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Non-cancellation of groups from module action

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Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

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Abstract. In [Hilton, P. Non-cancellation properties for certain finitely presented groups. Classical and categorical algebra (Durban, 1985). Quaestiones Math. 9 (1986), no. 1-4, 281-292], Hilton carried out a construction of specific groups expressed in the form of semidirect products $A \rtimes C$, where A is a C-module whose underlying abelian group structure is $\mathbb{Z}_n$ and C a free cyclic group. This enable him to determine the structure and the exponent of the genus group. From [14, Theorem 1.4], it appeared that these groups carried some remarkable property which gives rise to the exploration of the non-cancellation phenomenon. In this paper, we generalize it to R-modules, where R is some commutative ring. To achieve this, we form a category of metabelian groups $\mathcal{C}_R$ whose objects G are semidirect products obtained from R-modules. For an object G, a group structure is induced on the non-cancellation set of the localization of G at the set of primes $\{3, 7\}$. An example of trivial non-cancellation set in $\mathcal{C}_R$ is given.

1 Introduction

Until the mid-seventies, research on the cancellation question focused mainly on finitely generated projective modules over polynomial rings of algebraic varieties. The reason for this focus was the celebrated question raised by J.P. Serre [23] in 1955: *If R is the polynomial ring in a finite number of variables over a field, is every finitely generated projective module free*? This question reduces to the cancellation question. *If P and Q are finitely generated R-modules such that $P \oplus R \cong Q \oplus R$, are P and Q necessarily isomorphic?* Serre's question inspired a huge amount of research on the cancellation question for projective modules, from the Bass cancellation Theorem [1, Chap IV, (3.5)] (which gives a positive answer if the rank of P is greater than the number of variables) to the eventual solution by Quillen [20] and Suslin [24].
Efforts to answer Serre's problem resulted in many important results on cancellation and failure of cancellation for projective modules over more general rings (see [19, 28, 27]). Moreover, research on these questions continued long after the solution of Serre's problem (see [1, 18, 25, 26]). The modules we are going to consider will not in general be projective. Much less is known about cancellation in general. Among the first results was the theorem, due to Vasconcelos [29], that says cancellation holds for finitely generated modules over commutative semilocal rings. Other early results on cancellation are the following:

(1) Let R be any ring and let $\mathcal{C}$ be the class of R-modules with finite length. Then cancellation holds for $\mathcal{C}$.

(2) Let R be a local Noetherian commutative ring. Then cancellation holds for the class of finitely generated R-modules.

(3) Let R be a Dedekind domain. Then cancellation holds for the class of finitely generated R-modules.

The concept of cancellation is closely related to those of genus and localization of groups and modules. For example, cancellation holds for the class of finitely generated R-modules where R is a local noetherian commutative ring. The localization of a ring results to local ring. A *local ring* is a ring with just one maximal ideal. Ever since Krull's paper (1938), local rings have occupied a central position in commutative algebra. The technique of localization reduces many problems in commutative algebra to problems about local rings. This often turns out to be extremely useful. Most of the problems with which commutative algebra has been successful are those that can be reduced to the local case. Despite this, localization as a general procedure was defined rather late. In the case of integral domains, it was described by Grell, a student of Noether in 1927. It was not defined for arbitrary commutative rings until the work of Chevalley (1944) and Uzkov (1948). The process of localization does not lose much information about the ring. For example, if R is an integral domain, the fields of fractions of R and R_p (the localization of R at a prime ideal p) are the same. This process permits operations which, from a geometrical point of view, provide information about the neighborhood of p in $\mathrm{Spec}(R)$ (the set of all primes ideals of R). The localization of an R-module M at a prime ideal p is the R_p-module M_p obtained by tensoring M with R_p. Modules having isomorphic localization are said to be in the same genus. Cancellation holds also in every genus of R-modules if R is commutative and has no nilpotent elements, see [7].

The theory of P-localization of groups, where P is a family of primes, appears to have been first discussed in [15, 16] by Mal'cev and Lazard . In their work emphasis was placed on the explicit construction of the localization and properties of the localization G_P of the nilpotent group G were deduced from the construction, utilizing nilpotent group theory. Baumslag in [2] has given a comprehensive treatment of the main properties of nilpotent groups as they relate to the problem of localization. He has explicitly shown in [3] how to construct G_P in the case of an arbitrary nilpotent group G and an arbitrary family of primes P. Thus extending the generality of Malcev's original construction. Bousfield-Kan [4] exploit this general Mal'cev construction in their study of completion and localization.
In the 1970s, Hilton and Mislin became interested, through their work on the localization of nilpotent spaces, in the localization of nilpotent groups. Mislin [17] define the genus $\mathcal{G}(N)$ of a finitely generated nilpotent group N to be the set of isomorphism classes of finitely generated nilpotent groups M such that the localizations M_p and N_p are isomorphic at every prime p. This version of genus became known as the ***Mislin genus***, and other very useful variations of this concept came into being. In [9] Hilton and Mislin define an abelian group structure on the genus set $\mathcal{G}(N)$ of a finitely generated nilpotent group N with finite commutator subgroup.
For nilpotent groups which belong to class $\mathcal{K}$ (of semidirect products of the form $T \rtimes \mathbb{Z}^k$, where T is a finite abelian group and k is a positive integer), many computations of the genus groups appear in the literature. Indeed, the groups considered in [5, 11, 13, 21] all belong to this class.
When localizing non-nilpotent groups, it may happen that the kernel of the localizing homomorphism is bigger than what we would require. So for a non-nilpotent finitely generated group G with finite commutator subgroup, rather than considering localization, the idea of the genus is generalized through *non-cancellation*. For a group G, the *non-cancellation set*, denoted by $\chi(G)$, is the set of isomorphism classes of groups H such that $G \times \mathbb{Z} \cong H \times \mathbb{Z}$.

Investigations into non-cancellation phenomena on groups in the class $\mathcal{K}$ (of semidirect products of the form $T \rtimes_\omega \mathbb{Z}^k$, where T is a finite abelian group and k is a positive integer) appear in [31, 32], and reveal some similarities with genus computations for nilpotent groups in $\mathcal{K}$. One such instance is observed in [31, Theorem 4.2] on triviality of the non-cancellation set of a group in $\mathcal{K}$, which is a direct generalization of [21, Theorem 4.1] on triviality of the genus of a nilpotent group in $\mathcal{K}$. Such similarities are of course not completely surprizing in view of Warfield's result [30, Theorem 3.6], which asserts that for finitely generated nilpotent groups N and M having finite commutator subgroups, the condition $N \times \mathbb{Z} \cong M \times \mathbb{Z}$ is equivalent to $N_p \cong M_p$ for every prime number p, in other words, $\mathcal{G}(N) = \chi(N)$.

In [22], Scevenels and Witbooi give an alternate description of the non-cancellation group of a group $G = T \rtimes_\omega \mathbb{Z}^k$ where T is a finite abelian group and ω the action $\mathbb{Z}^k \to \mathrm{Aut}T$ with k a positive integer. This enable them to make some computations.
Hilton in [14] carried out a construction of specific groups expressed in the form of semidirect products $A \rtimes C$, where A is a C-module whose underlying abelian group structure is $\mathbb{Z}_n$ and C a free cyclic group.
This enable him to determine the structure and the exponent of the genus group.
From [14, Theorem 1.4], it appeared that these groups carried some remarkable property which gives rise to the exploration of the non-cancellation phenomenon.
We generalize this for R-modules, where R is some commutative ring. We form a category of metabelian groups denoted by $\mathcal{C}_R$ whose objects are semidirect products $T \rtimes_\zeta F$ where T is a finitely generated torsion R-module, F a free R-module and $\zeta : F \to \mathrm{Aut}_R T$ an action of F on T. For such object G, we describe the non-cancellation set $\chi_R(G)$, which is the set of all isomorphism classes of groups H such that $G \times R \cong H \times R$.

The remainder of this paper is organized as follows:

In Section 2, we describe morphisms in $\mathcal{C}_R$. Morphisms in this category are group homomorphisms with some specific properties.
Section 3 consists of a characterization of isomorphisms in $\mathcal{C}_R$. This is presented in Propositions 3.2 and 3.3. Proposition 3.4 describes the non-cancellation set for objects in $\mathcal{C}_R$. This section ends with two examples illustrating the usefulness of this construction. In the first one, we establish in Theorem 3.2, a surjective function between the non-cancellation sets $\chi(H)$ and $\chi_R(\bar{H})$, where $R = \mathbb{Z}[\frac{1}{3}, \frac{1}{7}]$ and $\bar{H}$ the localization of H at the set of primes $\{3, 7\}$. This is showing that a group structure can be induced on $\chi_R(\bar{H})$. In the second one, we present an example of trivial non-cancellation set in $\mathcal{C}_{\mathcal{R}}$.

When T is a finite group and F a free abelian group, in Section 4, we compute the non-cancellation of $\mathbb{Z}_n^k \rtimes_\mu \mathbb{Z}^{k+1}$ which turns to be trivial. This is an illustration of the known result that the non-cancellation set is always trivial in the case where the rank of the image of ζ in $\mathrm{Aut}(T)$ is less than the free rank of F.

2 A category of metabelian groups

Let us fix some commutative ring R. We observe some properties of the groups G of the form $T \rtimes_\zeta F$ where T is a finitely generated torsion R-module and F is a finite rank free R-module. For $G = T \rtimes_\zeta F$, let $C_G(F)$ be the centralizer of F in G.

$$C_G(F) = \{(a, x) \in G : (a, x) * (0, y) = (0, y) * (a, x) \ \text{ for all } y \in F\}$$

In order to describe the morphisms we first observe that there is a (natural) module structure on $C_G(F)$.

Proposition 2.1. *Let $G = T \rtimes_\zeta F$. Then*

$$C_G(F) = \{(a, x) \in G : \zeta_y(a) = a \ \textit{for all } y \in F\}.$$

Proof. Given $(a, x) \in C_G(F)$, for all $y \in F$, $(a, x) * (0, y) = (0, y) * (a, x)$
i.e. $(a, x + y) = (\zeta_y(a), x + y)$
i.e. $a = \zeta_y(a)$. ■

Remark 2.1. For $(a, x), (b, y) \in C_G(F)$, $(a, x) * (b, y) = (a + b, x + y)$.

Proposition 2.2. *Let $G = T \rtimes_\zeta F$. Then the centralizer of F in G, $C_G(F)$ becomes an R-module with the multiplication $R \times C_G(F) \to C_G(F)$ given by*
$r(a, x) = (ra, rx)$ for $r \in R$, $(a, x) \in C_G(F)$.

Proof. We first show that $(C_G(F), *)$ is an abelian group.

Given $(a, x), (b, y) \in C_G(F)$

$$\begin{aligned} (a, x) * (b, y) &= (a + b, x + y) \text{ by Remark 5.2.1} \\ (b, y) * (a, x) &= (b + a, y + x) \text{ by Remark 5.2.1} \\ &= (a + b, x + y) \text{ since } T \text{ and } F \text{ are abelian} \end{aligned}$$

Thus $(a, x) * (b, y) = (b, y) * (a, x)$.

Let $r, s \in R$ and $(a, x), (b, y) \in C_G(F)$.

$$\begin{aligned} r[(a, x) * (b, y)] &= r(a + b, x + y) \\ &= [r(a + b), r(x + y)] \\ &= (ra + rb, rx + ry) \text{ since } T \text{ and } F \text{ are } R\text{-modules.} \\ &= (ra, rx) * (rb, ry) \\ &= r(a, x) * r(b, y) \end{aligned}$$

$$\begin{aligned} (r + s)(a, x) &= [(r + s)a, (r + s)x] \\ &= (ra + sa, rx + sx) \text{ since } T \text{ and } F \text{ are } R\text{-modules} \\ &= (ra, rx) * (sa, sx) \\ &= r(a, x) * s(a, x) \end{aligned}$$

$$\begin{aligned}(rs)(a,x) &= [(rs)a,(rs)x]\\ &= [r(sa), r(sx)] \text{ since } T \text{ and } F \text{ are } R\text{-modules}\\ &= r[s(a,x)].\end{aligned}$$

Thus $C_G(F)$ is an R-module. ■

Now let $\mathcal{C}_R$ be the category having as its objets all groups of the form $G(\omega) = T \rtimes_\omega F$. The morphisms in $\mathcal{C}_R$ will be group homomorphisms

$$h : G(\zeta) = T_1 \rtimes_\zeta F_1 \to G(\xi) = T_2 \rtimes_\xi F_2$$

satisfying the following conditions:

(i) $h(T_1) \subseteq T_2$

(ii) $h(F_1) \subseteq C_{G_\xi}(F_2)$

(iii) The restrictions $h_1 = h|_{T_1}$ and $h_0 = h|_{F_1}$ are R-homomorphisms onto their images.

3 Isomorphisms in $\mathcal{C}_R$

The morphisms in $\mathcal{C}_R$ can be expressed in terms of simpler components. Let h be such morphism. We define h_2 and h_3 as follows :

$$h(0,x) = h_0(0,x) = (h_2(x), h_3(x))$$

Let $(a,x) \in G(\zeta)$. Then we can calculate:

$$\begin{aligned}h(a,x) &= h\left[(a,0) * (0,x)\right]\\ &= h(a,0) * h(0,x)\\ &= (h_1(a),0) * h_0(0,x) \text{ where } h_1, h_0 \text{ are homomorphisms defined previously}\\ &= (h_1(a),0) * (h_2(x), h_3(x))\\ &= (h_1(a) + \zeta_0 h_2(x), 0 + h_3(x))\\ &= (h_1(a) + h_2(x), h_3(x))\,,\end{aligned}$$

thus

$$h(a,x) = (h_1(a) + h_2(x), h_3(x))\,. \qquad (2.A)$$

Proposition 3.1. *Let h be an morphism in $\mathcal{C}_R$. Then*

$$h_1 \circ \zeta_x(a) = \xi_{h_3(x)} \circ h_1(a) \qquad (2.B)$$

Proof. Let h be a morphism of $\mathcal{C}_R$ from $G(\zeta)$ to $G(\xi)$ and $(a,x) \in G(\zeta)$. We have,

$$\begin{aligned}(0,x) * (a,0) &= (0 + \zeta_x(a), x)\\ &= (\zeta_x(a), x)\\ &= (\zeta_x(a), 0) * (0,x)\end{aligned}$$

$$\begin{aligned}h[(0,x) * (a,0)] &= h[(\zeta_x(a),0) * (0,x)]\\ &= h(\zeta_x(a),0) * h(0,x)\\ &= (h_1 \circ \zeta_x(a), 0) * (h_2(x), h_3(x))\\ &= (h_1 \circ \zeta_x(a) + h_2(x), h_3(x))\end{aligned}$$

On the other hand,

$$\begin{aligned}h[(0,x) * (a,0)] &= h(0,x) * h(a,0)\\ &= (h_2(x), h_3(x)) * (h_1(a), 0)\\ &= \big(h_2(x) + \xi_{h_3(x)} \circ h_1(a), h_3(x)\big)\end{aligned}$$

Therefore $h_1 \circ \zeta_x(a) + h_2(x) = h_2(x) + \xi_{h_3(x)} \circ h_1(a)$ and since T_2 is abelian, we can cancel $h_2(x)$ in the latter equation and obtain :

$$h_1 \circ \zeta_x(a) = \xi_{h_3(x)} \circ h_1(a)$$ ■

Proposition 3.2. *Suppose h is determined by h_1, h_2 and h_3. Then h is an isomorphism if and only if h_1 and h_3 are isomorphisms.*

Proof. Let $l \in \mathrm{Mor}_{\mathcal{C}_R}(G(\zeta), G(\xi))$ defined by :

$$l(b,y) = \left(h_1^{-1}(b) - h_1^{-1} \circ h_2 \circ h_3^{-1}(y), h_3^{-1}(y)\right).$$

Then

$$\begin{aligned} l \circ h(a,x) &= l[(h_1(a) + h_2(x), h_3(x))] \\ &= \left(h_1^{-1}(h_1(a) + h_2(x)) - h_1^{-1} \circ h_2 \circ h_3^{-1}(h_3(x)), h_3^{-1}(h_3(x))\right) \\ &= \left(a + h_1^{-1} \circ h_2(x) - h_1^{-1} \circ h_2(x), x\right) \\ &= (a,x), \end{aligned}$$

and

$$\begin{aligned} h \circ l(b,y) &= h[h_1^{-1}(b) - h_1^{-1} \circ h_2 \circ h_3^{-1}(y), h_3^{-1}(y)] \\ &= \left(h_1(h_1^{-1}(b) - h_1^{-1} \circ h_2 \circ h_3^{-1}(y)) + \lambda(h_3^{-1}(y)), h_3(h_3^{-1}(y))\right) \\ &= \left(b + h_2 \circ h_3^{-1}(y) - \lambda \circ h_3^{-1}(y), y\right) \\ &= (b,y). \end{aligned}$$

Hence l is a left and right inverse of h, thus h is an isomorphism.

Conversely suppose h is determined by h_1, h_2 and h_3 and that h is an isomorphism. Let l be the inverse of h.
l is determined by h'_1, h'_2 and h'_3. We then have,

$$l \circ h(a,x) = [h'_1 \circ h_1(a) + h'_1 \circ h_2(x) + h'_2 \circ h_3(x), h'_3 \circ h_3(x)] = (a,x)$$

and

$$h \circ l(b,y) = [h_1 \circ h'_1(b) + h_1 \circ h'_2(y) + h_2 \circ h'_3(y), h_3 \circ h'_3(y)] = (b,y)$$

We have by identification
$h'_1 \circ h_1(a) + h'_1 \circ h_2(x) + h'_2(x) \circ h_3(x) = a$
$h'_3 \circ h_3(x) = x$
$h_1 \circ h'_1(b) + h_1 \circ h'_2(y) + h_2 \circ h'_3(y) = b$
$h_3 \circ h'_3(y) = y$
That is
$h_3 \circ h'_3 = h'_3 \circ h_3 = Id$
$h'_1 \circ h_1 = h_1 \circ h'_1 = Id$
Therefore h_1 and h_3 are isomorphisms with inverses h'_1 and h'_3 respectively and $h'_2 = -h'_1 \circ h_2 \circ h'_3$.
■

Proposition 3.3. *If $G(\zeta) \cong G(\xi)$, then the image of ξ is a conjugate of the image of ζ.*

Proof. Given an isomorphism $h : G(\zeta) \to G(\xi)$, then h_1 and h_3 are isomorphisms. Given any $x \in F_1$, by formula (2), we have $\zeta_x = h_1^{-1} \circ \xi_{h_3(x)} \circ h_1$.
Since h_3 is an epimorphism, $\mathrm{Im}\xi = \mathrm{Im}(\xi \circ h_3)$ and the result follows. ■

Remark 3.1. We note that if $\zeta : F \to \mathrm{Aut}_R T$ is a homomorphism of groups and if $\zeta' = \zeta \circ q$ where $q : F \times R \to F$ is the projection onto the first factor. Then

$$T \rtimes_{\zeta'} (F \times R) \cong (T \rtimes_\zeta F) \times R$$

Proposition 3.4. *Let $G(\zeta) = T \rtimes_\zeta F$, $G(\xi) = T \rtimes_\xi F$ be objects of $\mathcal{C}_R$.*
The following conditions are equivalent :

(i) $G(\zeta) \times R \cong G(\xi) \times R$ *(isomorphic in $\mathcal{C}_R$)*

(ii) *There exist $\nu \in \mathrm{Aut}_R T$, $\varphi \in \mathrm{Aut}_R(F \times R)$ such that the following diagram commutes.*

$$\begin{tikzcd} F \times R \arrow[r, "\varphi"] \arrow[d, "\zeta'"'] & F \times R \arrow[d, "\xi'"] \\ \mathrm{Im}\zeta \arrow[r, "\theta_\nu"] & \mathrm{Im}\xi \end{tikzcd}$$

where $\theta_\nu : \gamma \mapsto \nu^{-1}\gamma\nu$ induced by the inner automorphism, $\zeta' = \zeta \circ q$ and $\xi' = \xi \circ q$ where $q : F \times R \to F$ is the projection onto the first factor.

Definition 3.1. Let G be an object of $\mathcal{C}_R$. The *non-cancellation set* denoted by $\chi_R(G)$, is the set of all isomorphism classes of groups $H \in \mathrm{Obj}(\mathcal{C}_R)$ such that $G \times R \cong H \times R$.

In the following example, let R be a subring of the rational number $\mathbb{Q}$ containing $\mathbb{Z}$ as a subset, $\mathbb{Z} \subset R \subset \mathbb{Q}$. We take $T = \mathbb{Z}_{11^2}$ and $F = R = \mathbb{Z}[\frac{1}{3}, \frac{1}{7}]$.
Let $H = \mathbb{Z}_{11^2} \rtimes_\omega \mathbb{Z}$ and $\bar{H} = \mathbb{Z}_{11^2} \rtimes_{\omega'} R$ where $\omega' = \omega \circ \eta$, with η the floor function from R to $\mathbb{Z}$. We want to show that there is a surjective function from $\chi(H)$ to $\chi(\bar{H})$.

Example 3.1. We will consider the particular case $h_3 = Id$. Let $H = \mathbb{Z}_{11^2} \rtimes_\omega \mathbb{Z}$ and let $\omega(1)$ be of order 110. Then from [22, Theorem 3.8], $\chi(H) \cong \mathbb{Z}^*_{110}/\pm 1$.
Let $K = \mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z}$ and $K \ncong H$.
Let $h : \bar{H} \to \bar{K}$ be a homomorphism from $\bar{H}$ to $\bar{K}$, where $\bar{K} = \mathbb{Z}_{11^2} \rtimes_{\mu'} R$ with $\mu' = \mu \circ \eta$. From $(2.A)$, h is given by $h(a, x) = (h_1(a) + h_2(x), h_3(x))$ for $a \in \mathbb{Z}_{11^2}$ and $x \in R$. We will consider the special case $h_3 = Id$.

Lemma 3.1. *If $K \ncong H$, then $\bar{K} \ncong \bar{H}$.*

Proof. Suppose $\bar{K} \cong \bar{H}$ and $K \ncong H$. Then by Proposition 3.1,
$\omega'(x) = h_1^{-1} \circ \mu'(x) \circ h_1$ where $\mu' = \mu \circ \eta$.
Therefore
$\omega(z) = h_1^{-1} \circ \mu(z) \circ h_1$ where $z = \eta(x)$.
Thus $K \cong H$ by [22, Theorem 3.2], which is in contradiction with the hypothesis. ■

Theorem 3.1. *If $K \times \mathbb{Z} \cong H \times \mathbb{Z}$ then $\bar{K} \times R \cong \bar{H} \times R$*

Proof. Suppose $K \times \mathbb{Z} \cong H \times \mathbb{Z}$. Then by [22, Theorem 3.1],
$\omega(\mathbb{Z}) = h_1^{-1} \circ \mu(\mathbb{Z}) \circ h_1$ for some h_1 in $\mathrm{Aut}(\mathbb{Z}_{11^2})$.
Since $\eta(R) = \mathbb{Z}$, the above equation implies that $\omega \circ \eta(R) = h_1^{-1} \circ \mu \circ \eta(R) \circ h_1$
therefore
$\omega'(R) = h_1^{-1} \circ \mu'(R) \circ h_1$ for some h_1 in $\mathrm{Aut}(\mathbb{Z}_{11^2})$.
Thus by taking $\nu = h_1$ and $\varphi = Id$, the result follows by Proposition 3.4. ■

Theorem 3.2. *Fix $\mathbb{Z} \leq R \leq \mathbb{Q}$. If H is a group in $\mathcal{K}$ (That is of the form of semidirect product) and $\bar{H} = H_R$, then there is a surjective function*
$\phi : \chi(H) \to \chi_R(\bar{H})$.

Proof. ϕ is well-defined by Theorem 3.1.
Let $[\bar{K}] \in \chi(\bar{H})$ where $\bar{K} = \mathbb{Z}_{11^2} \rtimes_\zeta R$.
Then $K = \mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z}$ with $\mu = \zeta \circ i$ where i is the inclusion map, is a suitable candidate. ■

Fixing any natural number k. Let us assume that in R the elements 2, 3, $\cdots$, $k - 1$ are multiplicatively invertible.
For $r \in R$, we define $\alpha(r) = 1 + (rx) + (2!)^{-1}(rx)^2 + \cdots + ((k-1)!)^{-1}(rx)^{k-1}$.
In the next example, we consider $R = \mathbb{Z}_{p^n}[x]$ where p is a prime number and $n \geq 1$. We are going to present here an object $G = T \rtimes_\zeta F$ of $\mathcal{C}_R$ with trivial non-cancellation set $\chi_R(G)$.

Example 3.2. Taking cyclic R-modules $F = R$, $T = R/J$ where $J =< x^k >$. We assume that n is relatively prime to the numbers $\{2, 3, \cdots , k - 1\}$.
Let $\zeta : R \to \mathrm{Aut}_R T$ given by $r \mapsto \zeta_r$ with $\zeta_r(a) = \alpha(r).a$

ζ is well-defined : For $r, s \in R$ and $a \in T$,

$$\begin{aligned} \zeta_{r+s}(a) &= \alpha(r+s).a \\ &= \alpha(r).\,(\alpha(s).a) \\ &= \zeta_r\,(\zeta_s(a)) \\ &= \zeta_r \circ \zeta_s(a) \end{aligned}$$

hence $\zeta_{r+s} = \zeta_r \circ \zeta_s$ and $\zeta_r \circ \zeta_{-r} = \zeta_{-r} \circ \zeta_r = Id_T$.

Moreover, for $\lambda \in R$,

$$\begin{aligned} \zeta_r(\lambda\, a) &= \alpha(r)(\lambda\, a) \\ &= (\alpha(r)\lambda)\, a && \text{since } T \text{ is an } R\text{-module} \\ &= (\lambda\alpha(r))\, a && \text{by commutativity of } R \\ &= \lambda\,(\alpha(r)a) && \text{since } T \text{ is an } R\text{-module} \\ &= \lambda\zeta_r(a) \end{aligned}$$

Thus $\zeta_r \in \mathrm{Aut}_R T$.

Let m be a natural number. Consider another action ξ of the form : $\xi_r = \zeta_r^m$.

Consider $F = R =< 1 >$ and $T =< 1 + J >$, let t be the order of the image of $\alpha(1)$ in R/J.

Lemma 3.2. *Let ζ and ξ be as given above. If $G(\zeta) \cong G(\xi)$, then there exists an integer u such that $um \equiv \pm 1 \mathrm{mod}(t)$.*

Proof. If $h : G(\zeta) \cong G(\xi)$, then there exists a commutative diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "h_1"', "\cong"] & G \arrow[r] \arrow[d, "h"'] & R \arrow[d, "h_3"', "\cong"] \arrow[dl, "h_0"'] \\
T \arrow[r] & H \arrow[r] & R
\end{tikzcd}$$

where h_1, $h_0 = (h_2, h_3)$ are induced by h.

For $F = R$ being cyclic, $h_3(1) = u$ or $-u$ where u is some unit. Following the formula (2.B), we have in particular,
$h_1 \circ \zeta_1(1 + J) = \xi_{h_3(1)} \circ h_1(1 + J)$,
i.e. $h_1(\alpha(1) + J) = \alpha(u)^m h_1(1 + J)$ or $\alpha(-u)^m h_1(1 + J)$
i.e. $h_1(\alpha(1) + J) = h_1(\alpha(u)^m + J)$ or $h_1(\alpha(-u)^m + J)$
i.e. $\alpha(1) + J = \alpha(u)^m + J$ or $\alpha(-u)^m + J$ since h_1 is injective.
Therefore $\alpha(1) \equiv \alpha(u)^m \bmod J$ or $\alpha(-u)^m \bmod J$.
Hence $\alpha(1)^{um-1} \equiv 1 \bmod J$ or $\alpha(1)^{um+1} \equiv 1 \bmod J$ so that $um \equiv \pm 1 \mathrm{mod}(t)$.

Now let u, v be units, $R \times R =< u, v >$ and consider the actions $\zeta' = \zeta \circ q$, $\xi' = \xi \circ q$ where q is the projection onto the first factor.

Let c, d be positive integers relatively prime. Let $\begin{pmatrix} m & t \\ c & d \end{pmatrix}$ be the unimodular matrix (such exists since c, d are relatively prime). Then the map $\phi : R \times R \to R \times R$ given by : $u \mapsto mu + cv$, $v \mapsto tu + dv$, is an automorphism.

ϕ is compatible with the identity map $Id_T : T \to T$ that is $\xi'_u(a) = \zeta'_{\phi(u)}(a)$ and $\xi'_v(a) = \zeta'_{\phi(v)}(a)$.

For $\xi'_u(a) = \alpha(u)^m.a$ in T, we have

$$\begin{aligned} \zeta'_{\phi(u)}(a) &= \zeta_{mu}(a) \\ &= \alpha(mu).a \\ &= \alpha(u)^m.a \ \text{ in } T \end{aligned}.$$

For $\xi'_{(v)}(a) = a$ in T, we have

$$\begin{aligned} \zeta'_{\phi(v)}(a) &= \zeta_{tu}(a) \\ &= \alpha(tu).a \\ &= \alpha(u)^t.a \\ &= \alpha(1)^{ut}.a \\ &= a \ \text{ in } T \end{aligned}$$

Thus $\phi : R \times R \to R \times R$, $Id_T : T \to T$ together determine an isomorphism of $T \rtimes_{\zeta'} (R \times R)$ with $T \rtimes_{\xi'} (R \times R)$. But by Proposition 3.4, $T \rtimes_{\zeta'} (R \times R) \cong (T \rtimes_{\zeta} R) \times R$ and $T \rtimes_{\xi'} (R \times R) \cong (T \rtimes_{\xi} R) \times R$
Thus $G(\zeta) \times R \cong G(\xi) \times R$. ∎

Theorem 3.3. *Let t be the order of the image of $\alpha(1)$ in R/J. For a positive integer m, let $G(m) = T \rtimes_{\xi} R$ where the action is given by : $\xi_1 : a \mapsto \alpha(1)^m.a$. The following conditions are equivalent :*

(i) For any $u \in (\mathbb{Z}_t)^*$, $u.m \not\equiv \pm 1 \bmod t$,

(ii) $G(1) \times R \cong G(m) \times R$.

Proof. $(1) \Rightarrow (2)$ has been shown above. Now let us look at $(2) \Rightarrow (1)$:
Suppose $G(1) \times R \cong G(m) \times R$ and $u.m \equiv \pm 1 \bmod t$ for some $u \in (\mathbb{Z}_t)^*$. Then by Proposition 3.4, there exist $\nu \in \mathrm{Aut}_R T$, $\varphi \in \mathrm{Aut}_R(R \times R)$ such that $\theta_\nu \circ \zeta'_u = \xi' \circ \varphi(u)$. Let $\begin{pmatrix} m & t \\ c & d \end{pmatrix}$

be the unimodular matrix (such exists since $m,\ t$ are relatively prime) defining φ. The equality $\theta_\nu \circ \zeta'_u = \xi' \circ \varphi(u)$ implies that $\nu^{-1} \circ \zeta_u \circ \nu(1+J) = \xi_{mu}(1+J)$ for $a \in T$. Therefore $\alpha(u) + J = \alpha(mu)^m + J$
i.e. $\alpha(1)^{u(m^2-1)} \equiv 1 \bmod J$
i.e. $u(m^2-1) \equiv 0 \bmod t$. Hence $m \equiv \pm 1 \bmod t$ which is in contradiction with $u.m \equiv \pm 1 \bmod t$.
■

Therefore we have the following proposition

Proposition 3.5. *If $n = p$ is a prime number, then $t = p$ and the non-cancellation set $\chi_R(G(1))$ is trivial. More precisely for any $m \in (\mathbb{Z}_p)^*$, $G(1) \cong G(m)$.*

4 Calculating the non-cancellation set of a group $H^{(k,k+1)}$

In [17] Mislin introduced the concept of the genus of a finitely generated nilpotent group N. This is the set of isomorphism classes of finitely generated nilpotent groups M such that the localizations M_p and N_p are isomorphic at every prime p. In [5, 8] the Mislin genus $\mathcal{G}(N)$ of a finitely generated nilpotent group N belonging to a certain class $\mathcal{N}_1$. The class $\mathcal{N}_1$ consists of those nilpotent groups N, given in terms of the associated short exact sequence

$$(1.1) \qquad TN \rightarrowtail N \twoheadrightarrow FN,$$

where TN is the torsion subgroup and FN the torsion-free quotient, by the conditions

(a) TN and FN are commutative

(b) Relation (1.1) splits for the action $\omega : FN \to \mathrm{Aut}TN$

(c) $\omega(FN)$ lies in the center of AutTN.
(It was observed in [5] that, in the presence of (a), (c) is equivalent to (c'))

(c') For all $\xi \in FN$, there exists an integer u such that $\xi \cdot a = \omega(\xi)(a) = ua$ for all $a \in TN$.
(Here, TN is written additively.)

It is shown in [13] that the genus $\mathcal{G}(N)$ of a group N in $\mathcal{N}_1$ is trivial unless FN is cyclic. Suppose then that FN is cyclic generated by ξ, and let d be the multiplicative order of u (see (c')) modulo m, where m is the exponent of TN. Then the calculation of the genus yields

$$(1.2) \qquad \mathcal{G}(N) \cong (\mathbb{Z}_d)^*/\pm 1$$

where $(\mathbb{Z}_d)^*$ is the multiplicative group of units in the ring $\mathbb{Z}_d$.
In [13], to study the case FN is not cyclic, Hilton and Scevenels generalized the procedure used in [10]. Thus we have, as previously presented, a finitely generated nilpotent group N belonging to $\mathcal{N}_1$ and fitting into a split short exact sequence

$$(1.3) \qquad TN \rightarrowtail N \twoheadrightarrow FN,$$

but now FN is free abelian of rank $r \geq 2$. We then know that we can write

$$(1.4) \qquad FN = <\xi_1, \xi_2, \cdots, \xi_r>,$$

where for $i = 1, 2, \cdots, r$,

$$\xi_i \cdot a = u_i a, \quad \text{for all } a \in TN \qquad (1.5)$$

and the multiplicative order of u_i modulo m is d_i, where m is the exponent of TN and

$$d_1 | d_2 | \cdots | d_r \qquad (1.6)$$

As in [5], it is known that the genus of N is given by

$$\mathcal{G}(N) \cong (\mathbb{Z}_{d_1})^*/\pm 1. \qquad (1.7)$$

However, [13, Theorem 2.2] tells us that, in fact,

$$d_1 = d_2 = \cdots = d_{r-2} = 1, \quad d_{r-1} = 1 \text{ or } 2. \qquad (1.8)$$

Note that, since FN is commutative, the commutator subgroup of N is finite.

Let $\mathcal{X}_0$ be the class of finitely generated groups having a finite commutator subgroup. For a group $H \in \mathcal{X}_0$, recall that $\chi(H)$ is the set of all isomorphism classes of groups G with the property that $G \times \mathbb{Z} \cong H \times \mathbb{Z}$. By a theorem of Warfield [30], $\chi(H) = \mathcal{G}(H)$ if H is nilpotent.
We recall the following notation, used in [6]. If (n, u) is a relatively prime pair of natural numbers, then the group

$$< a, b \mid a^n = 1,\ bab^{-1} = a^u >$$

is denoted by $G(n, u)$. $G(n, u)$ happens to be (isomorphic to) a semidirect product $\mathbb{Z}_n \rtimes_\mu \mathbb{Z}$, where $\mu : \mathbb{Z} \to \text{Aut}(\mathbb{Z}_n)$ is the action for which $\mu(1)$ is the automorphism $t \mapsto ut$ of $\mathbb{Z}_n$. Now let $H = G(n, u)$ and let d be the multiplicative order of u modulo n. From [22], we have that $\chi(H) \cong \mathbb{Z}_d^*/ \pm 1$. By $\mathbb{Z}_d^*/ \pm 1$ we mean the quotient group $\mathbb{Z}_d^*/ \left\{-\bar{1}, \bar{1}\right\}$.
For a natural number k, note that there is an obvious action $\omega : \mathbb{Z}^k \to \text{Aut}(\mathbb{Z}_n^k)$ for which $H^k = \mathbb{Z}_n^k \rtimes_\omega \mathbb{Z}^k$. From [33] we have an epimorphism $\theta : \mathbb{Z}_d^*/ \pm 1 \to \chi(H^k)$ and [32, Theorem 2.2] gives conditions under which θ is an isomorphism. Now we change the notation.
Let us denote $\mathbb{Z}_n \rtimes_\lambda \mathbb{Z} = \mathbb{Z}_n^1 \rtimes_\lambda \mathbb{Z}^1$ by $H^{(1,1)}$. For a natural number k, note that there is an obvious action $\omega : \mathbb{Z}^k \to \text{Aut}(\mathbb{Z}_n^k)$ for which $(H^{(1,1)})^k = H^{(k,k)} = \mathbb{Z}_n^k \rtimes_\omega \mathbb{Z}^k$. We want to calculate the genus of $H^{(k,k+1)} = \mathbb{Z}_n^k \rtimes_\mu \mathbb{Z}^{k+1}$.
Let u be a fixed natural number which is relatively prime to n and d a multiplicative order of u modulo n. In [32], $\text{Aut}(\mathbb{Z}_n^k)$ was identified with $\mathcal{M} = GL(\mathbb{Z}_n, k)$ the group of invertible $k \times k$-matrices with coefficients in $\mathbb{Z}_n$ and $\text{Im}\,\mu$ with the subgroup $\mathcal{U} =< U_1, U_2, \cdots, U_k >$ where U_i is a matrix in $\mathcal{M}$ for which the (i,i)-entry is u with 1 elsewhere on the main diagonal, and 0 everywhere off the main diagonal. $\mathcal{U}$ is an abelian group with exponenet d.
In the specific case of the group $H^{(k,k+1)} = \mathbb{Z}_n^k \rtimes_\mu \mathbb{Z}^{k+1}$, there exits $i, j \in \{1, 2, \cdots, k+1\}$, such that $U_j = U_i^l$ where l is a divisor of d. Otherwise, if $(l, d) = 1$ then U_j would describe a direct factor of a (isomorphism class) group in the $\chi(H^{(k,k+1)})$ as we know for the simple case of group in $\mathcal{N}_1$. Thus we can formulate it as follows.

Lemma 4.1. *Let $\xi_1, \xi_2, \cdots, \xi_{k+1}$ be the generators of the free abelian group $\mathbb{Z}^{k+1}$ in $H^{(k,k+1)} = \mathbb{Z}_n^k \rtimes_\mu \mathbb{Z}^{k+1}$ such that $\mu(\xi_i) = U_i$ for $i \in \{1, 2, \cdots, k+1\}$. Then there exist $i, j \in \{1, 2, \cdots, k+1\}$, $i \neq j$ such that $U_j = U_i^l$ where l is a divisor of d.*

Starting with $k = 1$. Let $H^{(1,2)} = \mathbb{Z}_n \rtimes_\gamma \mathbb{Z}^2$ where $\gamma : \mathbb{Z}^2 \to \text{Aut}(\mathbb{Z}_n)$. Let $\xi_1 = (1, 0)$ and $\xi_2 = (0, 1)$ be the generators of $\mathbb{Z}^2$. Then by Lemma 4.1, there exists l such that $\xi_1 \cdot a = ua$ and $\xi_2 \cdot a = u^l a$ for all $a \in \mathbb{Z}_n$. Let t be the multiplicative order of u^l modulo n. Then by [12, Theorem 2.2], $t|d$ and $t = 1$ or 2. Therefore $\chi(H^{(1,2)})$ is trivial.
In the general case, there is an action $\delta : \mathbb{Z}^{k-1} \to \text{Aut}(\mathbb{Z}_n^{k-1})$ such that $H^{(k,k+1)} = H^{(1,2)} \times H^{(k-1,k-1)}$. Since $\chi(H^{(k-1,k-1)}) \cong (\mathbb{Z}_d^*/ \{\pm 1\}) / \text{Ker}\theta$ where θ is an epimorphism $\chi(H^{(k-1,k-1)}) \to \mathbb{Z}_d^*/ \pm 1$ (see [33]) and remember that $\chi(H^{(1,2)})$ is trivial, therefore it results that $\chi(H^{(k,k+1)})$ is trivial. Thus we have the following theorem:

Theorem 4.1. *For a natural number k, $\chi(H^{(k,k+1)})$ is trivial.*

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