Quantum Codes Over $F_2 + uF_2 + vF_2$

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Abstract In this paper, it was constructed quantum codes from cyclic codes over finite ring $S = F_2 + uF_2 + vF_2$, $u^2 = u$, $v^2 = v$, $uv = vu = 0$ for arbitrary length $n$. It was given a new Gray map $\Psi$ which is both an isometry and weight preserving map. It was shown that $C$ is self orthogonal codes over $S$, so is $\Psi(C)$. It was given a necessary and sufficient condition for cyclic codes over $S$ that contains its dual and it was determined the parameters of quantum codes which are obtained from cyclic codes over $S$.

1 Introduction

Although the theory quantum error correcting codes has striking differences from the theory classical error correcting codes, Calderbank et al. gave a way to construct quantum error correcting codes from classical error correcting codes in [3].

Many good quantum codes have been constructed by using classical cyclic codes over $F_q$ with self orthogonal (or dual containing) properties.

Some authors constructed quantum codes by using linear codes over finite rings. For example, in [5], J. Qian et al. gave a new method to obtain self-orthogonal codes over $F_2$. They gave a construction for quantum error correcting codes starting from cyclic codes over finite ring, $F_2 + uF_2$, $u^2 = 0$. X. Kai, S. Zhu gave construction for quantum codes from linear and cyclic codes over $F_2 + uF_2$, $u^2 = 0$ in [6]. They derived Hermitian self-orthogonal codes over $F_4$ as Gray images of linear and cyclic codes over $F_4 + uF_4$. In [7], X. Yin and W. Ma gave an existence condition of quantum codes which are derived from cyclic codes over finite ring $F_2 + uF_2 + vF_2$, $v^3 = 0$ with Lee metric. J. Qian gave a new method of constructing quantum error correcting codes from cyclic codes over finite ring $F_2 + vF_2$, $v^2 = v$, for arbitrary length $n$ in [4]. A. Dertli et al. gave quantum codes over the finite ring in [1, 2].

This paper is organized as follows. In section 2, we give some basic knowledges about the finite ring $S$, cyclic code, dual code. In section 3, we define a new Gray map from $S$ to $F_2^2$, Lee weights of elements of $S$. We show that if $C$ is self orthogonal so is $\Psi(C)$. In section 4, a necessary and sufficient condition for cyclic code over $S$ that contains its dual is given. The parameters of quantum error correcting codes are obtained from cyclic codes over $S$. In section 5, we give some examples.

2 Preliminaries

Let $S$ be the ring $F_2 + uF_2 + vF_2$ where $u^2 = u$, $v^2 = v$, $uv = vu = 0$ and $F_2 = \{0, 1\}$ a finite commutative ring with 8 elements. $S$ is semi local ring with three maximal ideals and a principal ideal ring. It is not finite chain ring. Let $w = 1 + u + v$. Addition and multiplication over $S$ are given in the following tables:

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u & 0 & u & u & 0 & u & u & 0 & 0 \\
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u+v & 0 & 1+v & u & 0 & w & 1+v & u & w \\
1+u+v & 0 & w & 0 & 0 & w & w & 0 & w \\
\end{array}
\]

The ideals are follows:

\[
\begin{align*}
I_0 &= \{0\}, \\
I_u &= \{0, u\}, \\
I_v &= \{0, v\}, \\
I_{u+v} &= \{0, u, v, u+v\}, \\
I_{1+u} &= \{0, 1+u\}, \\
I_{1+u+v} &= \{0, 1+u+v\}
\end{align*}
\]

A linear code \(C\) over \(S\) length \(n\) is a \(S\)–submodule of \(S^n\). An element of \(C\) is called a codeword.

For any \(x = (x_0, x_1, \ldots, x_{n-1})\), \(y = (y_0, y_1, \ldots, y_{n-1})\) the inner product is defined as

\[
x.y = \sum_{i=0}^{n-1} x_i y_i
\]

If \(x.y = 0\) then \(x\) and \(y\) are said to be orthogonal. Let \(C\) be linear code of length \(n\) over \(S\), the dual code of \(C\)

\[
C^\perp = \{x : \forall y \in C, x.y = 0\}
\]

which is also a linear code over \(S\) of length \(n\). A code \(C\) is self orthogonal if \(C \subseteq C^\perp\) and self dual if \(C = C^\perp\).

A cyclic code \(C\) over \(S\) is a linear code with the property that if \(c = (c_0, c_1, \ldots, c_{n-1}) \in C\) then \(\sigma (C) = (c_{n-1}, c_0, \ldots, c_{n-2}) \in C\). A subset \(C\) of \(S^n\) is a linear cyclic code of length \(n\) iff it is polynomial representation is an ideal of \(S[x]/(x^n - 1)\).

Let \(C\) be code over \(F_2\) of length \(n\) and \(\hat{c} = (\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_{n-1})\) be a codeword of \(C\). The Hamming weight of \(\hat{c}\) is defined as \(w_H(\hat{c}) = \sum_{i=0}^{n-1} w_H(\hat{c}_i)\) where \(w_H(\hat{c}_i) = 1\) if \(\hat{c}_i = 1\) and \(w_H(\hat{c}_i) = 0\) if \(\hat{c}_i = 0\). Hamming distance of \(C\) is defined as \(d_H(C) = \min d_H(c, \hat{c})\), where for any \(c \in C\), \(c \neq \hat{c}\) and \(d_H(c, \hat{c})\) is Hamming distance between two codewords with \(d_H(c, \hat{c}) = w_H(c - \hat{c})\).

Let \(a \in F_2^{3n}\) with \(a = (a_0, a_1, \ldots, a_{3n-1}) = (a^{(0)}, a^{(1)}, a^{(2)})\), \(a^{(i)} \in F_2^n\) for \(i = 0, 1, 2\). Let \(\varphi\) be a map from \(F_2^{3n}\) to \(F_2^{3n}\) given by \(\varphi (a) = (\sigma (a^{(0)}), \sigma (a^{(1)}), \sigma (a^{(2)}))\) where \(\sigma\) is a cyclic shift from \(F_2^n\) to \(F_2^n\) given by \(\sigma (a^{(i)}) = ((a^{(i+1)}), (a^{(i+2)}), (a^{(i+3)}), \ldots, (a^{(i+n-1)}))\) for every \(a^{(i)} = (a^{(i,0)}, \ldots, a^{(i, n-1)})) \in F_2^n\), \(0 \leq j \leq n - 1\). A code of length \(3n\) over \(F_2\) is said to be quasi cyclic code of index 3 if \(\varphi (C) = C\).

3 \ Gray Map And Gray Images Of Cyclic Codes Over \(S\)

Let \(x = a + ub + uc\) be an element of \(S\) where \(a, b, c \in F_2\). We define Gray map \(\Psi\) from \(S\) to \(F_2^3\) by

\[
\Psi : S \to F_2^3 \\
\Psi (a + ub + vc) = (a, a + b, a + c)
\]

The Lee weight of elements of \(S\) are defined \(w_L (a + ub + vc) = w_H (a, a + b, a + c)\) where \(w_H\) denotes the ordinary Hamming weight for binary codes. Hence, there is one element whose weight is 0, there are \(u, v, 1 + u + v\) elements whose weights are 1, there are \(1 + u, 1 + v, u + v\) elements whose weights are 2, there is one element whose weight is 3.
Let $C$ be a linear code over $S$ of length $n$. For any codeword $c = (e_0, \ldots, e_{n-1})$ the Lee weight of $c$ is defined as $w_L(c) = \sum_{i=0}^{n-1} w_L(e_i)$ and the Lee distance of $C$ is defined as $d_L(C) = \min_{c \neq \hat{c}} d_L(c, \hat{c})$, where for any $\hat{c}$ in $C$, $c \neq \hat{c}$ and $d_L(c, \hat{c})$ is Lee distance between two codewords with $d_L(c, \hat{c}) = w_L(c - \hat{c})$. Gray map $\Psi$ can be extended to map from $S^n$ to $F_2^n$.

**Theorem 3.1.** The Gray map $\Psi$ is a weight preserving map from $(S^n, \text{Lee weight})$ to $(F_2^n, \text{Hamming weight})$. Moreover it is an isometry from $S^n$ to $F_2^n$.

**Theorem 3.2.** If $C$ is an $[n, k, d_L]$ linear codes over $S$ then $\Psi(C)$ is a $[3n, k, d_H]$ linear codes over $F_2$ where $d_H = d_L$.

**Proof.** Let $x_1 = a_1 + ub_1 + vc_1, x_2 = a_2 + ub_2 + vc_2 \in S, \alpha \in F_2$ then
$$\Psi(x_1 + x_2) = \Psi(a_1 + a_2 + u(b_1 + b_2) + v(c_1 + c_2)) = (a_1 + a_2, a_1 + b_1 + b_2, a_1 + a_2 + c_1 + c_2) = (a_1 + b_1, a_1 + b_2, a_2 + c_1 + c_2)$$

$$= a_1 \Psi(x_1) + a_2 \Psi(x_2)$$

Therefore, we have $d_H = d_L$.

**Theorem 3.3.** If $C$ is self orthogonal, so is $\Psi(C)$.

**Proposition 3.4.** Let $\Psi$ be Gray map from $S^n$ to $F_2^n$, let $\sigma$ be cyclic shift and let $\varphi$ be a map as in the preliminaries. Then $\Psi \varphi = \varphi \Psi$.

**Proposition 3.5.** Let $\sigma$ and $\varphi$ be as in the preliminaries. A code $C$ of length $n$ over $S$ is cyclic code if and only if $\Psi(C)$ is quasi cyclic code of index 3 over $F_2$ with length $3n$.

**Proof.** Similar to proof of in [8].

We denote that $A_1 \otimes A_2 \otimes A_3 = \{(a_1, a_2, a_3) : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3\}$ and $A_1 \oplus A_2 \otimes A_3 = \{a_1 + a_2 + a_3 : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3\}$.

Let $C$ be a linear code of length $n$ over $S$. Define

$$C_1 = \{a \in F_2^n : \exists b, c \in F_2^n, a + ub + vc \in C\}$$

$$C_2 = \{a + b \in F_2^n : \exists c \in F_2^n, a + ub + vc \in C\}$$

$$C_3 = \{a + c \in F_2^n : \exists b \in F_2^n, a + ub + vc \in C\}$$

Then $C_1, C_2$ and $C_3$ are binary linear codes of length $n$. Moreover, the linear code $C$ of length $n$ over $S$ can be uniquely expressed as $C = (1 + u + v)C_1 \oplus (u)C_2 \oplus (v)C_3$.

**Theorem 3.6.** Let $C$ be a linear code of length $n$ over $S$. Then $\Psi(C) = C_1 \otimes C_2 \otimes C_3$ and $|C| = |C_1||C_2||C_3|$.

**Proof.** For any $(a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_0, c_1, \ldots, c_{n-1}) \in \Psi(C)$. Let $r_i = a_i + u(a_i + b_i) + v(a_i + c_i), i = 0, 1, \ldots, n - 1$. Since $\Psi$ is a bijection $r = (r_0, r_1, \ldots, r_{n-1}) \in C$. By definitions of $C_1, C_2$ and $C_3$ we have $(a_0, a_1, a_{n-1}) \in C_1, (b_0, b_1, \ldots, b_{n-1}) \in C_2, (c_0, c_1, \ldots, c_{n-1}) \in C_3$. So, $(a_0, a_1, a_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_0, c_1, \ldots, c_{n-1}) \in C_1 \otimes C_2 \otimes C_3$. That is $\Psi(C) \subseteq C_1 \otimes C_2 \otimes C_3$.

On the other hand, for any $(a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_0, c_1, \ldots, c_{n-1}) \in C_1 \otimes C_2 \otimes C_3$ where $(a_0, a_1, \ldots, a_{n-1}) \in C_1, (b_0, b_1, \ldots, b_{n-1}) \in C_2, (c_0, c_1, \ldots, c_{n-1}) \in C_3$. There are $x = (a_0, a_1, \ldots, a_{n-1}), y = (b_0, b_1, \ldots, b_{n-1}), z = (c_0, c_1, \ldots, c_{n-1}) \in C$ such that $x_i = a_i + (a_i + v)q_i, y_i = b_i + (1 + u)q_i, z_i = c_i + (1 + v)q_i$ where $p_i, q_i \in F_2$ and $0 \leq i \leq n - 1$. Since $C$ is linear we have $r = (1 + u + v)x + (u)y + (v)z = a + u(a + b) + v(a + c)$ in $C$. It follows then $\Psi(r) = (a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_0, c_1, \ldots, c_{n-1})$, which gives $C_1 \otimes C_2 \otimes C_3 \subseteq \Psi(C)$. Therefore, $\Psi(C) = C_1 \otimes C_2 \otimes C_3$. The second result is easy to verify.
Corollary 3.7. If $\Psi(C) = C_1 \otimes C_2 \otimes C_3$, then $C = (1 + u + v)C_1 \oplus (u)C_2 \oplus (v)C_3$.

It is easy to see that,

$$|C| = |C_1| |C_2| |C_3| = 2^{n - \deg(f_1)} 2^{n - \deg(f_2)} 2^{n - \deg(f_3)}$$

where $f_1$, $f_2$ and $f_3$ are the generator polynomials of $C_1$, $C_2$ and $C_3$, respectively.

Corollary 3.8. If $G_1, G_2, G_3$ and $G_4$ are generator matrices of binary linear codes $C_1, C_2$ and $C_3$ respectively, then the generator matrix of $C$ is

$$G = \begin{bmatrix} (1 + u + v)G_1 \\ (u)G_2 \\ (v)G_3 \end{bmatrix}$$

We have

$$\Psi(G) = \begin{bmatrix} \Psi((1 + u + v)G_1) \\ \Psi((u)G_2) \\ \Psi((v)G_3) \end{bmatrix} = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix}.$$ 

Let $d_L$ minimum Lee weight of linear code $C$ over $S$. Then,

$$d_L = d_L(\Psi(C)) = \min\{d_L(C_1), d_L(C_2), d_L(C_3)\}$$

where $d_H(C_i)$ denotes the minimum Hamming weights of binary codes $C_1, C_2$ and $C_3$, respectively.

4 Quantum Codes From Cyclic Codes Over $S$

Theorem 4.1. (CSS Construction) Let $C$ and $\hat{C}$ be two binary codes with parameters $[n, k_1, d_1]$ and $[n, k_2, d_2]$, respectively. If $C^\perp \subseteq \hat{C}$, then an $[[n, k_1 + k_2 - n, \min\{d_1, d_2\}]]$ quantum code can be constructed. Especially, if $C^\perp \subseteq C$, then there exists an $[[n, 2k_1 - n, d_1]]$ quantum code.

Proposition 4.2. Let $C = (1 + u + v)C_1 \oplus (u)C_2 \oplus (v)C_3$ be a linear code over $S$. Then $C$ is a cyclic code over $S$ iff $C_1, C_2$ and $C_3$ are binary cyclic codes.

Proof. Let $(a_0, a_1, \ldots, a_{n-1}) \in C_1$, $(b_0, b_1, \ldots, b_{n-1}) \in C_2$ and $(c_0, c_1, \ldots, c_{n-1}) \in C_3$. Assume that $m_i = (1 + u + v)v_i + (u) b_i + (v) c_i$ for $i = 0, 1, \ldots, n-1$. Then $(m_0, m_1, \ldots, m_{n-1}) \in C$. Since $C$ is a cyclic code, it follows that $(m_{n-1}, m_0, \ldots, m_{n-2}) \in C$. Note that $(m_{n-1}, m_0, \ldots, m_{n-2}) = (1 + u + v)(a_{n-1}, a_0, \ldots, a_{n-2}) + (u)(b_{n-1}, b_0, \ldots, b_{n-2}) + (v)(c_{n-1}, c_0, \ldots, c_{n-2})$. Hence $(a_{n-1}, a_0, \ldots, a_{n-2}) \in C_1, (b_{n-1}, b_0, \ldots, b_{n-2}) \in C_2$ and $(c_{n-1}, c_0, \ldots, c_{n-2}) \in C_3$. Therefore, $C_1, C_2$ and $C_3$ are cyclic codes over $F_2$.

Conversely, suppose that $C_1, C_2$ and $C_3$ cyclic codes over $F_2$. Let $(m_0, m_1, \ldots, m_{n-1}) \in C$ where $m_i = (1 + u + v)v_i + (u) b_i + (v) c_i$ for $i = 0, 1, \ldots, n-1$. Then $(m_0, m_1, \ldots, m_{n-1}) \in C_1, (b_0, b_1, \ldots, b_{n-1}) \in C_2$ and $(c_0, c_1, \ldots, c_{n-1}) \in C_3$. Note that $(m_{n-1}, m_0, \ldots, m_{n-2}) = (1 + u + v)(a_{n-1}, a_0, \ldots, a_{n-2}) + (u)(b_{n-1}, b_0, \ldots, b_{n-2}) + (v)(c_{n-1}, c_0, \ldots, c_{n-2}) \in C$. Therefore, $C$ is cyclic code over $S$.

Proposition 4.3. Suppose $C = (1 + u + v)C_1 \oplus (u)C_2 \oplus (v)C_3$ is a cyclic code of length $n$ over $S$. Then

$$C = \langle (1 + u + v)f_1, (u)f_2, (v)f_3 \rangle$$

and $|C| = 2^{3n - (\deg f_1 + \deg f_2 + \deg f_3)}$ where $f_1, f_2$ and $f_3$ generator polynomials of $C_1, C_2$ and $C_3$ respectively.

Proposition 4.4. Suppose $C$ is a cyclic code of length $n$ over $S$, then there is a unique polynomial $f(x)$ such that $C = \langle f(x) \rangle$ and $f(x) \mid x^n - 1$ where $f(x) = (1 + u + v)f_1(x) + (u)f_2(x) + (v)f_3(x)$.

Proposition 4.5. If $C = (1 + u + v)C_1 \oplus (u)C_2 \oplus (v)C_3$ is a cyclic code of length $n$ over $S$. Then

$$C^\perp = \langle (1 + u + v)h_1^* + (u)h_2^* + (v)h_3^* \rangle$$

and $|C^\perp| = 2^{\deg f_1 + \deg f_2 + \deg f_3}$ where for $i = 1, 2, 3$, $h_i^*$ are the reciprocal polynomials of $h_i$ i.e.,

$h_i(x) = (x^n - 1)/f_i(x), h_i^*(x) = x^{\deg h_i} h_i(x^{-1})$ for $i = 1, 2, 3$. 
Lemma 4.6. A binary linear cyclic code $C$ with generator polynomial $f(x)$ contains its dual code iff

$$x^n - 1 \equiv 0 \quad (mod\ f^*)$$

where $f^*$ is the reciprocal polynomial of $f$.

Theorem 4.7. Let $C = \langle (1+u+v)f_1, (u)f_2, (v)f_3 \rangle$ be a cyclic code of length $n$ over $S$. Then $C^\perp \subseteq C$ iff $x^n - 1 \equiv 0 \quad (mod\ f_i f_i^*)$ for $i = 1, 2, 3$.

Proof. Let $x^n - 1 \equiv 0 \quad (mod\ f_i f_i^*)$ for $i = 1, 2, 3$. Then $C_1^\perp \subseteq C_1, C_2^\perp \subseteq C_2, C_3^\perp \subseteq C_3$. By using

$(1+u+v)C_1^\perp \subseteq (1+u+v)C_1, (u)C_2^\perp \subseteq (u)C_2, (v)C_3^\perp \subseteq (v)C_3$. We have $(1+u+v)C_1^\perp \oplus (u)C_2^\perp \oplus (v)C_3^\perp \subseteq (1+u+v)C_1 \oplus (u)C_2 \oplus (v)C_3$. So, $(1+u+v)h_1 + (u)h_2 + (v)h_3 \leq (1+u+v)f_1 (u)f_2 (v)f_3$. That is $C^\perp \subseteq C$.

Conversely, if $C^\perp \subseteq C$, then $(1+u+v)C_1^\perp \oplus (u)C_2^\perp \oplus (v)C_3^\perp \subseteq (1+u+v)C_1 \oplus (u)C_2 \oplus (v)C_3$. By thinking $mod\ (1+u+v), mod\ (u)$ and $mod\ (v)$ respectively we have $C_i^\perp \subseteq C_i$ for $i = 1, 2, 3$. Therefore, $x^n - 1 \equiv 0 \quad (mod\ f_i f_i^*)$ for $i = 1, 2, 3$.

Corollary 4.8. $C = \langle (1+u+v)C_1 \oplus (u)C_2 \oplus (v)C_3 \rangle$ is a cyclic code of length $n$ over $S$. Then $C^\perp \subseteq C$ iff $C_i^\perp \subseteq C_i$ for $i = 1, 2, 3$.

Example 4.9. Let $n = 7, S = F_2 + uF_2 + vF_2$ and $x^7 - 1 = (x+1)(x^3 + x^2 + 1)(x^3 + x^2 + 1) = f_1 f_2 f_3$ in $F_2[x]$. Hence,

$$f_1^* = x+1 = f_1$$
$$f_2^* = x^3 + x^2 + 1 = f_3$$
$$f_3^* = x^3 + x + 1 = f_2$$

Let $C = \langle (1+u+v)f_1, (u)f_2, (v)f_3 \rangle$. Obviously $x^n - 1$ is divisibly by $f_i f_i^*$ for $i = 2, 3$. Thus we have $C^\perp \subseteq C$.

Using Theorem 4.1 and Theorem 4.7 we can construct quantum codes.

Theorem 4.10. Let $C = \langle (1+u+v)C_1 \oplus (u)C_2 \oplus (v)C_3 \rangle$ be a cyclic code of arbitrary length $n$ over $S$ with type $8^n 4^k 2^k$. If $C_i^\perp \subseteq C_i$ where $i = 1, 2, 3$ then $C^\perp \subseteq C$ and there exists a quantum error-correcting code with parameters $[[3n, 3k_1 + 2k_2 + k_3 - 3n, d_L]]$ where $d_L$ is the minimum Lee weights of $C$.

5 Examples

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<th>$C_1$</th>
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<th>$C_3$</th>
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