Negatively Indexed Pell Numbers as the Permanent of Tridiagonal Matrix

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Communicated by Faruk Uygul

MSC 2010 Classifications:

Keywords and phrases: Pell numbers, contraction method, Laplace expansion formula.

Abstract. In this paper, we obtain negatively indexed Pell numbers as the permanents of a tridiagonal matrix sequence. We prove an identity for this number sequence by using Laplace expansion formula.

1 Introduction

Pell numbers are defined as

\[ P_n = 2P_{n-1} + P_{n-2} , \quad n \geq 2 \]  

with the initial conditions \( P_0 = 0 \), \( P_1 = 1 \). The first few Pell numbers are 0, 1, 2, 5, 12, 29, 70, 169, 408, 985. The recurrence relation (1.1) can be used to extend the sequence backward, thus

\[ P_{-n} = -2P_{-n+1} + P_{-n+2}. \]  

In [1], some relationships between Pell and Perrin numbers and permanents of special Hessenberg matrices are obtained as the determinant of the Hadamard product of two matrices. Some Fibonacci-Hessenberg matrices are derived and using the elementary row operations of the matrices, the Pell and Perrin numbers are obtained in a different way in [2]. In [3], the authors consider the relationship between the generalized Fibonacci numbers and the permanent of a \((0,1)\)-matrix. In [4], the authors develop the relationships between the second order linear recurrences and the permanent and determinants of the tridiagonal matrices.

In [5], an identity of Fibonacci numbers is proved via the determinant of tridiagonal matrix.

Let \( A \) be an \( n \times n \) matrix, \( A([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k]) \) be a \( k \times k \) (\( 1 \leq k < n \)) submatrix of \( A \) and \( \hat{A}([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k]) \) be the \((n-k) \times (n-k)\) submatrix of \( A \) obtained from \( A \) by deleting the rows \( i_1, i_2, \ldots, i_k \) and the columns \( j_1, j_2, \ldots, j_k \). We will call the submatrices \( \hat{A}([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k]) \) the corresponding submatrices. The permanent of the matrix \( A \) is

\[ \text{per} \ (A) = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n} \text{per} \ (A([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_k])) \]  

The expansion of the permanent in (1.3) is called the Laplace expansion by rows \( i_1, i_2, \ldots, i_k \). [6].

Let \( A = [a_{ij}] \) be an \( m \times n \) matrix with row vectors \( r_1, r_2, \ldots, r_m \). We call \( A \) contractible on column \( k \), if column \( k \) contains exactly two nonzero elements. Suppose that \( A \) is contractible on column \( k \) with \( a_{ik} \neq 0 \neq a_{jk} \) and \( i \neq j \). Then the \((m-1) \times (n-1)\) matrix \( A_{ij} : k \) obtained from \( A \) replacing row \( i \) by \( a_{jk} r_i + a_{ik} r_j \) and deleting row \( j \) and column \( k \) is called the contraction of \( A \) on column \( k \) relative to rows \( i \) and \( j \). If \( A \) is contractible on row \( k \) with \( a_{ki} \neq 0 \neq a_{kj} \) and \( i \neq j \), then
the matrix $A_{k;i,j} = \left[ A^T_{ij;k} \right]^T$ is called the contraction of $A$ on row $k$ relative to columns $i$ and $j$ [7].

In this paper, we obtain the permanents of $n \times n$ square tridiagonal matrix depending on $P_{-n}$ negatively indexed Pell numbers by using contraction method. The calculation of contraction will be on the first column. Then we give a proof of an identity for negatively indexed Pell numbers using Laplace expansion formula.

2 Main Result

Theorem 2.1. Let $A_n$ be an $n \times n$ tridiagonal matrix as in the following:

$$A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 1 & 0 & \cdots \\ 0 & 1 & -2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 & -2 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}.$$ 

Then the following equality holds

$$\text{per} (A_n) = P_{-n}. \tag{2.1}$$

Proof. The contraction method can be applied the matrix $A_n$ on the second column. Let $A^1_n$ be the $k$th contraction of $A_n$ for $1 \leq k \leq n - 3$. For $k = 1$ the first contraction of $A_n$ is

$$A^1_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & P_{-3} & P_{-2} \\ & 1 & -2 & 1 \\ & & 1 & \ddots \\ & & & \ddots & 1 \\ & & & & 1 & -2 \end{bmatrix}.$$ 

For $k = 2$ the second contraction can be found as in the following by using the contraction method according to the second column of $A^1_n$:

$$A^2_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & P_{-4} & P_{-3} \\ & -1 & -2 & 1 \\ & & 1 & \ddots \\ & & & \ddots & 1 \\ & & & & 1 & -2 \end{bmatrix}.$$ 

Continuing like this, we have the $(n - 2)^{th}$ contraction as

$$A^{n-2}_n = \begin{bmatrix} 1 & 0 \\ 0 & P_{-n} \end{bmatrix}.$$ 

then

$$\text{per} (A_n) = \text{per} (A^{n-2}_n) = P_{-n}.$$ 

So, this shows that equality (2.1) is true. \qed
We prove the identity $P_{-n} = P_{-k}P_{-n+k} + P_{-k+1}P_{-n+k}$ of negatively indexed Pell numbers using the relation between the tridiagonal matrix $A_r$ and Laplace expansion formula given in (1.3). Before proving this identity, we give the following theorem.

**Theorem 2.2.** The matrix $A_n$ has only $k$ submatrices which are $k \times k$ and for $r = 1, 2, \ldots, k$ the permanent of these submatrices can be obtained as in the following

$$
\text{per} \left( A \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \right) = P_{(k-r+1)}
$$

where $\alpha = [1, 2, \ldots, k]$ and $\beta = [1, \ldots, k-r+1, k-r+3, \ldots, k+1]$.

**Proof.** We will prove the equality (2.2) using the induction method on $k$. For $k = 1$ then

$$
\text{per} \left( A \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right) = P_{-1}.
$$

Now, we assume that the equality (2.2) holds for $k = t$ ($t \geq 2$). Then we will show that the equality (2.2) holds for $k = t+1$. So, we devide the $(t+1) \times (t+1)$ submatrix $A \left( \begin{array}{c} [1, \ldots, t+1], [1, \ldots, t-r+2, t-r+4, \ldots, t+2] \end{array} \right)$ into four block matrices as

$$
A \left( \begin{array}{c} [1, \ldots, t+1], [1, \ldots, t-r+2, t-r+4, \ldots, t+2] \end{array} \right) = \begin{bmatrix}
P & R \\
N & M
\end{bmatrix}
$$

where $P$ is $(t-r+2) \times (t-r+2)$ submatrix of $A_n$, $R$ is $(t-r+2) \times (r-1)$ zero matrix, $N$ is $(r-1) \times (t-r+2)$ matrix as in the following

$$
N = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0 \\
\vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0
\end{bmatrix}
$$

and $M$ is $(r-1) \times (r-1)$ lower triangular matrix as

$$
M = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-2 & 1 & 0 & & \\
1 & -2 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & -2 & 1
\end{bmatrix}
$$

Using the following fact

$$
\text{per} \left( \begin{bmatrix}
P & R \\
N & M
\end{bmatrix} \right) = \text{per} \left( A \right) \text{per} \left( M - NP^{-1}R \right)
$$

the permanent of the submatrix $A \left( \begin{array}{c} [\alpha_1], [\beta_1] \end{array} \right)$ obtained as

$$
\text{per} \left( A \left( \begin{array}{c} [\alpha_1], [\beta_1] \end{array} \right) \right) = \text{per} \left( A \left( [1, \ldots, t-r+2], [1, \ldots, t-r+2] \right) \right) = P_{-(t-r+2)}
$$

where $\alpha_1 = [1, \ldots, t+1]$ and $\beta_1 = [1, \ldots, t-r+2, t-r+4, \ldots, t+2]$.

The permanents of the corresponding submatrices are obtained as

$$
\text{per} \left( A \left( \begin{array}{c} [\alpha], [\beta] \end{array} \right) \right) = \text{per} \left( A[k+1, k+2, \ldots, n], [k-r+2, k+2, k+3, \ldots, n] \right).
$$

But for the reason the first column of $A[k+1, k+2, \ldots, n], [k-r+2, k+2, k+3, \ldots, n]$ is $\begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}^T$ for $r = 3, 4, \ldots, k$, all the permanents of the corresponding submatrices are zero except for $r = 1$ and $r = 2$. 

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For \( r = 1 \) the permanent of the corresponding submatrix is
\[
\text{per} \left( \hat{A} \left( \left[ 1, \ldots, k \right], \left[ 1, \ldots, k \right] \right) \right) = P_{-(n-k+1)}
\]
and \( r = 2 \) the permanent of the corresponding submatrix is
\[
\text{per} \left( \hat{A} \left( \left[ 1, \ldots, k \right], \left[ 1, \ldots, k - 1, k + 1 \right] \right) \right) = P_{-(n-k)}.
\]

**Theorem 2.3.** For \( 1 \leq k \leq n \), the sequence given in (1.2) satisfies the following identity
\[
P_{-n} = P_{-k}P_{-n+k-1} + P_{-k+1}P_{-n+k}.
\] (2.3)

**Proof.** We obtain the permanent of the matrix \( A_n \) using the first \( k \) rows. We know from Theorem 2 that only the permanents of the corresponding submatrices for \( r = 1 \) and \( r = 2 \) are nonzero. So the permanent of the matrix \( A_n \) is obtained as
\[
\text{per} \left( A_n \right) = P_{-k}P_{-n+k-1} + P_{-k+1}P_{-n+k}.
\]

If we combine the equality (2.1), the proof is completed. \( \square \)

**References**


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Received: May 22, 2015.

Accepted: October 21, 2015