Palestine Journal of Mathematics

Vol. 3(2) (2014) , 235–239

Generalized Derivations on Semiprime Gamma Rings with Involution

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16N60. Secondary 16W25,16U80, 16W10.

Keywords and phrases: prime Γ-ring with involution, generalized I-derivation, generalized reverse I-derivation.

Abstract The purpose of this article is to define notions of generalized I-derivation and generalized reverse I-derivation on Γ-rings and to prove some remarkable results involving these mappings.

1 Introduction

An extensive generalized concept of classical ring set forth the notion of a gamma ring theory. As an emerging field of research, the research work of classical ring theory to the gamma ring theory has been drawn interest of many algebraists and prominent mathematicians over the world to determine many basic properties of gamma ring and to enrich the world of algebra. The different researchers on this field have been doing a significant contributions to this field from its inception. In recent years, a large number of researchers are engaged to increase the efficacy of the results of gamma ring theory over the world.

The notion of a Γ-ring was first introduced by Nobusawa[13] and also shown that Γ-rings, more general than rings. Bernes[2] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Bernes[2], Kyuno[11], Luh[12], Ceven[4], Hoque and Paul[6,8,9] and others were obtained a large numbers of improtant basic properties of Γ-rings in various ways and determined some more remarkable results of Γ-rings. We start with the following necessary definitions.

Let M and Γ be additive abelian groups. If there exists a mapping $(a, \alpha, b) \rightarrow a\alpha b$ of $M \times \Gamma \times M \rightarrow M$, which satisfies the conditions
(i) $a\alpha b \in M$
(ii) $(a+b)\alpha c$=$a\alpha c$+$b\alpha c$, $a(\alpha+\beta)c$=$a\alpha c$+$a\beta c$, $a\alpha(b+c)$=$a\alpha b$+$a\alpha c$
(iii) $(a\alpha b\beta c$=$a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,
then M is called a Γ-ring.
Let M be a Γ-ring. Then M is said to be prime if $a\Gamma M\Gamma b$=(0) with $a, b \in M$, implies a=0 or b=0 and semiprime if $a\Gamma M\Gamma a$=(0) with $a \in M$ implies a=0. Furthermore, M is said to be commutative Γ-ring if $a\alpha b$=$b\alpha a$ for all $a, b \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M)$ =$\{a \in M : a\alpha b = b\alpha a$ for all $\alpha \in \Gamma, b \in M\}$ is called the centre of the Γ-ring M.
If M is a Γ-ring, then $[a, b]_\alpha$=$a\alpha b - b\alpha a$ is known as the commutator of a and b with respect to α, where $a, b \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:
$[a\alpha b, c]_\beta$=$[a, c]_\beta\alpha b + a[\alpha, \beta]_c b + a\alpha[b, c]_\beta$
and $[a, b\alpha c]_\beta$=$[a, b]_\beta\alpha c + b[\alpha, \beta]_a c + b\alpha[a, c]_\beta$,
for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. We consider the following assumption:
(A)................$a\alpha b\beta c$=$a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, which extensively used in through out the paper.
According to the assumption (A), the above two identites reduce to
$[a\alpha b, c]_\beta$=$[a, c]_\beta\alpha b + a\alpha[b, c]_\beta$
and $[a, b\alpha c]_\beta$=$[a, b]_\beta\alpha c + b\alpha[a, c]_\beta$,
which we extensively used.

Note that during the last some decades many authors have studied derivations in the context of prime and semiprime rings and Γ-rings with involution (viz., [1], [3], [7],[15], [16]). The notion of derivation and Jordan derivation on a Γ-ring were defined by Sapanci and Nakajima[14].

Let M be a Γ-ring. An additive mapping $D : M \to M$ is called a derivation if $D(a\alpha b) = D(a)\alpha b + a\alpha D(b)$ holds for all $a, b \in M$ and $\alpha \in \Gamma$. An additive mapping $T : M \to M$ is called a left (right) centralizer if $T(a\alpha b) = T(a)\alpha b$ (resp. $T(a\alpha b) = a\alpha T(b)$) for all $a, b \in M$ and $\alpha \in \Gamma$.

Let D be a derivation on M. Then the additive mapping $F : M \to M$ is called a generalized derivation if $F(a\alpha b) = F(a)\alpha b + a\alpha D(b)$ holds for all $a, b \in M$ and $\alpha \in \Gamma$.

The additive mapping $I : M \to M$ is called an involution if

(i) $II(a) = a$

(ii) $I(a + b) = I(a) + I(b)$

and (iii) $I(a\alpha b) = I(b)\alpha I(a)$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Let M be a Γ-ring with involution I. An additive mapping $D : M \to M$ is called an I-derivation if $D(a\alpha b) = D(a)\alpha I(b) + a\alpha D(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$ and D is called a reverse I-derivation if $D(a\alpha b) = D(b)\alpha I(a) + b\alpha D(a)$ for all $a, b \in M$ and $\alpha \in \Gamma$. An additive mapping $T : M \to M$ is called a left (right) I-centralizer if $T(a\alpha b) = T(a)\alpha I(b)$ (resp. $T(a\alpha b) = I(a)\alpha T(b)$) for all $a, b \in M$ and $\alpha \in \Gamma$. An additive mapping $F : M \to M$ is called a generalized I-derivation if $F(a\alpha b) = F(a)\alpha I(b) + a\alpha D(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$, D an I-derivation on M. An additive mapping $F : M \to M$ is called a generalized reverse I-derivation if $F(a\alpha b) = F(b)\alpha I(a) + b\alpha D(a)$ for all $a, b \in M$ and $\alpha \in \Gamma$, D a reverse I-derivation on M.

The goal of this article is to develop some remarkable results involving generalized I-derivations and generalized reverse I-derivations on Γ-rings.

2 The Main Results

Theorem 2.1. *Suppose that M is a semiprime Γ-ring with involution I and $D : M \to M$ is a I-derivation. If F is a generalized I-derivation on M, then F maps M into $Z(M)$.*

Proof. By definition of F, we have

$$F(a\alpha b) = F(a)\alpha I(b) + a\alpha D(b) \tag{2.1}$$

for all $a, b \in M$ and $\alpha \in \Gamma$. Putting $b = b\beta c$ in (2.1), we have

$$F(a\alpha b\beta c) = F(a)\alpha I(c)\beta I(b) + a\alpha D(b\beta c)$$

$$\Rightarrow \quad F(a\alpha b\beta c) = F(a)\alpha I(c)\beta I(b) + a\alpha D(b)\beta I(c) + a\alpha b\beta D(c) \tag{2.2}$$

Also, we can write

$$\begin{aligned} F(a\alpha b\beta c) &= F((a\alpha b)\beta c) \\ &= F(a)\alpha I(b)\beta I(c) + a\alpha D(b)\beta I(c) + a\alpha b\beta D(c) \end{aligned} \tag{2.3}$$

Hence, from (2.2) and (2.3), we obtain

$$F(a)\alpha[I(c), I(b)]_\beta = 0 \tag{2.4}$$

For $I(b) = b$ and $I(c) = c$, (2.4) becomes

$$F(a)\alpha[c, b]_\beta = 0 \tag{2.5}$$

Putting $c = c\gamma F(a)$ in (2.5), we have

$$F(a)\alpha c\gamma[F(a), b]_\beta + F(a)\alpha[c, b]_\beta\gamma F(a) = 0$$

$$\Rightarrow \quad F(a)\alpha c\gamma[F(a), b]_\beta = 0 \tag{2.6}$$

Left multiplication of (2.6) by $b\beta$, we get

$$b\beta F(a)\alpha c\gamma[F(a), b]_\beta = 0 \tag{2.7}$$

Putiing $c = b\beta c$ in (2.6), we have

$$F(a)\alpha b\beta c\gamma[F(a), b]_\beta = 0 \tag{2.8}$$

Subtracting (2.7) from (2.8) and using (A), we obtain

$$[F(a), b]_\beta \alpha c \gamma [F(a), b]_\beta \quad = 0 \tag{2.9}$$

for all $a, b, c \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Hence, by semiprimeness of M, we have $[F(a), b]_\beta = 0$ for all $a, b \in M$ and $\beta \in \Gamma$. Therefore F maps M into $Z(M)$. Hence the theorem is complete. □

Theorem 2.2. *Suppose that M is a semiprime Γ-ring with involution I. If the additive mapping $T : M \to M$ is defined by $T(a\alpha b) = T(a)\alpha I(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$, then T maps M into $Z(M)$*

Proof. By the hypothesis, we get

$$T(a\alpha b) \quad = T(a)\alpha I(b) \tag{2.10}$$

Putting $b = c\beta b$ in (2.10), we have

$$T(a\alpha c\beta b) \quad = T(a)\alpha I(b)\beta I(c) \tag{2.11}$$

Also, we can write

$$T(a\alpha c\beta b) \quad = T((a\alpha c)\beta b) = T(a\alpha c)\beta I(b) = T(a)\alpha I(c)\beta I(b) \tag{2.12}$$

Hence from (2.11), (2.12) and using (A), we obtain

$$T(a)\beta[I(c), I(b)]_\alpha \quad = 0 \tag{2.13}$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. The equation (2.13) is similar to the equation (2.4) with the exception that the left I-centralizer T instead of generalized I-derivation F. Thus the same approch, we have used after the equation (2.4) in Theorem-2.1, we obtain the required result $[T(a), b]_\alpha = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$. Hence the theorem is proved. □

Corollary 2.3. *Suppose that M is a prime Γ-ring with involution I and D an I-derivation on M. If F is a generalized I-derivation on M, then either $F = 0$ or M is commutative.*

Proof. According to Theorem-2.1, we have $F(a)\beta[b, c]_\alpha = 0$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Putting $b = b\gamma e$, we obtain $F(a)\beta b\gamma[e, c]_\alpha + F(a)\beta[b, c]_\alpha\gamma e = 0$ for all $a, b, c, e \in M$ and $\alpha, \beta, \gamma \in \Gamma$, which implies $F(a)\beta b\gamma[e, c]_\alpha = 0$ for all $a, b, c, e \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Hence by the primeness of M, we have $F(a) = 0$ or $[e, c]_\alpha = 0$, that is, $F = 0$ or M is commutative. □

Corollary 2.4. *Suppose that M is a semisimple Γ-ring with involution I and D an I-derivation on M. If F is a generalized I-derivation on M, then F maps M into $Z(M)$*

Proof. Since every semisimple Γ-ring with involution is semiprime Γ-ring with the involution, so according to the Theorem-2.1, the corollary is nothing to prove. □

Corollary 2.5. *Suppose that M is a Γ-ring with involution I. If D is a nonzero I-derivation on M, then D maps M into $Z(M)$.*

Proof. The corollary is nothing to prove if we consider $F = D$ in the proof of Theorem-2.1. □

Theorem 2.6. *Suppose that M is a semiprime Γ-ring with involution I and D a reverse I-derivation on M. If F is a generalized reverse I-derivation on M, then $[D(a), c]_\alpha = 0$ for all $a, c \in M$ and $\alpha \in \Gamma$.*

Proof. By the definition of generalized I-derivation F on M, we have

$$F(a\alpha b) \quad = F(b)\alpha I(a) + b\alpha D(a) \tag{2.14}$$

for all $a, b \in M$ and $\alpha \in \Gamma$. Replacing a by $a\beta c$ in (2.14), we have

$$F(a\beta c\alpha b) \quad = F(b)\alpha I(c)\beta I(a) + b\alpha D(c)\beta I(a) + b\alpha c\beta D(a) \tag{2.15}$$

Also, we can write

$$\begin{aligned} F(a\beta c\alpha b) &= F(a\beta(c\alpha b)) \\ &= F(b)\alpha I(c)\beta I(a) + b\alpha D(c)\beta I(a) + c\alpha b\beta D(a) \end{aligned} \tag{2.16}$$

Comparing (2.15), (2.16) and using (A), we have

$$[b,c]_\alpha \beta D(a) \quad = 0 \tag{2.17}$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Putting $b = D(a)\gamma b$ in (2.17), we have

$$D(a)\gamma[b,c]_\alpha \beta D(a) + [D(a), c]_\alpha \gamma b\beta D(a) \quad = 0 \tag{2.18}$$

Using (2.17), we obtain

$$[D(a), c]_\alpha \gamma b\beta D(a) \quad = 0 \tag{2.19}$$

Putting $b = b\alpha c$ in (2.19), we have

$$[D(a), c]_\alpha \gamma b\alpha c\beta D(a) \quad = 0 \tag{2.20}$$

Right multiplication of (2.19) by αc, we have

$$[D(a), c]_\alpha \gamma b\beta D(a)\alpha c \quad = 0 \tag{2.21}$$

Subtracting (2.20) from (2.21) and using (A), we obtain

$$[D(a), c]_\alpha \beta b\gamma [D(a), c]_\alpha \quad = 0 \tag{2.22}$$

Hence by semiprimeness of M, we have $[D(a), c]_\alpha = 0$ for all $a, c \in M$ and $\alpha \in \Gamma$ and the theorem is complete. □

The above theorem gives us two interesting corollaries-

Corollary 2.7. *Suppose that M is a noncommutative prime Γ-ring with involution I and D a reverse I-derivation on M. If F is a generalized reverse I-derivation on M, then F is a reverse left I-centralizer on M.*

Proof. If we replace b by $a\gamma b$, the relation (2.17) gives $a\gamma[b,c]_\alpha \beta D(a) + [a,c]_\alpha \gamma b\beta D(a) = 0$ and using (2.17), the relation implies $[a,c]_\alpha \gamma b\beta D(a) = 0$ for all $a, b, c \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Hence by primeness of M, either $[a,c]_\alpha = 0$ or $D(a) = 0$. If we consider, U=$\{a \in M : [a,c]_\alpha = 0$ for all $c \in M$, $\alpha \in \Gamma\}$ and V=$\{a \in M : D(a) = 0\}$. Then clearly U and V are additive subgroups of M and $U \cup V = M$. Therefore by Brauer's trick, either $U = M$ or $V = M$. If $U = M$, then $[a,c]_\alpha = 0$ for all $a, c \in M$ and $\alpha \in \Gamma$. That is, M is commutative which gives a contradiction. On the other hand, if $V = M$, then $D(a) = 0$ for all $a \in M$. Therefore by the definition of F gives $F(a\alpha b) = F(b)\alpha I(a)$ for all $a, b \in M$ and $\alpha \in \Gamma$. Hence the proof is complete. □

Corollary 2.8. *Suppose that M is a semiprime Γ-ring with involution I. If D is a reverse I-derivation on M, then D maps M into $Z(M)$.*

Proof. If we consider $F = D$, Theorem-2.6 gives the result. □

References

[1] M. Ashraf and Shakir, Ali, , On $(\alpha,\beta)^*$-derivations in $H*$-algebras, Advances in Algebra, Vol. 2(1)(2009), 23-31.

[2] W.E. Bernes, On the Γ-rings of Nobusawa, Pacific J. Math., Vol. 18(1966), 411-422.

[3] M. Bresar and J. Vukman, On some additive mappings in rings with involution, Aequationes Math., Vol. 38(1989), 178-185.

[4] Y. Ceven, Jordan left derivations on completely prime gamma rings, C.U.Fen-Edebiyat Fakultesi, Fen Bilimleri Dergisi(2002) Cilt 23 Sayi 2.

[5] I.N. Herstein, Rings with involution, The Univ. of Chicago Press, Chicago 1976.

[6] M.F. Hoque and A.C. Paul, On Centralizers of Semiprime Gamma Rings, International Mathematical Forum, Vol.6(2011), No.13, 627-638.

[7] M.F. Hoque and N. Rahman, The Jordan θ-Centralizers of Semiprime Gamma Rings with Involution, International J. of Math. Combin., Vol. 4(2013), 16-31.

[8] M.F. Hoque and A.C. Paul, An Equation Related to Centralizers in Semiprime Gamma Rings, Annals of Pure and Applied Mathematics, Vol.1(1)(2012), 84-90.

[9] M.F. Hoque and A.C. Paul, Prime Gamma Rings with Centralizing and Commuting Generalized Derivations, International Journal of Algebra, Vol.7(13)(2013), 645-651.

[10] M.F. Hoque and A.C. Paul, The θ-Centralizers of Semiprime Gamma Rings, Res. J. Appl. Sci. Technol., Vol. 6(22)(2013), 4129-4137.

[11] S. Kyuno, On prime Gamma ring, Pacific J. Math., Vol. 75(1978), 185-190.

[12] L. Luh, On the theory of simple Gamma rings, Michigan Math. J. Vol. 16(1969), 65-75.

[13] N. Nobusawa, On the Generalization of the Ring Theory, Osaka J. Math., Vol. 1(1964), 81-89.

[14] M. Sapanci and A. Nakajima, Jordan derivations on completely prime gamma rings, Math. Japonica, Vol. 46(1)(1997), 47-51.

[15] Shakir Ali and A.Fosner, On Jordan $(\alpha, \beta)^*$-derivations in rings, International J. Algebra Vol. 1(4)(2010), 99-108.

[16] J. Vukman and I. Kosi-Ulbl, On centralizers of semiprime rings with involution, Studio. Scientiarum Mathematicarum Hungarica, Vol. 43(2006), 61-67.

[17] B. Zalar, On Centralizers of semiprime rings, Comment. Math. Univ. Carolinae, Vol.32(1991), 609-614.

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Received: July 20, 2013

Accepted: January 8, 2014.