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# A subclass of Meromorphically Multivalent Functions Associated with an Operator

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**Abstract.** In this paper, we introduce a new subclass of meromorphic functions and investigate inclusion relationships and properties of a certain class of meromorphically p-valent functions, which are defined by means of the familiar operator involving the generalized hypergeometric function in the paper.

## 1 Introduction

Let $\Sigma_p$ denote the class of meromorphically multivalent functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \ \ (n \in N = 1, 2, 3, ...), \tag{1.1}$$

which are analytic in the punctured open unit disk $\mathbb{U}^* = \{z : 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$.

For functions $f \in \Sigma_p$ given by (1.1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p},$$

we define the Hadamard product (or convolution) of $f$ and $g$ by

$$(f * g)(z) = z^{-p} + \sum_{n=1}^{\infty} a_n b_n z^{n-p}. \tag{1.2}$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We say that the function $g(z)$ is subordinate to $f(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $g(z) = f(w(z))$. In such a case, we write $g(z) \prec f(z)$. If the function $f$ is univalent in $\mathbb{U}$, then $g(z) \prec f(z)$ if and only if $g(0) = f(0)$ and $g(\mathbb{U}) \subset f(\mathbb{U})$.

For complex parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ ($\beta_j \neq \mathbb{Z}_0^- := \{0, -1, -2, ...\}; j = 1, ..., s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ as follows

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n, ..., (\alpha_q)_n}{(\beta_1)_n, ..., (\beta_s)_n} \frac{z^n}{n!} \tag{1.3}$$

$$(q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where $(\lambda)_n$ is the Poehhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, (n=0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)...(\lambda+n-1), (n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases} \tag{1.4}$$

Corresponding to a function $h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$, given by

$$h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-p} \cdot_q F_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z),$$

we consider a linear operator

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) : \Sigma_p \mapsto \Sigma_p,$$

which is defined by the following Hadamard product (convolution):

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) f(z) = h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z) \tag{1.5}$$
$$(q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}^*),$$

so that, for a function $f$ of form (1.1), we have

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) f(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n, ..., (\alpha_q)_n}{(\beta_1)_n, ..., (\beta_s)_n} a_n \frac{z^{n-p}}{n!}. \tag{1.6}$$

In order to make the notation simple, we write

$$H_p^{q,s}(\alpha_1) f(z) := H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z),$$

then one can easily verify from definition (1. 5) that

$$z(H_p^{q,s}(\alpha_1) f(z))' = \alpha_1 H_p^{q,s}(\alpha_1+1) f(z) - (\alpha_1+p) H_p^{q,s}(\alpha_1) f(z). \tag{1.7}$$

In recent years, Raina and Srivastava [1], and Aouf [2] obtained many interesting results involving the linear operator $H_p^{q,s}(\alpha_1)$. Especially $q=2$, $s=1$, $\alpha_1 = a$, $\beta_1 = c$, and $\alpha_2 = 1$, we obtain the following linear operator:

$$\mathscr{L}_p(a,c) f(z) = H_p(\alpha_1, 1; \beta_1) f(z),$$

which was introduced and studied by Liu and Srivastava [3], and was further studied in a subsequent investigation by Srivastava [4]. Some interesting subclasses of analytic functions associated with the generalized hypergeometric function, were considered recently by Dziok and Srivastava [5,6].

Throughout this paper, we assume that $p, k \in \mathbb{N}, q, s \in \mathbb{N}_0, \varepsilon_k = exp(\frac{2\pi i}{k})$,

$$f_{p,k}^{q,s}(\alpha_1; z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{jp} (H_p^{q,s}(\alpha_1) f)(\varepsilon_k^j z) = z^{-p} + \cdots (f \in \Sigma_p). \tag{1.8}$$

Let $\mathscr{P}$ denote the class of functions of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

which are analytic and convex in $\mathbb{U}$ and satisfy the following condition : $\Re(p(z)) > 0, (z \in \mathbb{U})$.

**Definition 1.** A function $f(z) \in \Sigma_p$ is said to be in the class $\mathscr{M}(\lambda, \alpha, \alpha_1; h)$, if it satisfies the subordination condition

$$-\lambda \frac{z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(z) + \alpha(H_p^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z) + \alpha f_{p,k}^{q,s}(\alpha_1+1;z)]} - (1-\lambda)\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)} \prec h(z). \tag{1.9}$$

For simplicity, we write $\mathscr{M}(0, \alpha, \alpha_1; h) = \mathscr{M}(\alpha, \alpha_1; h)$.

## 2 Some Lemmas

In order to prove our main results, we need the following lemmas.

**Lemma 1** (see [7]). Let $\beta, \gamma \in \mathbb{C}$. Suppose also that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with

$$\phi(0) = 1, \Re(\beta\phi(z) + \gamma) > 0, (z \in \mathbb{U}).$$

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then the following subordination:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \phi(z), (z \in \mathbb{U}),$$

implies that $p(z) \prec \phi(z), (z \in \mathbb{U})$.

**Lemma 2** (see [8]). Let $\beta, \gamma \in \mathbb{C}$. Suppose that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with

$$\phi(0) = 1, \Re(\beta\phi(z) + \gamma) > 0; (z \in \mathbb{U}).$$

Also let

$$q(z) \prec \phi(z); (z \in \mathbb{U}).$$

If $p(z) \in \mathscr{P}$ and satisfies the following subordination:

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \phi(z),$$

then $p(z) \prec \phi(z)$.

**Lemma 3** Let $f \in \mathscr{M}(\lambda, \alpha, \alpha_1; \phi(z))$. Then

$$-\lambda \frac{z[(1+\alpha)(f_{p,k}^{q,s}(\alpha_1)f)'(z) + \alpha(f_{p,k}^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z) + \alpha f_{p,k}^{q,s}(\alpha_1+1;z)]} - (1-\lambda)\frac{z(f_{p,k}^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)} \prec \phi(z). \tag{2.1}$$

Furthermore, if $\phi(z) \in \mathscr{P}$ with

$$\Re(\frac{1}{\lambda}(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\phi(z)) > 0, (\alpha > 0, z \in \mathbb{U})),$$

then

$$-\frac{z(f_{p,k}^{q,s}(\alpha_1;z))'}{pf_{p,k}^{q,s}(\alpha_1;z)} \prec \phi(z), (z \in \mathbb{U}).$$

**Proof .** By (1.8), we have

$$\begin{aligned} f_{p,k}^{q,s}(\alpha_1;\varepsilon_k^j z) &= \frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{np}(H_p^{q,s}(\alpha_1)f)(\varepsilon_k^{n+j}z) \\ &= \varepsilon_k^{-jp}\frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{(n+j)p}(H_p^{q,s}(\alpha_1)f)(\varepsilon_k^{n+j}z) \\ &= \varepsilon_k^{-jp}f_{p,k}^{q,s}(\alpha_1;z), \end{aligned} \tag{2.2}$$

$$(j \in \{0, 1, \cdots, k-1\}),$$

and

$$(f_{p,k}^{q,s}(\alpha_1; z))' = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{j(p+1)}(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z). \tag{2.3}$$

Replacing $\alpha_1$ by $\alpha_1 + 1$ in (2.2) and (2.3), respectively, we can get

$$f_{p,k}^{q,s}(\alpha_1+1; \varepsilon_k^j z) = \varepsilon_k^{-jp} f_{p,k}^{q,s}(\alpha_1+1; z), (j \in \{0, 1, \cdots, k-1\}), \tag{2.4}$$

and

$$(f_{p,k}^{q,s}(\alpha_1+1; z))' = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{j(p+1)}(H_p^{q,s}(\alpha_1+1)f)'(\varepsilon_k^j z). \tag{2.5}$$

From (2.2) to (2.5), we can get

$$-\lambda\frac{z[(1+\alpha)(f_{p,k}^{q,s}(\alpha_1)f)'(z) + \alpha(f_{p,k}^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; z)]} - (1-\lambda)\frac{z(f_{p,k}^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1; z)}$$

$$= -\frac{1}{k}\sum_{j=0}^{k-1}\{\lambda\frac{\varepsilon_k^{j(p+1)}z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z) + \alpha(H_p^{q,s}(\alpha_1+1)f)'(\varepsilon_k^j z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; z)]} + (1-\lambda)\frac{\varepsilon_k^{j(p+1)}z(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z)}{pf_{p,k}^{q,s}(\alpha_1; z)}\}$$

$$= -\frac{1}{k}\sum_{j=0}^{k-1}\{\lambda\frac{\varepsilon_k^{j}z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z) + \alpha(H_p^{q,s}(\alpha_1+1)f)'(\varepsilon_k^j z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; \varepsilon_k^j z)]} + (1-\lambda)\frac{\varepsilon_k^{j}z(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z)}{pf_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z)}\}. \tag{2.6}$$

Moreover, since $f \in \mathscr{M}(\lambda, \alpha, \alpha_1; h)$, it follows that

$$-\lambda\frac{\varepsilon_k^{j}z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z) + \alpha(H_p^{q,s}(\alpha_1+1)f)'(\varepsilon_k^j z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; \varepsilon_k^j z)]} - (1-\lambda)\frac{\varepsilon_k^{j}z(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z)}{pf_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z)} \prec \phi(z). \tag{2.7}$$

By noting that $\phi(z)$ is convex and univalent in $\mathbb{U}$, we conclude from (2.6) and (2.7) that the assertion (2.1) of Lemma 3 holds true.

Next, making use of the relationships (1.7) and (1.8), we have

$$\begin{aligned} z(f_{p,k}^{q,s}(\alpha_1; z))' + (\alpha_1+p)f_{p,k}^{q,s}(\alpha_1; z) &= \frac{\alpha_1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{jp}(H_p^{q,s}(\alpha_1+1)f)(\varepsilon_k^j z) \\ &= \alpha_1 f_{p,k}^{q,s}(\alpha_1+1; z). \end{aligned} \tag{2.8}$$

Let $f \in \mathscr{M}(\lambda, \alpha, \alpha_1; \phi)$ and suppose that

$$\psi(z) = -\frac{z(f_{p,k}^{q,s}(\alpha_1; z))'}{pf_{p,k}^{q,s}(\alpha_1; z)}, (z \in \mathbb{U}). \tag{2.9}$$

Then $\psi(z)$ is analytic in $\mathbb{U}$ and $\psi(0) = 1$. It follows from (2.8) and (2.9) that

$$\alpha_1 + p - p\psi(z) = \alpha_1\frac{f_{p,k}^{q,s}(\alpha_1+1; z)}{f_{p,k}^{q,s}(\alpha_1; z)}. \tag{2.10}$$

By (2.9) and (19), we get

$$z(f_{p,k}^{q,s}(\alpha_1+1;z))' = -\frac{p}{\alpha_1}(z\psi'(z)+[\alpha_1+p-p\psi(z)]\psi(z))f_{p,k}^{q,s}(\alpha_1;z), (z\in\mathbb{U}^*). \tag{2.11}$$

It now follows from (2.1) and (2.9)-(2.11) that

$$\begin{aligned}
& -\lambda\frac{z[(1+\alpha)(f_{p,k}^{q,s}(\alpha_1)f)'(z)+\alpha(f_{p,k}^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z)+\alpha f_{p,k}^{q,s}(\alpha_1+1;z)]}-(1-\lambda)\frac{z(f_{p,k}^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)} \\
= {} & \lambda\frac{p(1+\alpha)\psi(z)f_{p,k}^{q,s}(\alpha_1;z)+\frac{\alpha}{\alpha_1}p(z\psi'(z)+[\alpha_1+p-p\psi(z)]\psi(z))f_{p,k}^{q,s}(\alpha_1;z)}{p(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z)+\frac{\alpha}{\alpha_1}p[\alpha_1+p-p\psi(z)]f_{p,k}^{q,s}(\alpha_1;z)}+(1-\lambda)\psi(z) \\
= {} & \lambda\frac{(1+\alpha)\psi(z)+\frac{\alpha}{\alpha_1}(z\psi'(z)+[\alpha_1+p-p\psi(z)]\psi(z))}{(1+\alpha)+\frac{\alpha}{\alpha_1}[\alpha_1+p-p\psi(z)]}+(1-\lambda)\psi(z) \\
= {} & \psi(z)+\frac{z\psi'(z)}{\frac{1}{\lambda}(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\psi(z))}\prec\phi(z), (z\in\mathbb{U}).
\end{aligned} \tag{2.12}$$

Since

$$\Re(\frac{1}{\lambda}(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z))>0, (\alpha>0, z\in\mathbb{U})),$$

by means of (2.12) and Lemma 1, we find that

$$\psi(z) = -\frac{z(f_{p,k}^{q,s}(\alpha_1;z))'}{pf_{p,k}^{q,s}(\alpha_1;z)}\prec\phi(z), (z\in\mathbb{U}).$$

This completes the proof of Lemma 3 .

## 3 Main results

**Theorem 1.** Let $\phi(z)\in\mathscr{P}$ with

$$\Re(\frac{1}{\lambda}(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z)))>0; (\alpha>0; z\in\mathbb{U}).$$

Then $\mathscr{M}(\lambda,\alpha,\alpha_1;\phi(z))\subset\mathscr{M}(\alpha,\alpha_1;\phi(z))$.

**Proof .** Let $f\in\mathscr{M}(\lambda,\alpha,\alpha_1;\phi(z))$ and suppose that

$$q(z) = -\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)}; (z\in\mathbb{U}). \tag{3.1}$$

Then $q(z)$ is analytic in $\mathbb{U}$ and $q(0)=1$, It follows from (1.7) and (3.1) that

$$q(z)f_{p,k}^{q,s}(\alpha_1;z) = -\frac{\alpha_1}{p}H_p^{q,s}(\alpha_1+1)f(z)+\frac{\alpha_1+p}{p}H_p^{q,s}(\alpha_1)f(z). \tag{3.2}$$

Differentiating both sides of (3.2) with respect to z and using (3.1), we have

$$zq'(z)+(\alpha_1+p+\frac{z(f_{p,k}^{q,s}(\alpha_1;z))'}{f_{p,k}^{q,s}(\alpha_1;z)})q(z) = -\frac{\alpha_1}{p}\frac{z(H_p^{q,s}(\alpha_1+1)f)'(z)}{f_{p,k}^{q,s}(\alpha_1;z)}. \tag{3.3}$$

It now follows from (1.9),(2.9),(2.10),(3.1)and (3.3) that

$$-\lambda\frac{z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(z)+\alpha(H_p^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z)+\alpha f_{p,k}^{q,s}(\alpha_1+1;z)]}-(1-\lambda)\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)}$$

$$\begin{aligned}
&= \lambda\frac{p(1+\alpha)q(z)f_{p,k}^{q,s}(\alpha_1;z)+\frac{\alpha}{\alpha_1}p[zq'(z)+[\alpha_1+p-p\psi(z)]q(z)]f_{p,k}^{q,s}(\alpha_1;z)}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z)]+\frac{\alpha}{\alpha_1}[\alpha_1+p-p\psi(z)]f_{p,k}^{q,s}(\alpha_1;z)}+(1-\lambda)q(z)\\
&= \lambda\frac{(1+\alpha)q(z)+\frac{\alpha}{\alpha_1}[zq'(z)+[\alpha_1+p-p\psi(z)]q(z)]}{(1+\alpha)+\frac{\alpha}{\alpha_1}[\alpha_1+p-p\psi(z)]}+(1-\lambda)q(z)\\
&= q(z)+\frac{zq'(z)}{\frac{1}{\lambda}(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\psi(z))}\prec\phi(z). \qquad (3.4)
\end{aligned}$$

Moreover, since

$$\Re(\frac{1}{\lambda}(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z)))>0;(\alpha>0;z\in\mathbb{U}).$$

by Lemma 3, we have

$$\psi(z)=-\frac{z(f_{p,k}^{q,s}(\alpha_1;z))'}{pf_{p,k}^{q,s}(\alpha_1;z)}\prec\phi(z),(z\in\mathbb{U}).$$

Thus, by (3.4) and Lemma 2, we find that

$$q(z)\prec\phi(z),(z\in\mathbb{U}),$$

that is, that $f\in\mathscr{M}(\alpha,\alpha_1;\phi(z))$. This implies that

$$\mathscr{M}(\lambda,\alpha,\alpha_1;\phi(z))\subset\mathscr{M}(\alpha,\alpha_1;\phi(z)).$$

The proof of Theorem 1 is completed.

**Theorem 2.** Let $h\in\mathscr{P}$, and $0\le\lambda_1<\lambda_2$,

$$\Re(\frac{1}{\lambda_2}(\frac{\alpha_1}{\alpha}+2\alpha_1+p-ph(z))>0,(\alpha>0,z\in\mathbb{U})),$$

then

$$\mathscr{M}(\lambda_2,\alpha,\alpha_1;h)\subset\mathscr{M}(\lambda_1,\alpha,\alpha_1;h).$$

**Proof .** For $f\in\mathscr{M}(\lambda_2,\alpha,\alpha_1;h)$, we have

$$-\lambda_2\frac{z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(z)+\alpha(H_p^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z)+\alpha f_{p,k}^{q,s}(\alpha_1+1;z)]}-(1-\lambda_2)\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)}\prec h(z). \qquad (3.5)$$

Put

$$q(z)=-\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)},(z\in\mathbb{U}).$$

By Theorem 1, we get

$$\mathscr{M}(\lambda_2,\alpha,\alpha_1;h(z))\subset\mathscr{M}(\alpha,\alpha_1;h(z)).$$

Hence,

$$q(z)\prec h(z),(z\in\mathbb{U}). \qquad (3.6)$$

Since $0\le\frac{\lambda_1}{\lambda_2}<1$, and since $h(z)$ is convex univalent in $\mathbb{U}$, we deduce from (3.5) and (3.6) that

$$-\lambda_1\frac{z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(z)+\alpha(H_p^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z)+\alpha f_{p,k}^{q,s}(\alpha_1+1;z)]}-(1-\lambda_1)\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)}$$

$$
\begin{aligned}
&= \frac{\lambda_1}{\lambda_2}(-\lambda_2\frac{z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(z)+\alpha(H_p^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z)+\alpha f_{p,k}^{q,s}(\alpha_1+1;z)]}-(1-\lambda_2)\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)}) \\
&+ (1-\frac{\lambda_1}{\lambda_2})q(z)\prec h(z),(z\in\mathbb{U}).
\end{aligned} \tag{3.7}
$$

Thus $f\in\mathscr{M}(\lambda_1,\alpha,\alpha_1;h)$ and the proof of Theorem 2 is completed.

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