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ON MEROMORPHIC FUNCTIONS THAT SHARE ONE SMALL FUNCTION WITH THEIR DERIVATIVES

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Abstract In this paper, we study the problem of meromorphic functions sharing a small function with its derivative and prove one theorem. The theorem improves the results of Lahiri-Sarkar [J.Ineq.Pure Appl.Math.5(1)(2004) Art.20.], Zhang[J.Inequal.Pure Appl.Math.6(4)(2005) Art.116].

1 Introduction and results

Let f be a nonconstant meromorphic function defined in the whole complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f), N(r,f)$ and so on, that can be found, for instance in [1,2].

Let f and g be two nonconstant meromorphic functions. Let a be a finite complex number. We say that f and g share the value a CM(counting multiplicities) if $f-a$ and $g-a$ have the same zeros with the same multiplicities and we say that f and g share the value a IM(ignoring multiplicities) if we do not consider the multiplicities. When f and g share 1 IM, Let z_0 be a 1-points of f of order p, a 1-points of g of order q, we denote by $N_{11}(r,\frac{1}{f-1})$ the counting function of those 1-points of f and g where $p=q=1$; and $N_E^{(2}(r,\frac{1}{f-1})$the counting function of those 1-points of f and g where $p=q\geq 2$. $\overline{N}_L(r,\frac{1}{f-1})$ is the counting function of those 1-points of both f and g where $p>q$. In the same way, we can define $N_{11}(r,\frac{1}{g-1})$, $N_E^{(2}(r,\frac{1}{g-1})$ and $\overline{N}_L(r,\frac{1}{g-1})$. If f and g share 1 IM, it is easy to see that

$$\begin{aligned}\overline{N}(r,\frac{1}{f-1}) &= N_{11}(r,\frac{1}{f-1})+\overline{N}_L(r,\frac{1}{f-1})+\overline{N}_L(r,\frac{1}{g-1})+N_E^{(2}(r,\frac{1}{g-1})\\ &=\overline{N}(r,\frac{1}{g-1})\end{aligned}$$

Let f be a nonconstant meromorphic function. Let a be a finite complex number, and k be a positive integer, we denote by $N_{k)}(r,\frac{1}{f-a})$(or$\overline{N}_{k)}(r,\frac{1}{f-a})$) the counting function for zeros of $f-a$ with multiplicity $\leq k$(ignoring multiplicities), and by $N_{(k}(r,\frac{1}{f-a})$(or$\overline{N}_{(k}(r,\frac{1}{f-a})$) the counting function for zeros of $f-a$ with multiplicity at least k(ignoring multiplicities). Set

$$N_k(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})+\overline{N}_{(2}(r,\frac{1}{f-a})+\ldots+\overline{N}_{(k}(r,\frac{1}{f-a})$$

$$\Theta(a,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)},\delta(a,f)=1-\overline{\lim_{r\to\infty}}\frac{N(r,\frac{1}{f-a})}{T(r,f)}.$$

We further define

$$\delta_k(a,f)=1-\overline{\lim_{r\to\infty}}\frac{N_k(r,\frac{1}{f-a})}{T(r,f)}$$

Clearly

$$0\leq\delta(a,f)\leq\delta_k(a,f)\leq\delta_{k-1}(a,f)\cdots\leq\delta_2(a,f)\leq\delta_1(a,f)=\Theta(a,f)$$

Definition 1.1. (see [3])Let k be a nonnegative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a, f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k; clearly if f, g share (a, k), then f, g share (a, p) for all integers p with $0 \le p \le k$. Also, we note that f, g share a value a IM or CM if and only if they share $(a, 0)$ or (a, ∞),respectively.

A meromorphic function a is said to be a small function of f where $T(r, a) = S(r, f)$, that is $T(r, a) = o(T(r, f))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure. Similarly, we can define that f and g share a small function a IM or CM or with weight k.

R.Brück [4] first considered the uniqueness problems of an entire function sharing one value with its derivative and proved the following result.

Theorem A. Let f be a non-constant entire function satisfying $N(r, \frac{1}{f'}) = S(r, f)$. If f and f' share the value 1 CM, then $\frac{f'-1}{f-1} \equiv c$ for some nonzero constant c.

Brück[4] further posed the following conjecture.

Conjecture 1.1 Let f be a non-constant entire function, $\rho_1(f)$ be the first iterated order of f. If $\rho_1(f)$ is not a positive integer or infinite, f and f' share the value 1 CM, then $\frac{f'-1}{f-1} \equiv c$ for some nonzero constant c.

Yang [5] proved that the conjecture is true if f is an entire function of finite order. Yu [6] considered the problem of an entire or meromorphic function sharing one small function with its derivative and proved the following two theorems.

Theorem B. Let f be a non-constant entire function and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.

Theorem C. Let f be a non-constant non-entire meromorphic function and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If

(i) f and a have no common poles.

(ii) $f - a$ and $f^{(k)} - a$ share 0 CM

(iii) $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$

then $f \equiv f^{(k)}$ where k is a positive integer.

In the same paper, Yu [6] posed the following open questions.

(i) Can a CM shared be replaced by an IM shared value?

(ii) Can the condition $\delta(0, f) > \frac{3}{4}$ of Theorem B be further relaxed?

(iii) Can the condition (iii) in Theorem C be further relaxed?

(iv) Can in general the condition (i) of Theorem C be dropped?

In 2004, Liu and Gu [7] improved Theorem B and obtained the following results.

Theorem D. Let f be a non-constant entire function and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0, f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.

Lahiri and Sarkar [8] gave some affirmative answers to the first three questions imposing some restrictions on the zeros and poles of a. They obtained the following results.

Theorem E. Let f be a non-constant meromorphic function, k be a positive integer, and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If

(i) a has no zero(pole) which is also a zero(pole) of f or $f^{(k)}$ with the same multiplicity.

(ii) $f - a$ and $f^{(k)} - a$ share $(0, 2)$

(iii) $2\delta_{2+k}(0, f) + (4 + k)\Theta(\infty, f) > 5 + k$

then $f \equiv f^{(k)}$.

In 2005, Zhang [9] improved the above results and proved the following theorems.

Theorem F. Let f be a non-constant meromorphic function, $k(\ge 1), l(\ge 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If
$l \ge 2$ and

$$(3 + k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4 \tag{1.1}$$

or $l = 1$ and

$$(4 + k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6 \tag{1.2}$$

or $l = 0$ and

$$(6+2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10 \tag{1.3}$$

then $f \equiv f^{(k)}$.

It is natural to ask can the conditions (1.1) and (1.2) and (1.3) in Theorem F be further relaxed? In the present paper, we shall answer the question and improve the above result by repalcing the conditions (1.1) and (1.2) and (1.3) by three weaker ones,and thus provide a better answer to the first question of Yu than that of Zhang.

The following theorem is the main result of the paper.

Theorem 1.2. *Let f be a non-constant meromorphic function, $k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If*
$l \geq 2$ and

$$(3+k)\Theta(\infty, f) + \delta_2(0, f) + \delta_{2+k}(0, f) > k + 4 \tag{1.4}$$

$l = 1$ and

$$(\frac{7}{2}+k)\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \delta_2(0, f) + \delta_{2+k}(0, f) > k + 5 \tag{1.5}$$

or $l = 0$ and

$$(6+2k)\Theta(\infty, f) + 2\Theta(\infty, f) + \delta_2(0, f) + \delta_{1+k}(0, f) + \delta_{2+k}(0, f) > 2k + 10 \tag{1.6}$$

then $f \equiv f^{(k)}$.

From Theorem 1.2 we have the following corollary.

Corollary 1.3. *Let f be a non-constant meromorphic function, $k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If*
$l \geq 2$ and $\delta_{2+k}(0, f) > \frac{1}{2}$
or $l = 1$ and $\delta_{2+k}(0, f) > \frac{3}{5}$
or $l = 0$ and $\delta_{2+k}(0, f) > \frac{4}{5} - \frac{1}{5}[2\Theta(\infty, f) + \delta_2(0, f) + \delta_{1+k}(0, f) - 4\delta_{2+k}(0, f)]$
then $f \equiv f^{(k)}$.

2 Lemmas

Lemma 2.1. *(see [10]) Let f be a nonconstant meromorphic function, k, p be two positve integers, then*

$$N_p(r\frac{1}{f^{(k)}}) \leq N_{p+k}(r\frac{1}{f}) + k\overline{N}(r, f) + S(r, f)$$

Clearly $\overline{N}(r\frac{1}{f^{(k)}}) = N_1(r\frac{1}{f^{(k)}})$

Lemma 2.2. *Let*

$$H = (\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1}) \tag{2.1}$$

where F and G are two nonconstant meromorphic functions. If F and G share 1 IM and $H \not\equiv 0$, then

$$N_{11}(r, \frac{1}{F-1}) \leq N(r, H) + S(r, F) + S(r, G)$$

Proof. If z_0 is a common simple 1-point of F and G, substituting their Taylor series at z_0 into (2.1), we see that z_0 is a zero of H. Thus, we get

$$N_{11}(r, \frac{1}{F-1}) \leq \overline{N}(r, \frac{1}{H}) \leq T(r, H) + O(1) \leq N(r, H) + S(r, F) + S(r, G)$$

□

3 Proof of the Theorem 1.2

Proof. Let $F = \frac{f}{a}$ and $G = \frac{f^{(k)}}{a}$. Then F and G share $(1, l)$, except the zeros and poles of $a(z)$. Let H be defined by (2.1)

Case 1. Let $H \not\equiv 0$.

By our assumptions, H have poles only at zeros of F' and G' and poles of F and G, and those 1-points of F and G whose multiplicities are distinct from the multiplicities of corresponding 1-points of G and F respectively. Thus, we deduce from (2.1)that

$$\begin{aligned} N(r, H) \leq & \overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{G}) + \overline{N}(r, F) + N_0(r, \frac{1}{F'}) \\ & + N_0(r, \frac{1}{G'}) + \overline{N}_L(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{G-1}) \end{aligned} \tag{3.1}$$

here $N_0(r, \frac{1}{F'})$ is the counting function which only counts those points such that $F' = 0$ but $F(F-1) \neq 0$.

Because F and G share 1 IM, it is easy to see that

$$\begin{aligned} \overline{N}(r, \frac{1}{F-1}) &= N_{11}(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{G-1}) + N_E^{(2}(r, \frac{1}{G-1}) \\ &= \overline{N}(r, \frac{1}{G-1}) \end{aligned} \tag{3.2}$$

By the second fundamental theorem, we see that

$$\begin{aligned} T(r, F) + T(r, G) \leq & \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) \\ & + \overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) - N_0(r, \frac{1}{F'}) \\ & - N_0(r, \frac{1}{G'}) + S(r, F) + S(r, G) \end{aligned} \tag{3.3}$$

Using Lemma 2.2 and (3.1) and (3.2) and (3.3) we get

$$\begin{aligned} T(r, F) + T(r, G) \leq & N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 3\overline{N}(r, F) \\ & + N_{11}(r, \frac{1}{F-1}) + 2N_E^{(2}(r, \frac{1}{G-1}) + 3\overline{N}_L(r, \frac{1}{F-1}) \\ & + 3\overline{N}_L(r, \frac{1}{G-1}) + S(r, F) + S(r, G) \end{aligned} \tag{3.4}$$

We discuss the following three subcases.

Subcase 1.1 $l \geq 2$. Obviously

$$\begin{aligned} & N_{11}(r, \frac{1}{F-1}) + 2N_E^{(2}(r, \frac{1}{G-1}) + 3\overline{N}_L(r, \frac{1}{F-1}) + 3\overline{N}_L(r, \frac{1}{G-1}) \\ & \leq N(r, \frac{1}{G-1}) + S(r, F) \\ & \leq T(r, G) + S(r, F) + S(r, G) \end{aligned} \tag{3.5}$$

Combing (3.4) and (3.5), we get

$$T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 3\overline{N}(r, F) + S(r, F) \tag{3.6}$$

that is

$$T(r, f) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{f^{(k)}}) + 3\overline{N}(r, f) + S(r, f)$$

By Lemma 2.1 for $p = 2$, we get

$$T(r, f) \leq (k+3)\overline{N}(r, f) + N_2(r, \frac{1}{f}) + N_{2+k}(r, \frac{1}{f}) + S(r, f)$$

So

$$(3+k)\Theta(\infty, f) + \delta_2(0, f) + \delta_{2+k}(0, f) \leq k + 4$$

which contradicts with (1.4).

Subcase 1.2 $l = 1$. It is easy to see that

$$N_{11}(r, \frac{1}{F-1}) + 2N_E^{(2}(r, \frac{1}{G-1}) + 2\overline{N}_L(r, \frac{1}{F-1}) + 3\overline{N}_L(r, \frac{1}{G-1})$$
$$\leq N(r, \frac{1}{G-1}) + S(r, F)$$
$$\leq T(r, G) + S(r, F) + S(r, G) \tag{3.7}$$

$$\overline{N}_L(r, \frac{1}{F-1}) \leq \frac{1}{2}N(r, \frac{F}{F'}) \leq \frac{1}{2}N(r, \frac{F'}{F}) + +S(r, F)$$
$$\leq \frac{1}{2}(\overline{N}(r, \frac{1}{F}) + \overline{N}(r, F)) + S(r, F) \tag{3.8}$$

Combing (3.4) and (3.7) and (3.8), we get

$$T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + \frac{7}{2}\overline{N}(r, F) + \frac{1}{2}\overline{N}(r, \frac{1}{F}) + S(r, F) \tag{3.9}$$

that is

$$T(r, f) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{f^{(k)}}) + \frac{7}{2}\overline{N}(r, f) + \frac{1}{2}\overline{N}(r, \frac{1}{f}) + S(r, f)$$

By Lemma 2.1 for $p = 2$, we get

$$T(r, f) \leq (k + \frac{7}{2})\overline{N}(r, f) + N_2(r, \frac{1}{f}) + N_{2+k}(r, \frac{1}{f}) + \frac{1}{2}\overline{N}(r, \frac{1}{f}) + S(r, f)$$

So

$$(k + \frac{7}{2})\Theta(\infty, f) + \delta_2(0, f) + \delta_{2+k}(0, f) + \frac{1}{2}\Theta(0, f) \leq k + 5$$

which contradicts with (1.5).

Subcase 1.3 $l = 0$. It is easy to see that

$$N_{11}(r, \frac{1}{F-1}) + 2N_E^{(2}(r, \frac{1}{G-1}) + \overline{N}_L(r, \frac{1}{F-1}) + 2\overline{N}_L(r, \frac{1}{G-1})$$
$$\leq N(r, \frac{1}{G-1}) + S(r, F)$$
$$\leq T(r, G) + S(r, F) + S(r, G) \tag{3.10}$$

$$\overline{N}_L(r, \frac{1}{F-1}) \leq N(r, \frac{1}{F-1}) - \overline{N}(r, \frac{1}{F-1})$$
$$\leq N(r, \frac{F}{F'}) \leq N(r, \frac{F'}{F}) + S(r, f)$$
$$\leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, F) + S(r, F) \tag{3.11}$$

Similarly, we have

$$\overline{N}_L(r, \frac{1}{G-1} \leq \overline{N}(r, \frac{1}{G}) + \overline{N}(r, G) + S(r, F)$$
$$\leq N_1(r, \frac{1}{G}) + \overline{N}(r, F) + S(r, G) \tag{3.12}$$

Combing (3.4) and (3.10) – (3.12), we get

$$T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 6\overline{N}(r, F) + 2\overline{N}(r, \frac{1}{F}) + N_1(r, \frac{1}{G}) + S(r, F) \tag{3.13}$$

that is

$$T(r, f) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{f^{(k)}}) + 6\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{f}) + N_1(r, \frac{1}{f^{(k)}}) + S(r, f)$$

By Lemma 2.1 for $p = 2$ and for $p = 1$ respectively, we get

$$T(r, f) \leq (2k + 6)\overline{N}(r, f) + N_2(r, \frac{1}{f}) + N_{2+k}(r, \frac{1}{f}) + 2\overline{N}(r, \frac{1}{f}) + N_{1+k}(r, \frac{1}{f}) + S(r, f)$$

So

$$(2k + 6)\Theta(\infty, f) + \delta_2(0, f) + \delta_{2+k}(0, f) + 2\Theta(0, f) + \delta_{1+k}(0, f) \leq 2k + 10$$

which contradicts with (1.6).
Case 2. Let $H \equiv 0$.
By using the argument of as in [9], we can obtain $f \equiv f^{(k)}$, we here omit the detail. □

References

[1] W.K.Hayman, *Meronorphic Functions*, Clarendon Press, Oxford(1964).

[2] L.Yang, *Distrbution Theory*, Spinger-Verlag, Berlin(1993).

[3] I.Lahiri, Weighted sharing and uniqueness of meromorphic function,*Nagoya Math. J.* **161**, 193–206 (2001).

[4] R.Brück, On entire functions which share one value CM with theire first derivative, *Result. Math.* **30**, 21–24 (1996).

[5] L.Z.Yang, Solution of a differential equation and its applications,*Kodai. Math. J.* **22**, 458–464(1999).

[6] K.W.Yu, On entire and meromorphic functions that share small functions with their derivatives, *J. Inequal. Pure Appl. Math.* **4(1)**(2003) Art.21(Online:http://jipam.vu.edu.au/).

[7] L.P. Liu and Y.X.Gu,Uniqueness of meromorphic functions that share one small function with their derivatives,*Kodai. Math. J.* **27**, 272–279(2004).

[8] I.Lahiri and A.Sarkar, Uniqueness of meromorphic function and its derivative, *J. Inequal. Pure Appl. Math.* **5(1)**(2004) Art.20(Online:http://jipam.vu.edu.au/).

[9] Q.C.Zhang, Meromorphic function that shares one small function with their derivatives, *J. Inequal. Pure Appl. Math.* **6(4)**(2005) Art.116(Online:http://jipam.vu.edu.au/).

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