coupled parabolic-hyperbolic partial differential equations [19, 2, 3, 20, 22]. These systems cover a wide range of control applications in structure-acoustics, thermoelasticity, fluid-structure interactions and composite beams [5, 13, 15, 8, 12, 4, 24].

In this context, a special class of control systems driven by strongly continuous semigroups have been observed whereby the kernel of the control-to-state map satisfies a singular estimate, and for which the prototype are special systems of parabolic-hyperbolic coupled partial differential equations with point or boundary control [5, 22]. However, the singular estimate property which generalizes the class of analytic systems does not hold mutatis mutandis for non-analytic semigroups and its verification occasionally requires proving special trace and interior regularity results for hyperbolic and parabolic partial differential equations [11, 12]. Of special interest in the literature of optimal control, has been Bolza type problems concerned with final time target states or which incorporate final time weights into the objective functionals [17, 18, 23]. It is notable that the Bolza type objective functional introduces possible singularities in the optimal state and control [18, 23, 25].

Further extensions of the theoretical framework have been developed to address a larger class of hybrid systems with a weaker degree of analyticity which is mathematically manifested as a weaker singular estimate property [3, 25, 29]. Such extensions were aimed to address a larger class of systems in thermoelasticity and fluid-structure interactions [3, 12, 24].

In this work, we apply the theoretical framework developed in [29] to a Bolza problem involving this system of thermoelasticity with clamped boundary conditions and interior point control. The application entails a formulation of the system into the abstract framework of semigroup language, and the verification of the main assumptions of the theory; most importantly the singular estimate property. Once the assumptions are verified, the main conclusions of the framework can be specialized to the system and in particular, regularity properties of the optimal solutions are deduced along with a feedback characterization of the optimal control via a self-adjoint operator solving a Riccati equation.

2 The Model

We consider a clamped thermoelastic plate system with rotational inertia ($\rho > 0$) subject to interior point control

$$\left.\begin{aligned} w_{tt} - \rho\Delta w_{tt} + \Delta^2 w + \Delta\theta = 0, \qquad & \Omega \times [0,T] \\ \theta_t - \Delta\theta - \Delta w_t = \delta(x - x_0)u(t), \qquad & \Omega \times [0,T] \end{aligned}\right\} \tag{2.1}$$

where $w(x,t)$ is the transversal displacement and $\theta(x,t)$ is the temperature of the plate which occupies the open domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$, subject to the clamped boundary conditions

$$w = \frac{\partial}{\partial\nu} w = 0, \quad \partial\Omega \times [0,T] \tag{2.2}$$

and Neumann thermal boundary conditions

$$\frac{\partial}{\partial\nu}\theta + \theta = 0, \quad \partial\Omega \times [0,T]. \tag{2.3}$$

Moreover, we impose the initial conditions

$$\theta(x,0) = \theta_0(x), \quad \text{in} \quad \Omega \tag{2.4}$$

$$w(x,0) = w_0(x), \quad \text{in} \quad \Omega \tag{2.5}$$

$$w_t(x,0) = w_1(x), \quad \text{in} \quad \Omega. \tag{2.6}$$

The interior temperature point control is described by the forcing term $\delta(x - x_0)u(t)$ where x_0 is an interior point in Ω and $u(t)$ is the control function.

It is well known that the uncontrolled dynamics for this model are driven by a c_0 semigroup on the state space

$$\mathcal{H} = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$$

which is the natural energy space [18, 11].

We are particularly interested in a Bolza type optimal control of this system with the objective of minimizing an energy functional

$$\begin{aligned} J(u, w, w_t, \theta) = \int_0^T & \|u(t)\|_U^2 + \|w(\cdot,t)\|_{H^{2-2\alpha}(\Omega)}^2 + \|w_t(\cdot,t)\|_{H^{1-\alpha}(\Omega)}^2 + \|\theta(\cdot,t)\|_{H^{-\alpha}(\Omega)}^2 \, dt \\ & + \|w(\cdot,T)\|_{H^{2-2\alpha}(\Omega)}^2 + \|w_t(\cdot,T)\|_{H^{1-\alpha}(\Omega)}^2, \end{aligned} \tag{2.7}$$

over all controls $u(t) \in L^2([0,T];U)$ where $U = \mathbb{R}$, given initial data in the $(w_0, w_1, \theta_0) \in \mathcal{H}$. Here, the parameter α is a positive parameter to be specified.

We note that the system, does not fall in the regular singular estimate control class of systems typically involving coupled parabolic hyperbolic components featuring the property

$$\|e^{At}Bu\|_{\mathcal{H}} \leq \frac{C}{t^{\gamma}} \tag{2.8}$$

for $\gamma < 1$ satisfied by the semigroup e^{At} which generates the solution to the uncontrolled system and a control operator $B : U \to [\mathcal{D}(A^{\star})]'$. For such class of systems, a complete linear quadratic theory including feedback characterization and associated Riccati equations were developed, along with an extension to Bolza type problems. However, this particular system does not possess a sufficient degree of analyticity, and satisfies instead a weaker form of the above property. This weaker form of the property is captured by the condition

$$\|Re^{At}Bu\|_{W} \leq \frac{C}{t^{\gamma}} \tag{2.9}$$

and

$$\|Ge^{At}Bu\|_{Z} \leq \frac{C}{t^{\gamma}} \tag{2.10}$$

for $\gamma < 1$ where G and R are observation operators and R and W are Hilbert observation spaces. This explains the inclusion of slightly weaker norms in the Bolza type energy functional. For the non-Bolza type objective (quadratic functional without final time penalization), a theoretical framework was developed in [2] to address such systems with weak singular estimate, under some additional conditions.

3 A Theoretical Framework of Optimal Control for Coupled Systems of PDEs

Our aim to apply the theoretical framework developed in [29] and formulated in the next theorem to this system. In particular, given the abstract linear differential equation

$$y_t = Ay + Bu \tag{3.1}$$

$$y(s) = y_s \tag{3.2}$$

defined on a Hilbert space Y and an objective functional

$$J(u, w, w_t, \theta) = \int_s^T \|u(t)\|_U^2 + \|Ry(t)\|_W^2 \, dt + \|Gy(T)\|_Z^2 \tag{3.3}$$

under the assumptions

(i) The linear operator A generates a c_0 semigroup on the space Y (State Space).

(ii) The control operator B is linear acting from Hilbert space U (Control Space) to $[D(A^*)]'$, or equivalently $A^{-1}B$ is bounded $U \to Y$.

(iii) The operators R and G are bounded linear operators acting from state space Y into Hilbert spaces W and Z, also known as the observation spaces.

(iv) There exists $\gamma_1, \gamma_2 \in [0,1]$ such that

$$\|Re^{At}Bu\|_{W} \leq \frac{C}{t^{\gamma_1}}\|u\|_U \tag{3.4}$$

and

$$\|Ge^{At}Bu\|_Z \leq \frac{C}{t^{\gamma_2}}\|u\|_U \tag{3.5}$$

for all $t \in [0,1]$ and any $u \in U$.

(v) The operator GL_T where $L_T f = \int_s^T e^{A(T-t)}Bf(t)\,dt$, is a closed operator acting from control space $L^2([s,T];U)$ to observation space Z.

3.1 Preliminaries and Main Theorems

The solution to the problem (3.1), can be expressed using the variation of parameters formula as

$$y(t) = e^{A(t-s)}y_s + \int_s^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

which is well defined in $C([s,T];[\mathcal{D}(A^\star)]')$ due to the well known properties of the control to state map L_s defined by

$$L_s f = \int_s^t Ae^{A(t-\tau)}A^{-1}Bf(\tau)\,d\tau \tag{3.6}$$

acting boundedly $L^p([s,T];U) \to C([s,T];[\mathcal{D}(A^\star)]')$. In addition, the map RL_s acts boundedly from $L^p([s,T];U) \to L^p([s,T];W)$ which is straightforward to show using the singular estimate property (3.4) and Young's inequality.

Under these assumptions, we invoke the following results established in [29]. The first Theorem delineates existence and regularity of optimal state and control trajectories, while the second theorem is concerned with the feedback characterization of the optimal control and the properties of the gain operator which satisfies a Riccati equation.

Theorem 3.1. *For any initial state $y_s \in Y$ there exists a unique optimal control $u^0(t,s,y_s) \in L_2([s,T];U)$ and a unique optimal trajectory $Ry^0(t,s,y_s) \in L_2([s,T];W)$ such that*

$$J(u^0, y^0, s, y_s) = min_{u\in L^2([s,T],U)} J(u, y(u), s, y_s).$$

Moreover, the optimal solutions satisfy

(i) The optimal control $u^0(t)$ is continuous on $[s,T)$ with values in U, but has a singularity of order $\gamma = max\{\gamma_1,\gamma_2\}$ at the terminal time. In particular, we have

$$\|u^0(t,s,y_s)\|_U \leq \frac{C}{(T-t)^\gamma}, \quad s \leq t < T. \tag{3.7}$$

(ii) The optimal state $y^0(\cdot,s;y_s) \in C([s,T];[D(A^\star)]')$ for every $y_s \in Y$.

(iii) The observed optimal output $Ry^0(t)$ is continuous on $[s,T]$ when $\gamma_1 \leq \gamma_2$ and $\gamma_1+\gamma_2 < 1$, but has a singularity of order $\gamma_1+\gamma-1$ otherwise. In particular, we have

$$\|Ry^0(t,s,y_s)\|_W \leq \frac{C}{(T-t)^{\gamma_1+\gamma-1+\epsilon}}, \quad s \leq t < T, \quad \epsilon > 0. \tag{3.8}$$

(iv) When $4\gamma_1 + 2\gamma_2 < 4$, the function $Ry^0(\cdot,s,Bu)$ corresponding to the feedback dynamics satisfies

$$\|Ry^0(t,s;Bu)\|_W \leq \frac{C_{T-s}\|u\|_U}{(T-t)^{\gamma_2}(t-s)^{\gamma_1}}, \quad s < t < T, \tag{3.9}$$

for any $u \in U$ where the constant C_{T-s} blows up when $T \to s$.

(v) *The observed optimal state at final time T, $Gy^0(T,s;y_s)$ satisfies*

$$\|Gy^0(T,s;y_s)\|_Z \le C_T\|y_s\|_Y$$

for any $y_s \in Y$ where the constant C_T is independent of s.

The second theorem is concerned with the feedback characterization of the optimal control via an operator solving a Riccati equation, and its properties.

Theorem 3.2. *Let $\gamma = max\{\gamma_1, \gamma_2\}$. Then, under the assumptions above, we have*

(i) **[Optimal cost and value function]**
With $J(u^0, y^0, s, y_s) \equiv \min_{u\in L^2([s,T],U)} J(u, y(u), s, y_s)$ we have that there exists a self–adjoint positive operator $P(t) \in \mathcal{L}(Y)$ with $t \in [0,T)$ such that $\langle P(s)x, x\rangle_Y = J(u^0, y^0, s, x)$.

(ii) **[Singular behavior of the feedback operator]**

(a) *$P(t)$ is continuous on $[s,T]$ and $P(t) \in \mathcal{L}(Y, C([s,T];Y))$.*

(b) *The feedback operator $B^*P(t) \in \mathcal{L}(Y, C([s,T), U))$ exhibits the singularity*

$$\|B^*P(t)x\|_U \le \frac{C\|x\|_Y}{(T-t)^\gamma}, \ \ s \le t < T. \tag{3.10}$$

and

$$\|B^*e^{A^*(z-t)}P(z)x\|_U \le \frac{C\|x\|_Y}{(T-t)^\gamma}, \ \ \forall t \le z < T. \tag{3.11}$$

(iii) **[The optimal synthesis]** *The optimal control u^0 is given by the feedback formula*

$$u^0(t,s;y_s) = -B^*P(t)y^0(t,s;y_s), \ \ s \le t < T. \tag{3.12}$$

(iv) **[Riccati Equation]** *If in addition $4\gamma_1 + 2\gamma_2 < 4$, the operator $P(t)$ satisfies the differential Riccati equation with $t < T$, $x, y \in \mathcal{D}(A)$*

$$\langle P_t x, y\rangle_Y + \langle P(t)x, Ay\rangle_Y + \langle P(t)Ax, y\rangle_Y + \langle Rx, Ry\rangle_Z = \langle B^*P(t)x, B^*P(t)y\rangle_U. \tag{3.13}$$

$$\lim_{t\to T} P(t)x = G^*Gx \ \ \forall x \in Y. \tag{3.14}$$

(v) **[Uniqueness of RE]** *The solution of the Riccati equation above is unique in the class of positive and self–adjoint operators $P(t)$ satisfying (3.10) with $\gamma < \frac{1}{2}$.*

4 Application of the Theory to System (2.1)

To demonstrate the applicability of Theorem 1 and Theorem 2 to the system (2.1) under boundary conditions (2.2) and control (2.3) with the objective of minimizing the cost functional (2.7), we must first formulate the system into the abstract framework of section 3 and verify the conditions of these theorems.

4.1 Abstract Formulation

Following [18, 11], we first rewrite the system (2.1) in the abstract linear form

$$y_t = Ay + Bu \tag{4.1}$$

on the state space $\mathcal{H}$

$$\mathcal{H} = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$$

which is the natural energy space for the system (2.1).

Following [18, 11], we introduce the self adjoint operator $\mathcal{A}$ on $L^2(\Omega)$ defined by

$$\mathcal{A}h = \Delta^2 h \tag{4.2}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \{h \in H^4(\Omega) : h|_{\partial\Omega} = \frac{\partial}{\partial\nu} h|_{\partial\Omega} = 0\}. \tag{4.3}$$

The fractional power $\mathcal{A}^{1/2}$ of this operator has a domain which can be identified with the space $H^2(\Omega) \times H_0^1(\Omega)$. We also introduce the self-adjoint operator A_N on $L^2(\Omega)$

$$A_N h = -\Delta h \tag{4.4}$$

with domain

$$\mathcal{D}(A_N) = \{h \in H^2(\Omega) : \frac{\partial}{\partial\nu} h + h = 0 \text{ on } \partial\Omega\}. \tag{4.5}$$

The operator $-A_N$ is well known to generate an analytic semigroup $e^{-A_N t}$ on the space $L^2(\Omega)$.

We also follow [11] in introducing the operator $\mathcal{M}$ on $L^2(\Omega)$ given by

$$\mathcal{M} = (I + \rho A_N) \tag{4.6}$$

with the well defined bounded inverse $\mathcal{M}^{-1}$.

Accordingly, we shall use the following equivalent norm on $\mathcal{H}$

$$\|(w, z, \theta)\|_{\mathcal{H}} = \|w\|_{\mathcal{D}(\mathcal{A}^{1/2})} + \|z\|_{\mathcal{D}(\mathcal{M}^{1/2})} + \|\theta\|_{L^2(\Omega)}.$$

This definition of A_N allows us to incorporate the boundary condition (2.3) into the heat equation in the form

$$\theta_t = -A_N w_t - A_N \theta \tag{4.7}$$

and then extend the action of the operator A_N to act from $L^2(\Omega)$ to $[\mathcal{D}(A_N)]'$.

Hence, the system (2.1) above can be expressed as the abstract differential equation

$$y_t = Ay + Bu$$

where

$$y(t) = \begin{pmatrix} w \\ w_t \\ \theta \end{pmatrix} \tag{4.8}$$

and

$$A = \begin{pmatrix} 0 & I & 0 \\ -\mathcal{M}^{-1}\mathcal{A} & 0 & \mathcal{M}^{-1}A_N \\ 0 & -A_N & -A_N \end{pmatrix} \tag{4.9}$$

On the other hand, the control operator B is the operator

$$B = \begin{pmatrix} 0 \\ 0 \\ \delta(x - x_0) \end{pmatrix} \tag{4.10}$$

The operator A is well known to be maximal dissipative on $\mathcal{H}$ which implies that A generates a c_0 semigroup by the Lumer-Philips Theorem. The operator A is indeed dissipative, since the inner product $\langle Ay, y\rangle_{\mathcal{H}} \leq 0$ for all $y \in \mathcal{H}$ which is straightforward to compute. That A is maximal, follows from the fact that the equation $Ay^\star = y$ has a solution $y^\star \in \mathcal{H}$ for any given right side $y \in \mathcal{H}$.

Indeed, given $y = (f, g, h) \in \mathcal{H}$, we seek $(w, z, \theta) \in \mathcal{H}$ satisfying

$$\begin{aligned} z = f \in \mathcal{D}(\mathcal{A}^{1/2}) \\ -\mathcal{M}^{-1}\mathcal{A}w + \mathcal{M}^{-1}A_N\theta = g \\ -A_N z - A_N\theta = h \end{aligned}$$

Therefore,

$$\begin{aligned} w = -\mathcal{A}^{-1}\mathcal{M}g - \mathcal{A}^{-1}(h + A_N f) \in \mathcal{D}(\mathcal{A}^{3/4}) \\ \theta = -A_N^{-1}(h + A_N f) \in \mathcal{D}(A_N) \end{aligned}$$

from which we conclude that the domain of A is

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}^{3/4}) \times \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(A_D). \tag{4.11}$$

while the bounded inverse A^{-1} of A on $\mathcal{H}$ is identified as

$$A^{-1} = \begin{pmatrix} -\mathcal{A}^{-1}A_N & -\mathcal{A}^{-1}\mathcal{M} & -\mathcal{A}^{-1} \\ I & 0 & 0 \\ -I & 0 & -A_N^{-1} \end{pmatrix}. \tag{4.12}$$

Hence, A generates a c_0 semigroup on the state space $\mathcal{H}$.

We claim that the control operator B acts from $\mathbb{R} \to [\mathcal{D}(A^\star)]'$ or equivalently $A^{-1}B$ is a bounded operator $\mathbb{R} \to \mathcal{H}$. Indeed, we have

$$A^{-1}Bu = \begin{pmatrix} -\mathcal{A}^{-1}\delta(x - x_0)u(t) \\ 0 \\ -A_N^{-1}\delta(x - x_0)u(t) \end{pmatrix} \tag{4.13}$$

and equivalently we must show $\delta(x - x_0)u(t) \in [\mathcal{D}(\mathcal{A}^{1/2})]'$ and $[\mathcal{D}(A_N)]'$. Since $\mathcal{D}(\mathcal{A}^{1/2})$ is identified with $H^2(\Omega) \cap H_0^1(\Omega)$ and $\mathcal{D}(A_N) \subset H^2(\Omega)$, while $H^2(\Omega) \subset C(\Omega)$ for $n \leq 3$, we conclude $[C(\Omega)]' \subset [\mathcal{D}(\mathcal{A}^{1/2})]'$ and $[C(\Omega)]' \subset [\mathcal{D}(A_N)]'$ which establishes the claim [18].

4.2 The Singular Estimate Property

The crucial condition that must be verified is the generalized singular estimate condition (3.4) and (3.5), (Assumption 4). We first define the space $\mathcal{H}^\alpha$ for $\alpha > 0$ by

$$\mathcal{H}^\alpha = \mathcal{D}(A^\alpha)$$

or equivalently

$$\mathcal{H}^\alpha = H^{2+\alpha}(\Omega) \cap H_0^1(\Omega) \times H_0^{1+\alpha}(\Omega) \times \mathcal{D}(A_N^\alpha).$$

In particular, we designate the observation operator $R = I$ on observation space $W \equiv \mathcal{H}^{-\alpha}$ defined as the dual space of $\mathcal{H}^\alpha$ with respect to the pivot space $\mathcal{H}$ and choose $\alpha > (n - 2)/2$. Equivalently

$$\mathcal{H}^{-\alpha} = H^{2-\alpha}(\Omega) \times H^{1-\alpha}(\Omega) \times \mathcal{D}(A_N^{-\alpha}).$$

We also designate the second observation operator G whose action is defined by $G(w,z,\theta) = (w,z)$ on observation space

$$Z \equiv \mathcal{D}(\mathcal{A}^{1/2-\alpha/2}) \times \mathcal{D}(\mathcal{M}^{1/2-\alpha/2}).$$

with the norm $\|(f_1, f_2)\|_Z$ equivalent to $\|f_1\|_{H^{2-2\alpha}(\Omega)} + \|f_2\|_{H^{1-\alpha}(\Omega)}$.

Hence, to establish the singular estimate assumption, we prove the following theorem

Theorem 4.1. *The kernel of the state to control map satisfies the singular estimate*

$$\|e^{At}Bu\|_{\mathcal{H}^{-\alpha}} \leq \frac{C}{t^{1/2+\epsilon}}\|u(t)\|_{L^2(\partial\Omega)} \tag{4.14}$$

for $\alpha > (n-2)/2$.

Proof. We prove instead the equivalent estimate

$$\|B^\star e^{A^\star t}y\|_{L^2(\partial\Omega)} \leq \frac{C}{t^{3/4+\epsilon}}\|y\|_{\mathcal{H}^\alpha} \tag{4.15}$$

Computing the adjoint of A with respect to the space $\mathcal{H}$ we have

$$A^\star = \begin{pmatrix} 0 & -I & 0 \\ \mathcal{M}^{-1}\mathcal{A} & 0 & -\mathcal{M}^{-1}A_N \\ 0 & A_N & -A_N \end{pmatrix} \tag{4.16}$$

Hence, $e^{A^\star t}y$ represents the solution (w, z, θ) to the following system of PDEs

$$w_t = -z \tag{4.17}$$

$$z_t - \rho\Delta z_t = \Delta^2 w + \Delta\theta \tag{4.18}$$

$$\theta_t = \Delta\theta - \Delta z \tag{4.19}$$

given initial data $y = (w_0, z_0, \theta_0)$ which is an equivalent system to (2.1).

Computing the adjoint of the kernel $B^\star e^{A^\star t}y$ we conclude

$$B^\star e^{A^\star t}y = \begin{pmatrix} 0 \\ 0 \\ \theta(x_0, t) \end{pmatrix} \tag{4.20}$$

Hence, it is enough to estimate the L^∞ norm of $\theta(\cdot, t)$. In particular, estimating using Sobolev embedding inequalities for 2 dimensions and using the variation of parameters formula which allows us to express the solution θ via the semigroup $e^{A_N t}$ we have

$$\begin{aligned} \|\theta(\cdot,t)\|_{L^\infty(\Omega)} &\leq C\|\theta\|_{H^{n/2+\epsilon}(\Omega)} \\ &\leq C\|A_N^{n/4+\epsilon}\theta\|_{L^2(\Omega)} \\ &\leq \|A_N^{n/4+\epsilon}e^{-A_N t}\theta_0\|_{L^2(\Omega)} + \left\|\int_0^t A_N^{n/4+\epsilon}e^{-A_N(t-s)}A_N w_t(\cdot,s)\,ds\right\|_{L^2(\Omega)} \\ &= I_1 + I_2 \end{aligned}$$

We estimate each of the two terms separately. The term I_1 is the source of the singular estimate. To estimate I_1, we use the analyticity of the semigroup $e^{-A_N t}$ to get

$$\begin{aligned} I_1 &= \|A_N^{n/4+\epsilon} e^{-A_N t}\theta_0\|_{L^2(\Omega)} \\ &\le \frac{C}{t^{1/2+\epsilon}} \|A_N^{(n-2)/4}\theta_0\|_{L^2(\Omega)} \\ &\le \frac{C}{t^{1/2+\epsilon}} \|A_N^{\alpha}\theta_0\|_{L^2(\Omega)} \\ &\le \frac{C}{t^{1/2+\epsilon}} \|y_0\|_{\mathcal{H}^\alpha} \end{aligned}$$

To estimate I_2, we again appeal to the analyticity of the semigroup $e^{-A_N t}$ and write

$$\begin{aligned} \int_0^t \|A_N^{n/4+\epsilon} e^{-A_N(t-s)} A_N w_t(\cdot,s)\|_{L^2(\Omega)}\, ds &\le \int_0^t \frac{C}{(t-s)^{1-\epsilon}} \|A_N^{n/4+2\epsilon} w_t(\cdot,s)\|_{L^2(\Omega)}\, ds \\ &\le Ct\|A_N^{n/4+2\epsilon} w_t\|_{L^\infty([0,T];L^2(\Omega))} \\ &\le CT\|w_t\|_{L^\infty([0,T];H^{n/2+\epsilon})} \\ &\le C\,\|y_0\|_{\mathcal{H}^\alpha} \\ &\le \frac{C_T}{(t-s)^{1/2+\epsilon}} \|y_0\|_{\mathcal{H}^\alpha} \end{aligned}$$

where $\alpha = (n-2)/2+\epsilon$ or $\alpha > 0$ for $n = 2$ and $\alpha > 1/2$ for $n = 3$. Here, we used the regularity of the solution map $\mathcal{D}(A^\alpha) \to C([0,T];\mathcal{D}(A^\alpha))$. This establishes the desired estimate.

□

A corollary of this theorem that

$$\|Ge^{A(t-s)}Bf\|_Z \le \|e^{A(t-s)}Bf\|_{\mathcal{H}^{-\alpha}} \le \frac{C}{(t-s)^{1/2+\epsilon}}|f|$$

and

$$\|Re^{A(t-s)}Bf\|_{\mathcal{H}^{-\alpha}} = \|e^{A(t-s)}Bf\|_{\mathcal{H}^{-\alpha}} \le \frac{C}{(t-s)^{1/2+\epsilon}}|f|.$$

4.3 Closability of GL_{sT}

To show that the operator $GL_{sT}f \equiv \int_s^T Ge^{A(T-t)}Bf(t)\,dt$ is closable from $L^2([s,T];\mathbb{R}) \to Z$, we compute the action of the operator $GL_{sT}f$. Note that L_{sT} is the solution map to (2.1) evaluated at the final time T due to zero initial conditions and control $f(t)$, while G is the projection onto the first two components. Hence, we express the solution (w, w_t) to the system (2.1) arising from zero initial conditions using the variation of parameters formula as

$$\begin{pmatrix} w(\cdot,T) \\ w_t(\cdot,T) \end{pmatrix} = \int_s^T e^{A_k(T-t)} \begin{pmatrix} 0 \\ \mathcal{M}^{-1}A_N\theta(u) \end{pmatrix} dt$$

where A_k is the closed operator

$$\begin{pmatrix} 0 & I \\ -\mathcal{M}^{-1}\mathcal{A} & 0 \end{pmatrix},$$

which generates the c_0 semigroup $e^{A_k t}$, and has a bounded inverse A_k^{-1}. The operator A_k can be shown to be closed on the space $Z = \mathcal{D}(\mathcal{A}^{1/2-\alpha/2}) \times \mathcal{D}(\mathcal{M}^{1/2-\alpha/2})$.

To show closability of GL_{sT} from $L^2([s,T];\mathbb{R}) \to Z$, it suffices to show that there exists a closed operator K on Z with a bounded inverse K^{-1} such that $K^{-1}GL_{sT}$ is bounded $L^2([s,T];\mathbb{R}) \to Z$. We claim that $A_k^{-1}GL_{sT}$ is bounded. Indeed, we have

$$A_k^{-1}GL_{sT}u = \int_s^T e^{A_k(T-t)} \begin{pmatrix} -\mathcal{A}^{-1}A_N\theta(u) \\ 0 \end{pmatrix} dt$$

and thus we estimate the norm of $A_k^{-1}GL_{sT}$ to obtain

$$\begin{aligned}\|A_k^{-1}GL_{sT}u\|_{\mathcal{D}(\mathcal{A}^{1/2-\alpha/2})\times\mathcal{D}(\mathcal{M}^{1/2-\alpha/2})} &\le C\|\mathcal{A}^{-1}A_N\theta(u)\|_{\mathcal{D}(\mathcal{A}^{1/2-\alpha/2})} \\ &\le C\|(w, w_t, \theta)\|_{L^2([s,T];\mathcal{H}^{-\alpha})} \\ &\le C\|u\|_{L^2([s,T];\mathbb{R})}\end{aligned}$$

where we used the continuity of the control to state map (3.6) from $L^2([s,T];\mathbb{R}) \to L^2([s,T];W)$ in the last step [29]. This establishes the closability of GL_{sT} on Z.

5 Application of the Theoretical Framework

Now, that we have demonstrated the validity of the conditions for our system, we apply theorems 3.1 and 3.2 to the control problem (2.1), with the objective functional (2.7), in dimension 2 and 3. This yields the following theorem

Theorem 5.1. *Let $\alpha > (n-2)/2$. Given initial data $(\theta_0, w_0, w_1) \in \mathcal{H}$, there exists a unique optimal control $u^0(t) \in L^2([0,T];\mathbb{R})$ and corresponding optimal solutions $(\theta^0, w^0, w_t^0) \in L^2([0,T];\mathcal{H})$ of the system* (2.1) *which minimizes the cost functional J and such that the following properties hold*

(i) The optimal control is continuous on $[0,T)$ with a singularity of order $\gamma = 1/2+\epsilon$ at time T and in particular

$$|u(t)| \le \frac{C}{t^{1/2+\epsilon}}\|y_0\|_{\mathcal{H}}$$

(ii) The optimal solutions $(\theta^0, w^0, w_t^0) \in C([0,T);\mathcal{H}^{-\alpha})$ with a singularity of order ϵ at time the final time T

$$\|\theta(t)\|_{H^{-2\alpha}(\Omega)} + \|w(t)\|_{H^{2-\alpha}(\Omega)} + \|w_t(t)\|_{H^{1-\alpha}(\Omega)} \le \frac{C}{t^\epsilon}\|(\theta_0, w_0, w_1)\|_{\mathcal{H}}$$

(iii) There exists a positive self adjoint operator $P(t)$ on $\mathcal{H}$ such that

$$u^0(t) = -B^\star P(t)y^0(t) = -p_3(t)|_{x=x_0}$$

where $P(t)(w^0, w_t^0, \theta^0) = (p_1(t), p_2(t), p_3(t))$.

(iv) The operator $B^\star P(t)$ satisfies the estimate

$$|B^\star P(t)f| \le \frac{C}{(T-t)^{1/2+\epsilon}}\|f\|_{\mathcal{H}} \tag{5.1}$$

(v) The operator $P(t)$ solves the Riccati equation

$$\langle P_t f, g\rangle_{\mathcal{H}} + \langle P(t)f, Ag\rangle_{\mathcal{H}} + \langle P(t)Af, g\rangle_{\mathcal{H}} + \langle A^{-\alpha} f, A^{-\alpha} g\rangle_{\mathcal{H}} = (P(t)f)_3|_{x_0} \cdot (P(t)g)_3|_{x_0},$$

for all $f, g \in \mathcal{D}(A)$ and

$$\lim_{t\to T} P(t)(f_1, f_2, f_3) = (f_1, f_2, 0) \ \ \forall (f_1, f_2, f_3) \in \mathcal{H}. \tag{5.2}$$

(vi) The minimum value of the cost functional (2.7) *is given by*

$$J = \int_\Omega \Delta p_1(0)\Delta w_0 \, dx + \int_\Omega \nabla p_2(0)\nabla w_0 \, dx + \int_\Omega p_2(0) w_0 \, dx + \int_\Omega p_3(0)\theta_0 \, dx. \tag{5.3}$$

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