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Negatively Indexed Pell Numbers as the Permanent of Tridiagonal Matrix

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Abstract. In this paper, we obtain negatively indexed Pell numbers as the permanents of a tridiagonal matrix sequence. We prove an identity for this number sequence by using Laplace expansion formula.

1 Introduction

Pell numbers are defined as

$$P_n = 2P_{n-1} + P_{n-2} \ , \ n \geq 2 \tag{1.1}$$

with the initial conditions $P_0 = 0$, $P_1 = 1$. The first few Pell numbers are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985$. The recurrence relation (1.1) can be used to extend the sequence backward, thus

$$P_{-n} = -2P_{-n+1} + P_{-n+2}. \tag{1.2}$$

In [1], some relationships between Pell and Perrin numbers and permanents of special Hessenberg matrices are obtained as the determinant of the Hadamard product of two matrices. Some Fibonacci-Hessenberg matrices are derived and using the elementary row operations of the matrices, the Pell and Perrin numbers are obtained in a different way in [2]. In [3], the authors consider the relationship between the generalized Fibonacci numbers and the permanent of a $(0, 1)-$matrix. In [4], the authors develop the relationships between the second order linear recurrences and the permanent and determinants of the tridiagonal matrices.

In [5], an identity of Fibonacci numbers is proved via the determinant of tridiagonal matrix.

Let A be an $n \times n$ matrix, $A\left([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k]\right)$ be a $k \times k$ $(1 \leq k < n)$ submatrix of A and $\mathring{A}\left([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k]\right)$ be the $(n-k) \times (n-k)$ submatrix of A obtained from A by deleting the rows $i_1, i_2, \ldots, i_k$ and the columns $j_1, j_2, \ldots, j_k$. We will call the submatrices $\mathring{A}\left([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k]\right)$ the corresponding submatrices. The permanent of the matrix A is

$$\begin{aligned} per\,(A) \quad = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n} & per\left(A\left([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k]\right)\right) \\ & per\left(\mathring{A}\left([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k]\right)\right). \end{aligned} \tag{1.3}$$

The expansion of the permanent in (1.3) is called the Laplace expansion by rows $i_1, i_2, \ldots, i_k$. [6].

Let $A = [a_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call A contractible on column k, if column k contains exactly two nonzero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $Aij : k$ obtained from A replacing row i by $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A^T_{ij:k}\right]^T$ is called the contraction of A on row k relative to columns i and j [7].

In this paper, we obtain the permanents of $n-$square tridiagonal matrix depending on P_{-n} negatively indexed Pell numbers by using contraction method. The calculation of contraction will be on the first column. Then we give a proof of an identity for negatively indexed Pell numbers using Laplace expansion formula.

2 Main Result

Theorem 2.1. *Let A_n be an $n \times n$ tridiagonal matrix as in the following:*

$$A_n = \begin{bmatrix} 1 & 0 & 0 & & \cdots & 0 \\ 0 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & & 0 & 1 & -2 \end{bmatrix}.$$

Then the following equality holds

$$per\left(A_n\right) = P_{-n}. \tag{2.1}$$

Proof. The contraction method can be applied the matrix A_n on the second column. Let A_n^k be the kth contraction of A_n for $1 \leq k \leq n-3$. For $k = 1$ the first contraction of A_n is

$$A_n^1 = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & P_{-3} & P_{-2} & & & \\ & 1 & -2 & 1 & & \\ & & 1 & & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & -2 \end{bmatrix}.$$

For $k = 2$ the second contraction can be found as in the following by using the conraction method according to the second column of A_n^1 :

$$A_n^2 = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & P_{-4} & P_{-3} & & & \\ & -1 & -2 & 1 & & \\ & & 1 & & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & -2 \end{bmatrix}.$$

Continuing like this, we have the $(n-2)^{th}$ contraction as

$$A_n^{n-2} = \begin{bmatrix} 1 & 0 \\ 0 & P_{-n} \end{bmatrix}.$$

then

$$per\left(A_n\right) = per\left(A_n^{n-2}\right) = P_{-n}.$$

So, this shows that equality (2.1) is true. □

We prove the identity $P_{-n} = P_{-k}P_{-n+k-1} + P_{-k+1}P_{-n+k}$ of negatively indexed Pell numbers using the relation between the tridiagonal matrix A_n and Laplace expansion formula given in (1.3). Before proving this identity, we give the following theorem.

Theorem 2.2. *The matrix A_n has only k submatrices which are $k \times k$ and for $r = 1, 2, \ldots, k$ the permanent of these submatrices can be obtained as in the following*

$$per\left(A\left([\alpha],[\beta]\right)\right) = P_{-(k-r+1)} \tag{2.2}$$

where $\alpha = [1, 2, \ldots, k]$ and $\beta = [1, \ldots, k-r+1, k-r+3, \ldots, k+1]$.

Proof. We will prove the equality (2.2) using the induction method on k. For $k = 1$ then

$$per\left(A\left([1],[1]\right)\right) = P_{-1}.$$

Now, we assume that the equality (2.2) holds for $k = t$ $(t \geq 2)$. Then we will show that the equality (2.2)holds for $k = t+1$. So, we devide the $(t+1) \times (t+1)$ submatrix $A\left([1, \ldots, t+1], [1, \ldots, t-r+2, t-r+4, \ldots, t+2]\right)$ into four block matrices as

$$A\left([1, \ldots, t+1], [1, \ldots, t-r+2, t-r+4, \ldots, t+2]\right) = \begin{bmatrix} P & R \\ N & M \end{bmatrix}$$

where P is $(t-r+2) \times (t-r+2)$ submatrix of A_n, R is $(t-r+2) \times (r-1)$ zero matrix, N is $(r-1) \times (t-r+2)$ matris as in the following

$$N = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

and M is $(r-1) \times (r-1)$ lower triangular matrix as

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -2 & 1 & 0 & & \vdots \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \end{bmatrix}.$$

Using the following fact

$$per\left(\begin{bmatrix} P & R \\ N & M \end{bmatrix}\right) = per\,(A)\,per\left(M - NP^{-1}R\right)$$

the permanent of the submatrix $A\left([\alpha_1],[\beta_1]\right)$ obtained as

$$\begin{aligned} per\left(A\left([\alpha_1],[\beta_1]\right)\right) &= per\left(A\left([1, \ldots, t-r+2], [1, \ldots, t-r+2]\right)\right) \\ &= P_{-(t-r+2)} \end{aligned}$$

where $\alpha_1 = [1, \ldots, t+1]$ and $\beta_1 = [1, \ldots, t-r+2, t-r+4, \ldots, t+2]$.

The permanents of the corresponding submatrices are obtained as

$$per\left(\mathring{A}\left([\alpha],[\beta]\right)\right) = per\left(A[k+1, k+2, \ldots, n], [k-r+2, k+2, k+3, \ldots n]\right).$$

But for the reason the first column of $A[k+1, k+2, \ldots, n], [k-r+2, k+2, k+3, \ldots n]$ is $\begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}^T$ for $r = 3, 4, \ldots k$, all the permanents of the corresponding submatrices are zero except for $r = 1$ and $r = 2$.

For $r=1$ the permanent of the corresponding submatrix is

$$per\left(\mathring{A}\left([1,\ldots,k],[1,\ldots,k]\right)\right)=P_{-(n-k+1)}$$

and $r=2$ the permanent of the corresponding submatrix is

$$per\left(\mathring{A}\left([1,\ldots,k],[1,\ldots,k-1,k+1]\right)\right)=P_{-(n-k)}.$$

□

Theorem 2.3. *For $1\leq k\leq n$, the sequence given in (1.2) satisfies the following identity*

$$P_{-n}=P_{-k}P_{-n+k-1}+P_{-k+1}P_{-n+k}. \tag{2.3}$$

Proof. We obtain the permanent of the matrix A_n using the first k rows. We know from Theorem 2 that only the permanents of the corresponding submatrices for $r=1$ and $r=2$ are nonzero. So the permanent of the matrix A_n is obtained as

$$per\left(A_n\right)=P_{-k}P_{-n+k-1}+P_{-k+1}P_{-n+k}.$$

If we combine the equality (2.1), the proof is completed. □

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