On Generalization Ostrowski Type Inequalities for Functions of Two Variables with Bounded Variation

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Abstract In this paper, we establish a generalization of the Ostrowski type integral inequalities for functions of two independent variables with bounded variation and we give some applications for general quadrature formulae.

1 Introduction

Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) whose derivative \( f' : (a, b) \to \mathbb{R} \) is bounded on \((a, b)\), i.e. \( \|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty \). Then we have the inequality

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{4} + \frac{(x-a+b)^2}{(b-a)^2} \|f'\|_\infty \tag{1.1}
\]

for all \( x \in [a, b] \). The constant \( \frac{1}{4} \) is the best possible. This inequality is well known in the literature as the Ostrowski inequality.

In [10], Dragomir proved following Ostrowski type inequalities related to functions of bounded variation:

**Theorem 1.1.** Let \( f : [a, b] \to \mathbb{R} \) be a mapping of bounded variation on \([a, b] \). Then

\[
\left| \int_a^b f(t) dt - (b-a) f(x) \right| \leq \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \sqrt{(f)}
\]

holds for all \( x \in [a, b] \). The constant \( \frac{1}{2} \) is the best possible.

In a recent years, many authors studied the well-known Ostrowski inequality in one variable for variant types of functions such as, Lipschitzian, absolutely continuous and \( n \)-differentiable functions as well as the functions of bounded variation. However, a small attention and a few works have been considered for functions of two variables with bounded variation (see, [3], [6], [7], [16]). Among others, in particular, Dragomir and his group studied a very interesting inequalities for functions of one variable. For more information and recent developments on inequalities for mappings of bounded variation, please refer to([1], [2], [5], [8]-[14], [17], [19]-[24]).

2 Preliminaries and Lemmas

In 1910, Fréchet [15] has given the following characterization for the double Riemann-Stieltjes integral. Assume that \( f(x, y) \) and \( \alpha(x, y) \) are defined over the rectangle \( Q = [a, b] \times [c, d] \); let \( R \) be the divided into rectangular subdivisions, or cells, by the net of straight lines \( x = x_i, y = y_j \),

\[ a = x_0 < x_1 < \ldots < x_n = b, \quad c = y_0 < y_1 < \ldots < y_m = d; \]

let \( \xi, \eta \) be any numbers satisfying \( \xi \in [x_{i-1}, x_i], \eta \in [y_{j-1}, y_j] \), \( i = 1, 2, \ldots, n; j = 1, 2, \ldots, m \); and for all \( i, j \) let

\[ \Delta_{ij}(x_i, y_j) = \alpha(x_{i-1}, y_{j-1}) - \alpha(x_{i-1}, y_j) - \alpha(x_i, y_{j-1}) + \alpha(x_i, y_j). \]

Then if the sum

\[ S = \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi, \eta) \Delta_{ij}(x_i, y_j) \]
tends to a finite limit as the norm of the subdivisions approaches zero, the integral of \( f \) with respect to \( \alpha \) is said to exist. We call this limit the restricted integral, and designate it by the symbol

\[
\int_{a}^{b} \int_{c}^{d} f(x, y)dydx \alpha(x, y).
\]  \hspace{1cm} (2.1)

If in the above formulation \( S \) is replaced by the sum

\[
S^* = \sum_{i=1}^{m} \sum_{j=1}^{n} f(\xi_{ij}, \eta_{ij}) \Delta_{11} \alpha(i, j),
\]

where \( \xi_{ij}, \eta_{ij} \) are numbers satisfying \( \xi_{ij} \in [x_{i-1}, x_i] \), \( \eta_{ij} \in [y_{j-1}, y_j] \), we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

\[
\int_{a}^{b} \int_{c}^{d} f(x, y)dydx \alpha(x, y).
\]  \hspace{1cm} (2.2)

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson ([7]) has shown that the existence of (2.1) does not imply the existence of (2.2).

In [6], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

### 2.1 Definitions

The function \( f(x, y) \) is assumed to be defined in rectangle \( R(a \leq x \leq b, c \leq y \leq d) \). By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

\[
\begin{align*}
x &= x_i (i = 0, 1, 2, \ldots, m), \quad a = x_0 < x_1 < \ldots < x_m = b; \\
y &= y_j (j = 0, 1, 2, \ldots, n), \quad c = y_0 < y_1 < \ldots < y_n = d.
\end{align*}
\]

Each of the smaller rectangles into which \( R \) is devided by a net will be called a *cell*. We employ the notation

\[
\begin{align*}
\Delta_{11} f(x_i, y_j) &= f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j), \\
\Delta f(x_i, y_j) &= f(x_{i+1}, y_{j+1}) - f(x_i, y_j).
\end{align*}
\]

The total variation function, \( \phi(\overline{x}) \ [\psi(\overline{y})] \), is defined as the total variation of \( f(\overline{x}, \overline{y}) \ [f(x, \overline{y})] \) considered as a function of \( y \) [\( x \)] alone in interval \( (c, d) \ [(a, b)] \), or as \( +\infty \) if \( f(\overline{x}, \overline{y}) \ [f(x, \overline{y})] \) is of unbounded variation.

**Definition 2.1.** (Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function \( f(x, y) \) is said to be of bounded variation if the sum

\[
\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\Delta_{11} f(x_i, y_j)|
\]

is bounded for all nets.

**Definition 2.2.** (Fréchet). The function \( f(x, y) \) is said to be of bounded variation if the sum

\[
\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \epsilon_i \overline{\epsilon_j} |\Delta_{11} f(x_i, y_j)|
\]

is bounded for all nets and all possible choices of \( \epsilon_i = \pm 1 \) and \( \overline{\epsilon_j} = \pm 1 \).

**Definition 2.3.** (Hardy-Krause). The function \( f(x, y) \) is said to be of bounded variation if it satisfies the condition of Definition 2.1 and if in addition \( f(\overline{x}, y) \) is of bounded variation in \( y \) (i.e. \( \phi(\overline{x}) \) is finite) for at least one \( \overline{x} \) and \( f(x, \overline{y}) \) is of bounded variation in \( y \) (i.e. \( \psi(\overline{y}) \) is finite) for at least one \( \overline{y} \).
Definition 2.4. (Arzelà). Let \((x_i, y_i) (i = 0, 1, 2, ..., m)\) be any set of points satisfying the conditions

\[
a = x_0 < x_1 < ... < x_m = b; \\
c = y_0 < y_1 < ... < y_m = d.
\]

Then \(f(x, y)\) is said to be of bounded variation if the sum

\[
\sum_{i=1}^{m} |\Delta f(x_i, y_i)|
\]

is bounded for all such sets of points.

Therefore, one can define the concept of total variation of a function of two variables, as follows:

Let \(f\) be of bounded variation on \(Q = [a, b] \times [c, d]\), and let \(\sum (P)\) denote the sum \(\sum_{i=1}^{n} \sum_{j=1}^{m} |\Delta t f(x_i, y_j)|\) corresponding to the partition \(P\) of \(Q\). The number

\[
\bigvee_Q (f) := \sqrt[2]{\int_c^b \int_a^d f(t,s) dtds} := \sup \left\{ \sum (P) : P \in P(Q) \right\},
\]

is called the total variation of \(f\) on \(Q\). Here \(P([a, b])\) denotes the family of partitions of \([a, b]\).

In [16], authors proved following Lemmas related double Riemann-Stieltjes integral:

Lemma 2.5. (Integrating by parts) If \(f \in RS(\alpha)\) on \(Q\), then \(\alpha \in RS(f)\) on \(Q\), and we have

\[
\begin{align*}
\int_c^b \int_a^d f(t,s) dtds &= \int_a^b \int_c^d \alpha(t, s) dtds + \int_a^b \int_c^d \alpha(t, s) dtds f(t,s) \\
&= f(b, d)\alpha(b, d) - f(b, c)\alpha(b, c) - f(a, d)\alpha(a, d) + f(a, c)\alpha(a, c).
\end{align*}
\]

Lemma 2.6. Assume that \(g \in RS(\alpha)\) on \(Q\) and \(\alpha\) is of bounded variation on \(Q\), then

\[
\left| \int_c^b \int_a^d g(x, y) dxdy \alpha(x, y) \right| \leq \sup_{(x,y) \in Q} |g(x, y)| \bigvee_Q (\alpha).
\]

Theorem 2.7. Let \(f : Q \to \mathbb{R}\) be a mapping of bounded variation on \(Q\). Then for all \((x, y) \in Q\), we have inequality

\[
\left| (b - a) (d - c) f(x,y) - \int_c^b \int_a^d f(t,s) dtds \right| \leq \left[ \frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right] \left[ \frac{1}{2} (d - c) + \left| y - \frac{c + d}{2} \right| \right] \bigvee_Q (f)
\]

where \(\bigvee_Q (f)\) denotes the total (double) variation of \(f\) on \(Q\).

The aim of this paper is to establish a generalization of the Ostrowski type integral inequalities for functions of two independent variables with bounded variation and we give some applications for general quadrature formulae.

3 Main Results

We first prove the following theorem:
Theorem 3.1. Let $f : Q \to \mathbb{R}$ be a mapping of bounded variation on $Q$. Then for all $(x, y) \in Q$, we have inequality

$$
\left| (b - a) (d - c) \left[ (1 - \lambda) (1 - \eta) f(x, y) + \frac{(1 - \lambda) \eta}{2} [f(a, y) + f(b, y)] \right] + \frac{\lambda (1 - \eta)}{2} [f(x, c) + f(x, d)] + \frac{\lambda \eta}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \right| - \int_a^b \int_c^d f(t, s) ds dt \right| 
$$

$$
\leq \max \left\{ \lambda \left( \frac{b - a}{2} \right), \left( x - \left( 2 - \lambda \right) \frac{a + \lambda b}{2} \right), \left( \frac{2 - \lambda}{2} b + \frac{\lambda a}{2} - x \right) \right\} 
\times \max \left\{ \eta \left( \frac{d - c}{2} \right), \left( y - \left( 2 - \eta \right) \frac{c + \lambda d}{2} \right), \left( \frac{2 - \eta}{2} d + \frac{\eta c}{2} - y \right) \right\} \bigvee_a^b \bigvee_c^d (f)
$$

for any $\lambda, \eta \in [0, 1]$ and $a + \lambda \frac{b - a}{2} \leq x \leq b - \lambda \frac{b - a}{2}$, $c + \eta \frac{d - c}{2} \leq y \leq d - \eta \frac{d - c}{2}$, where $\bigvee_a^b \bigvee_c^d (f)$ denotes the total variation of $f$ on $Q$.

Proof. Applying Lemma 2.5, we have

$$
\int_a^x \int_c^y \left( t - \left( a + \lambda \frac{b - a}{2} \right) \right) \left( s - \left( \frac{c + \eta d - c}{2} \right) \right) ds dt f(t, s) \hspace{1cm} (3.2)
$$

$$
= \left( x - a - \lambda \frac{b - a}{2} \right) \left( y - c - \eta \frac{d - c}{2} \right) f(x, y) 
\quad + \left( x - a - \lambda \frac{b - a}{2} \right) \left( \eta \frac{d - c}{2} \right) f(x, c) 
\quad + \left( \lambda \frac{b - a}{2} \right) \left( y - c - \eta \frac{d - c}{2} \right) f(a, y) 
\quad + \left( \lambda \frac{b - a}{2} \right) \left( \eta \frac{d - c}{2} \right) f(a, c) 
\quad - \int_a^x \int_c^y f(t, s) ds dt,
$$

and similarly

$$
\int_a^x \int_y^d \left( t - \left( a + \lambda \frac{b - a}{2} \right) \right) \left( s - \left( \frac{d - \eta d - c}{2} \right) \right) ds dt f(t, s) \hspace{1cm} (3.3)
$$

$$
= \left( x - a - \lambda \frac{b - a}{2} \right) \left( \frac{d - c}{\eta} \right) f(x, d) 
\quad + \left( x - a - \lambda \frac{b - a}{2} \right) \left( \eta \frac{d - c}{2} \right) f(x, y) 
\quad + \left( \lambda \frac{b - a}{2} \right) \left( \frac{d - c}{\eta} \right) f(a, d) 
\quad + \left( \lambda \frac{b - a}{2} \right) \left( \eta \frac{d - c}{2} \right) f(a, y) 
\quad - \int_a^x \int_y^d f(t, s) ds dt,
$$
Summing (3.2)-(3.5), we have

\[
\begin{align*}
\int_b^a \int_y^x \left( t - \left( b - \lambda \frac{b-a}{2} \right) \right) \left( s - \left( c + \eta \frac{d-c}{2} \right) \right) d_s d_t f(t, s) \\
= \left( \lambda \frac{b-a}{2} \right) \left( y - c - \eta \frac{d-c}{2} \right) f(b, y) \\
+ \left( \lambda \frac{b-a}{2} \right) \left( h - \frac{d-c}{2} \right) f(a, c) \\
+ \left( b - x - \lambda \frac{b-a}{2} \right) \left( y - c - \eta \frac{d-c}{2} \right) f(x, y) \\
+ \left( b - x - \lambda \frac{b-a}{2} \right) \left( h - \frac{d-c}{2} \right) f(x, c) \
\end{align*}
\]

\[
\int_b^a \int_y^x \left( t - \left( b - \lambda \frac{b-a}{2} \right) \right) \left( s - \left( d - \eta \frac{d-c}{2} \right) \right) d_s d_t f(t, s) \\
= \left( \lambda \frac{b-a}{2} \right) \left( h - \frac{d-c}{2} \right) f(b, d) \\
+ \left( \lambda \frac{b-a}{2} \right) \left( d - y - \eta \frac{d-c}{2} \right) f(b, y) \\
+ \left( b - x - \lambda \frac{b-a}{2} \right) \left( h - \frac{d-c}{2} \right) f(x, d) \\
+ \left( b - x - \lambda \frac{b-a}{2} \right) \left( d - y - \eta \frac{d-c}{2} \right) f(x, y) 
\]

Summing (3.2)-(3.5), we have

\[
\begin{align*}
\int_b^a \int_y^x P(x, t; y, s) d_s d_t f(t, s) \\
= (b-a)(d-c) \left\{ (1-\lambda)(1-\eta) f(x, y) + \frac{(1-\lambda)\eta}{2} [f(a, y) + f(b, y)] \\
+ \frac{\lambda(1-\eta)}{2} [f(x, c) + f(x, d)] + \frac{\lambda \eta}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \right\} - \int_b^a \int_y^x f(t, s) d_s d_t
\end{align*}
\]

where

\[
P(x, t; y, s) = \left\{ \begin{array}{ll}
(t - (a + \lambda \frac{b-a}{2})) \left( s - (c + \eta \frac{d-c}{2}) \right), & (t, s) \in [a, x] \times [c, y] \\
(t - (a + \lambda \frac{b-a}{2})) \left( s - (d - \eta \frac{d-c}{2}) \right), & (t, s) \in [a, x] \times (y, d) \\
(t - (b - \lambda \frac{b-a}{2})) \left( s - (c + \eta \frac{d-c}{2}) \right), & (t, s) \in (x, b) \times [c, y] \\
(t - (b - \lambda \frac{b-a}{2})) \left( s - (d - \eta \frac{d-c}{2}) \right), & (t, s) \in (x, b) \times (y, d)
\end{array} \right\
\]

for any \( \lambda, \eta \in [0, 1] \) and \( a + \lambda \frac{b-a}{2} \leq x \leq b - \lambda \frac{b-a}{2}, c + \eta \frac{d-c}{2} \leq y \leq d - \eta \frac{d-c}{2} \).
Now, taking the modulus in (3.6), we have
\[
\left| \int_a^b \int_c^d P(x, t; y, s)dsdt \right| f(t, s)
\]
\[
= \left| (b-a)(d-c) \left[ (1-\lambda)(1-\eta) f(x, y) + \frac{(1-\lambda)\eta}{2} [f(a, y) + f(b, y)] \right. \\
+ \left. \frac{\lambda(1-\eta)}{2} [f(x, c) + f(x, d)] + \frac{\lambda\eta}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \right] - \int_a^b \int_c^d f(t, s)dsdt \right|.
\]
On the other hand, using Lemma 2.6 it follows that
\[
\left| \int_a^b \int_c^d P(x, t; y, s)dsdt \right| f(t, s)
\]
\[
\leq \sup_{(t,s)\in\Omega} |P(x, y; t, s)| \bigg( \sup_{a\leq x\leq c} |P(x, y; t, s)|, \sup_{a\leq x\leq b} |P(x, y; t, s)| \bigg)
\]
\[
= \max \left\{ \sup_{a\leq x\leq c} \left( \lambda \frac{b-a}{2}, \left( x - \frac{(2-\lambda) a + \lambda b}{2} \right) \right), \sup_{a\leq x\leq d} \left( \lambda \frac{b-a}{2}, \left( x - \frac{(2-\lambda) a + \lambda b}{2} \right) \right) \right\}
\]
\[
\leq \max \left\{ \frac{\lambda}{2} \left( x - \frac{(2-\lambda) a + \lambda b}{2} \right), \left( \frac{(2-\lambda) b + \lambda a}{2} - x \right) \right\}
\]
\[
\times \max \left\{ \frac{\lambda}{2} \left( x - \frac{(2-\lambda) a + \lambda b}{2} \right), \left( \frac{(2-\lambda) b + \lambda a}{2} - x \right) \right\}
\]
\[
\leq \frac{1}{4} \left( \frac{b}{b-a} \right) \left( \frac{d}{d-c} \right) \int_a^b \int_c^d f(t, s)dsdt \leq \frac{1}{4} \left( \frac{b}{a} \right) \left( \frac{d}{c} \right) \left( \frac{f(t, s)}{f(t, s)} \right) \quad (3.7)
\]
which proved by Jawarneh and Noorani in [16]. The constant \( \frac{1}{4} \) is the best possible.
Proof. For proof of the sharpness of the constant, assume that (3.7) holds with a constant $A > 0$, that is,

$$\left| \frac{f(b, d) + f(a, d) + f(b, c) + f(a, c)}{4} - \frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f(t, s) ds dt \right| \leq A \bigvee_a^b \bigvee_c^d (f).$$

(3.8)

If we choose $f : Q \rightarrow \mathbb{R}$ with

$$f(x, y) = \begin{cases} 1 & \text{if } x = a, b \text{ and } y = c, d \\ 0 & \text{if } (x, y) \in (a, b) \times (c, d) \end{cases}$$

then $f$ is of bounded variation on $Q$, and

$$\frac{f(b, d) + f(a, d) + f(b, c) + f(a, c)}{4} = 1, \quad \int_a^b \int_c^d f(t, s) ds dt = 0, \quad \bigvee_Q (f) = 4,$$

giving in (3.8), $1 \leq 4A$, thus $A \geq \frac{1}{4}$.

Corollary 3.4. Under the assumptions of Theorem 3.1 with $\lambda = \frac{1}{3}$ and $\eta = \frac{1}{3}$, we have the inequality

$$\left\| (b - a) (d - c) \left[ \frac{4}{9} f(x, y) + f(a, y) + f(b, y) + f(x, c) + f(x, d) \right] \right\|$$

$$+ \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{36} - \int_a^b \int_c^d f(t, s) ds dt \right| \leq \max \left\{ \frac{b - a}{6}, \left( x - \frac{5a + b}{6} \right), \left( \frac{5b + a}{6} - x \right) \right\} \times \max \left\{ \frac{d - c}{6}, \left( y - \frac{5c + d}{6} \right), \left( \frac{5d + c}{6} - y \right) \right\} \bigvee_a^b \bigvee_c^d (f)$$

for $\frac{5a + b}{6} \leq x \leq \frac{5b + a}{6}$ and $\frac{5c + d}{6} \leq y \leq \frac{5d + c}{6}$.

Remark 3.5. If we choose $x = \frac{a + b}{2}$ and $y = \frac{c + d}{2}$ in Corollary 3.4, then we have the "Simpson’s rule inequality"

$$\left\| (b - a) (d - c) \left[ \frac{f(b, d) + f(b, c) + f(a, d) + f(a, c)}{36} \right. \right.$$

$$+ \left. \frac{f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + f \left( \frac{a + b}{2}, c \right) + f \left( b, \frac{c + d}{2} \right) + f \left( \frac{a + b}{2}, d \right)}{9} \right.$$  

$$+ \frac{4}{9} f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \bigvee_a^b \bigvee_c^d (f)$$

which is proved by Jawarneh and Noorani in [16].

Corollary 3.6. Under the assumptions of Theorem 3.1 with $\lambda = \frac{1}{2}$ and $\eta = \frac{1}{2}$, we have the
inequality
\[
\left| \frac{(b-a)(d-c)}{4} \left[ f(x,y) + f(a,y) + f(b,y) + f(x,d) + f(x,c) + f(x,d) \right] \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right|
\]
\[
+ \left| \int_a^b \int_c^d f(t,s) ds dt \right|
\]
\[
\leq \max \left\{ \frac{b-a}{4}, \left( x - \frac{3a+b}{4} \right), \left( \frac{3b+a}{4} - x \right) \right\}
\]
\times \max \left\{ \frac{d-c}{4}, \left( y - \frac{3c+d}{4} \right), \left( \frac{3d+c}{4} - y \right) \right\}
\]
\[
\bigg| \frac{b}{4} \bigg| \bigg| \bigg| f \bigg| \bigg|_Q
\]
\[
\text{for } \frac{3a+b}{4} \leq x \leq \frac{3b+a}{4} \text{ and } \frac{3c+d}{4} \leq y \leq \frac{3d+c}{4}.
\]

**Corollary 3.7.** If we take 
\[
x = \frac{a+b}{2} \text{ and } y = \frac{c+d}{2}
\]
in Corollary 3.6, then we get
\[
\left| \frac{(b-a)(d-c)}{4} \left[ f(b,d) + f(b,c) + f(a,d) + f(a,c) \right] \right|
\]
\[
+ \left| \int_a^b \int_c^d f(t,s) ds dt \right|
\]
\[
\leq \frac{(b-a)(d-c)}{16} \bigg| \bigg| f \bigg| \bigg|_Q
\]

The constant \( \frac{1}{16} \) is the best possible.

**Proof.** Assume that (3.10) holds with a constant \( C > 0 \), i.e.,
\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{16} \right|
\]
\[
+ \left| \int_a^b \int_c^d f(t,s) ds dt \right|
\]
\[
\leq C \bigg| \bigg| f \bigg| \bigg|_Q
\]

Define the set
\[
E := \left\{ \left( a,c \right), \left( a,d \right), \left( b,c \right), \left( b,d \right), \left( a, \frac{c+d}{2} \right), \left( b, \frac{c+d}{2} \right), \right\}
\]
\[
\left( \frac{a+b}{2}, c \right), \left( \frac{a+b}{2}, d \right), \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \right\}.
\]

If we choose \( f : Q \rightarrow \mathbb{R} \) with
\[
f(x,y) = \begin{cases} 
1 & \text{if } (x,y) \in E \\
0 & \text{if } (x,y) \notin [a,b] \times [c,d] \setminus E
\end{cases}
\]
then \( f \) is of bounded variation on \( Q \), and

\[
\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{16} = \frac{1}{4},
\]

\[
\frac{f(a, \frac{a+d}{2}) + f(b, \frac{c+d}{2}) + f(\frac{a+b}{2}, c) + f(\frac{a+b}{2}, d)}{8} = \frac{1}{4},
\]

\[
\frac{1}{4} f \left( \frac{a + b + c + d}{2}, \frac{1}{2} \right) = \frac{1}{4},
\]

\[
\int_a^b \int_c^d f(t, s) ds dt = 0,
\]

and \( \sqrt{Q} = 16 \).

Therefore, we get in (3.11), \( 1 \leq 16C \), thus \( C \geq \frac{1}{16} \), which implies the constant \( \frac{1}{16} \) is the best possible. This completes the proof. \( \square \)

4 Some Composite Quadrature Formula

Let us consider the arbitrary division \( I_n : a = x_0 < x_1 < \ldots < x_n = b \), and \( J_m : c = y_0 < y_1 < \ldots < y_m = d, h_i := x_{i+1} - x_i \), and \( l_j := y_{j+1} - y_j \),

\( v(h) := \max \{ |h_i| \ i = 0, \ldots, n - 1 \} \),

\( v(l) := \max \{ |l_j| \ j = 0, \ldots, m - 1 \} \).

Then the following Theorem holds.

**Theorem 4.1.** Let \( f : Q \rightarrow \mathbb{R} \) is of bounded variation on \( Q \) and \( \xi_i \in [x_i, x_{i+1}] \ (i = 0, \ldots, n - 1) \), \( \tau_j \in [y_j, y_{j+1}] \ (j = 0, \ldots, m - 1) \). Then we have the quadrature formula:

\[
\int_a^b \int_c^d f(t, s) ds dt = 4 \int_0^{n-1} \int_0^{m-1} f(\xi_i, \tau_j) h_i l_j
\]

\[
+ \frac{1}{9} \int_0^{n-1} \int_0^{m-1} [f(x_i, \tau_j) + f(x_{i+1}, \tau_j) + f(\xi_i, y_j) + f(\xi_i, y_{j+1})] h_i l_j
\]

\[
+ \frac{1}{36} \int_0^{n-1} \int_0^{m-1} [f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})] h_i l_j
\]

\[+ R(I_n, J_m, \xi, \tau, f) \]

The remainder \( R(I_n, J_m, \xi, \eta, f) \) satisfies

\[
| R(I_n, J_m, \xi, \eta, f) | \leq \max_{i \in \{0, \ldots, n-1\}} \left\{ \max \left\{ \frac{h_i}{6}, \left( \xi_i - \frac{5x_i + x_{i+1}}{6} \right), \left( \frac{5x_{i+1} + x_i}{6} - \xi_i \right) \right\} \right\} \]

\[
\times \max_{j \in \{0, \ldots, m-1\}} \left\{ \max \left\{ \frac{l_j}{6}, \left( \tau_j - \frac{5y_j + y_{j+1}}{6} \right), \left( \frac{5y_{j+1} + y_j}{6} - \tau_j \right) \right\} \right\}
\]

\[
\times \sqrt{Q} \int_a^b \int_c^d f(t, s) ds dt.
\]
Proof. Applying Corollary 3.4 to bidimensional interval \([x_i, x_{i+1}] \times [y_j, y_{j+1}]\) and \(\xi_i \in [x_i, x_{i+1}]\) \((i = 0, \ldots, n - 1)\), \(\tau_j \in [y_j, y_{j+1}]\) \((j = 0, \ldots, m - 1)\), we have the inequality

\[
\begin{align*}
|b - a| \sum_{i=0}^n \sum_{j=0}^m \left[ \frac{4}{9} f(\xi_i, y_j) + \frac{f(x_i, \tau_j) + f(x_{i+1}, \tau_j) + f(\xi_i, y_j) + f(\xi_i, y_{j+1})}{9} \right] \\
+ \frac{f(x_i, y_j) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1})}{36} - \int_a^b \int_c^d f(t, s) ds dt
\end{align*}
\]  

\[
\leq \max \left\{ \frac{h_i}{6}, \left( \xi_i - \frac{5x_i + x_{i+1}}{6} \right), \left( \frac{5x_{i+1} + x_i}{6} - \xi_i \right) \right\} \times \max \left\{ \frac{l_j}{6}, (\tau_j - \frac{5y_j + y_{j+1}}{6}), \left( \frac{5y_{j+1} + y_j}{6} - \tau_j \right) \right\}
\]

\[
\times \max \left\{ \frac{h_i}{6}, \left( \xi_i - \frac{5x_i + x_{i+1}}{6} \right), \left( \frac{5x_{i+1} + x_i}{6} - \xi_i \right) \right\} \times \max \left\{ \frac{l_j}{6}, (\tau_j - \frac{5y_j + y_{j+1}}{6}), \left( \frac{5y_{j+1} + y_j}{6} - \tau_j \right) \right\}
\]

Summing the inequality (4.1) over \(i\) from 0 to \(n - 1\) and \(j\) from 0 to \(m - 1\), then we get

\[
|R(I_n, J_m, \xi, \tau, f)|
\]

\[
\leq \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[ \frac{h_i}{6}, \left( \xi_i - \frac{5x_i + x_{i+1}}{6} \right), \left( \frac{5x_{i+1} + x_i}{6} - \xi_i \right) \right] \times \max \left\{ \frac{l_j}{6}, (\tau_j - \frac{5y_j + y_{j+1}}{6}), \left( \frac{5y_{j+1} + y_j}{6} - \tau_j \right) \right\}
\]

which is the required result.

Corollary 4.2. Let \(I_n, J_m\) and \(f\) be as above. If we choose \(\xi_i = \frac{x_i + x_{i+1}}{2}\) and \(\tau_j = \frac{y_j + y_{j+1}}{2}\), then we get

\[
|b - a| \sum_{i=0}^n \sum_{j=0}^m \left[ \frac{4}{9} f(\xi_i, y_j) + \frac{f(x_i, \tau_j) + f(x_{i+1}, \tau_j) + f(\xi_i, y_j) + f(\xi_i, y_{j+1})}{9} \right] \\
+ \frac{f(x_i, y_j) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1})}{36} - \int_a^b \int_c^d f(t, s) ds dt
\]  

\[
\leq \max \left\{ \frac{h_i}{6}, \left( \xi_i - \frac{5x_i + x_{i+1}}{6} \right), \left( \frac{5x_{i+1} + x_i}{6} - \xi_i \right) \right\} \times \max \left\{ \frac{l_j}{6}, (\tau_j - \frac{5y_j + y_{j+1}}{6}), \left( \frac{5y_{j+1} + y_j}{6} - \tau_j \right) \right\}
\]

\[
\times \max \left\{ \frac{h_i}{6}, \left( \xi_i - \frac{5x_i + x_{i+1}}{6} \right), \left( \frac{5x_{i+1} + x_i}{6} - \xi_i \right) \right\} \times \max \left\{ \frac{l_j}{6}, (\tau_j - \frac{5y_j + y_{j+1}}{6}), \left( \frac{5y_{j+1} + y_j}{6} - \tau_j \right) \right\}
\]

\[
\times \left[ \frac{b}{a} \sum_{i=0}^{b-1} \sum_{j=0}^{d-1} f(\xi_i, \tau_j) \right]
\]
Theorem 4.1, then we have the "Simpson’s rule"

\[
\int_a^b \int_c^d f(t, s)\, ds\, dt = \frac{4}{9} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[ f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right) h_i l_j \right] + \frac{1}{9} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[ f\left(\frac{x_i + x_{i+1}}{2}, y_j\right) + f\left(\frac{x_{i+1}}{2}, y_j + y_{j+1}\right)\right] h_i l_j \\
+ \frac{1}{9} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[ f\left(x_i, \frac{y_j + y_{j+1}}{2}\right) + f\left(x_{i+1}, \frac{y_j + y_{j+1}}{2}\right)\right] h_i l_j \\
+ \frac{1}{36} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[ f(x_i, y_j) + f(x_{i+1}, y_j) + f(x_{i+1}, y_j) + f(x_{i+1}, y_j)\right] h_i l_j \\
+ R_S(I_n, J_m, f).
\]

The remainder \(R_S(I_n, J_m, f)\) satisfies

\[
|R_S(I_n, J_m, f)| \leq \frac{1}{9} v(h) v(l) \sqrt{b-a} \sqrt{d-c} (f).
\]

References


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