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# A MULTIPLICATIVE (GENERALIZED)-$(\sigma, \sigma)$-DERIVATION ACTING AS (ANTI-)HOMOMORPHISM IN SEMIPRIME RINGS

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**Abstract**. Let $R$ be an associative ring and $\sigma, \tau$ two epimorphisms of $R$. A mapping $F : R \to R$ (not necessarily additive) is said to be a multiplicative (generalized)-$(\sigma, \tau)$-derivation if $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ holds for all $x, y \in R$, where $g$ is any mapping on $R$. The main object in this article is to study the situations (1) $F(xy) - F(x)F(y) = 0$, (2) $F(xy) + F(x)F(y) = 0$, (3) $F(xy) - F(y)F(x) = 0$ and (4) $F(xy) + F(y)F(x) = 0$; for all $x, y$ in some nonzero subsets of prime and semiprime rings, where $F$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation of $R$.

## 1 Introduction

Let $R$ be an associative ring with center $Z(R)$. For given $x, y \in R$, the symbol $[x, y]$ stands for commutator $xy - yx$. Recall that a ring $R$ is prime if for any $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. An additive mapping $d : R \to R$ is said to be a derivation of $R$ if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. The generalized derivation of $R$ is an additive mapping $F : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for any $x, y \in R$, where $d$ is a derivation of $R$. If $d = 0$, then we have $F(xy) = F(x)y$ for all $x, y \in R$, which is called a left multiplier mapping of $R$. Thus, generalized derivation generalizes both the concepts, derivations as well as left multiplier mappings of $R$. Let $S$ be a nonempty subset of a ring $R$ and $\sigma$ be an endomorphism of $R$. The mapping $F : R \to R$ is said to be a homomorphism (anti-homomorphism) acting on $S$ if $F(xy) = F(x)F(y)$ holds for all $x, y \in S$ (respectively $F(xy) = F(y)F(x)$ holds for all $x, y \in S$). An additive mapping $f : R \to R$ is said to be $\sigma$-commuting ($\sigma$-centralizing) on $S$, if $[f(x), \sigma(x)] = 0$ for all $x \in S$ (resp. $[f(x), \sigma(x)] \in Z(R)$ for all $x \in S$). For convenience, we shall write 1-commuting and 1-centralizing maps as commuting and centralizing maps respectively, where 1 means the identity map of $R$.

A multiplicative derivation of $R$ is a mapping $D : R \to R$ which satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. Of course these mappings are not additive. The concept of multiplicative derivations was introduced by Daif [6]. Further, the complete description of those maps was given by Goldmann and Semrl in [9].

The notion of multiplicative derivation was extended to multiplicative generalized derivation in [4]. A mapping $F : R \to R$ is called a multiplicative generalized derivation, if there exists a derivation $d$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In this definition, if we consider $d$ is any map on $R$ (not necessarily additive) then it is called as a multiplicative (generalized)-derivation. A mapping $F : R \to R$ (not necessarily additive) is said to be multiplicative (generalized)-derivation if $F(xy) = F(x)y + xg(y)$ holds for all $x, y \in R$, where $g$ is any mapping on $R$ (not necessarily a derivation nor an additive map). Hence, the concept of multiplicative (generalized)-derivation covers the concept of multiplicative derivation. Moreover, multiplicative (generalized)-derivation with $g = 0$ covers the concept of multiplicative centralizers (not necessarily additive).

Obviously, every generalized derivation is a multiplicative (generalized)-derivation on $R$. However, the converse is not true in general.

In the present paper, we generalize this concept of multiplicative (generalized)-derivations as follows: Let $\sigma, \tau$ be any two endomorphisms of $R$. A mapping $F : R \to R$ (not necessarily additive) is said to be multiplicative (generalized)-$(\sigma, \tau)$-derivation if $F(xy) = F(x)\sigma(y) + \tau(x)g(y)$ holds for all $x, y \in R$, where $g$ is any mapping on $R$ (not necessarily a $(\sigma, \tau)$-derivation nor an additive map). In particular, if $F$ is additive and $g$ is $(\sigma, \tau)$-derivation of $R$, then $F$ is said to be a generalized $(\sigma, \tau)$-derivation of $R$.

A series of papers have investigated some specific types of additive maps acting as homomorphism or anti-homomorphism on some subsets of $R$ and then determined the structure of ring as well as associated maps (see [1], [2], [3], [7], [8], [10], [12], [13]). In [12], Rehman proved the following result :

*Let $R$ be a $2$-torsion free prime ring and $I$ be a nonzero ideal of $R$. Suppose $F : R \to R$ is a nonzero generalized derivation associated with a derivation $d$.*
*(i) If $F$ acts as a homomorphism on $I$ and if $d \neq 0$, then $R$ is commutative.*
*(ii) If $F$ acts as an anti-homomorphism on $I$ and if $d \neq 0$, then $R$ is commutative.*

Then Gusic in [10] showed that the result of Rehman is not in complete form. Gusic [10] proved the result in more complete form as follows :

*Let $R$ be an associative prime ring, let $d$ be any function on $R$ (not necessary a derivation nor an additive function), let $F$ be any function on $R$ (not necessarily additive) satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, and let $I$ be a non-zero ideal in $R$.*
*(a) Assume that $F(xy) = F(x)F(y)$ for all $x, y \in I$. Then $d = 0$, and $F = 0$ or $F(x) = x$ for all $x \in R$.*
*(b) Assume that $F(xy) = F(y)F(x)$ for all $x, y \in I$. Then $d = 0$, and $F = 0$ or $F(x) = x$ for all $x \in R$ (in this case $R$ should be commutative).*

In [1], Ali and Huang studied the case when a generalized Jordan $(\alpha, \beta)$-derivation $F$ acts as homomorphism or anti-homomorphism on a square closed Lie ideal $U$ in prime ring $R$. More precisely, they obtained the following result :

*Let $R$ be a $2$-torsion free prime ring and $U$ a square-closed Lie ideal of $R$. Suppose that $\alpha$ and $\beta$ are two automorphisms of $R$ and $(F, d)$ is a generalized Jordan $(\alpha, \beta)$-derivation of $R$ such that $F \neq \beta$ on $U$. If $(F, d)$ acts as a homomorphism or anti-homomorphism on $U$, then either $d = 0$ or $U \subseteq Z(R)$.*

The above result of Gusic [10] for prime rings motivates us to study the situations when $F(xy) = \pm F(x)F(y)$ holds for all $x, y \in I$ and $F(xy) = \pm F(y)F(x)$ holds for all $x, y \in I$, where $F$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation, $I$ is a nonzero left sided ideal in semiprime ring $R$ and $\sigma$ is an epimorphism of $R$.

## 2 Preliminaries

Let $R$ be a ring. We need the following basic identities which will be used in the proof of our results. For any $x, y, z \in R$,

$$[xy, z] = x[y, z] + [x, z]y \quad \text{and} \quad [x, yz] = y[x, z] + [x, y]z.$$

Moreover,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

For any subset $S$ of $R$, we will denote by $r_R(S)$ the right annihilator of $S$ in $R$, that is, $r_R(S) = \{x \in R | Sx = 0\}$ and by $l_R(S)$ the left annihilator of $S$ in $R$, that is, $l_R(S) = \{x \in R | xS = 0\}$. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of $R$ and is written as $ann_R(S)$. We know that if $R$ is a semiprime ring and $I$ is an ideal of $R$, then $r_R(I) = l_R(I)$. A right ideal $\rho$ of $R$ is said to be *large*, if $\rho$ has zero left annihilator in $R$, that is $l_R(\rho) = 0$. Note that any nonzero ideal in prime rings is large.

The next statements are well-known results which will be used in the next section.

**Lemma 2.1.** *[5, Lemma 2] $(a)$ If $R$ is a semiprime ring, then the center of a nonzero one-sided ideal is contained in the center of $R$. In particular, any commutative one-sided ideal is contained in the center of $R$.*
$(b)$ *If $R$ is prime with a nonzero central ideal, then $R$ is commutative.*

**Lemma 2.2.** *[11, Corollary 2] Let $R$ be a semiprime ring and $\sigma, \tau$ two epimorphisms of $R$. If $d : R \to R$ is a nonzero $(\sigma, \tau)$-derivation of $R$ such that $[d(x), \sigma(x)] = 0$ for all $x \in R$, then $d(R)$ is contained in a central ideal of $R$.*

**Lemma 2.3.** *[11, Corollary 4] Let $R$ be a prime ring and $\rho$ a nonzero right ideal of $R$ and $\sigma, \tau$ two epimorphisms of $R$. Let $d : R \to R$ be a nonzero $(\sigma,\tau)$-derivation of $R$ such that $\sigma(\rho)$ and $\tau(\rho)$ contain some large right ideals of $R$. If $[d(x), \sigma(x)] = 0$ for all $x \in \rho$, then $R$ is commutative.*

**Lemma 2.4.** *Let $R$ be a prime ring and $I$ an ideal of $R$. Suppose that $\sigma$ is an epimorphism of $R$ such that $\sigma(I) \neq (0)$. If $d$ is a nonzero $(\sigma,\sigma)$-derivation of $R$ such that $[d(x), \sigma(x)] = 0$ for all $x \in I$, then $R$ is commutative.*

**Proof.** Since $\sigma$ is an epimorphism of $R$ such that $\sigma(I) \neq (0)$, we have $(0) \neq R\sigma(I)R = \sigma(RIR) \subseteq \sigma(I)$. Thus $\sigma(I)$ contains some nonzero ideals of $R$. Since each nonzero ideal in prime rings are large, we can apply Lemma 2.2 and conclude that $R$ must be commutative. □

## 3 Main Results

**Theorem 3.1.** *Let $R$ be a semiprime ring, $I$ a nonzero left ideal of $R$ and $\sigma$ any epimorphism of $R$. Suppose that $F : R \to R$ be a multiplicative (generalized)-$(\sigma,\sigma)$-derivation associated with the map $d$. If $F(xy) - F(x)F(y) = 0$ holds for all $x, y \in I$, then $\sigma(I)d(I) = (0)$ and $\sigma(I)[F(x), \sigma(x)] = (0)$ for all $x \in I$.*

**Proof.** We have

$$F(xy) = F(x)F(y) \tag{3.1}$$

for all $x, y \in I$. We put $y = yz$, in above relation and then obtain that

$$F(x(yz)) = F(x)F(yz) \tag{3.2}$$

for all $x, y, z \in I$; which gives

$$F(xy)\sigma(z) + \sigma(xy)d(z) = F(x)\{F(y)\sigma(z) + \sigma(y)d(z)\}. \tag{3.3}$$

By (3.1), above relation reduces to

$$(F(x) - \sigma(x))\sigma(y)d(z) = 0 \tag{3.4}$$

for all $x, y, z \in I$. Since $I$ is a left ideal and $\sigma$ an epimorphism of $R$, it follows that

$$F(y)(F(x) - \sigma(x))R\sigma(y)d(z) = 0 \tag{3.5}$$

for all $x, y, z \in I$.

Again, from (3.1) we can write $F(x)\sigma(y) + \sigma(x)d(y) = F(x)F(y)$, that is

$$F(x)(F(y) - \sigma(y)) = \sigma(x)d(y) \tag{3.6}$$

for all $x, y \in I$. Using (3.6), we can write from (3.5) that $\sigma(y)d(x)R\sigma(y)d(x) = 0$ for all $x, y \in I$ and $F(y)(F(x) - \sigma(x))RF(y)(F(x) - \sigma(x)) = 0$ for all $x, y \in I$. Since $R$ is semiprime, we conclude that $\sigma(y)d(x) = 0$ for all $x, y \in I$, that is $\sigma(I)d(I) = (0)$ and $F(y)(F(x) - \sigma(x)) = 0$ for all $x, y \in I$. Thus we have $F(xy) = F(x)\sigma(y) + \sigma(x)d(y) = F(x)\sigma(y)$ for all $x, y \in I$.

Now replacing $y$ with $yx$ and $x$ with $x^2$ in $F(y)(F(x) - \sigma(x)) = 0$ for all $x, y \in I$, we get

$$F(y)\sigma(x)(F(x) - \sigma(x)) = 0 \tag{3.7}$$

and

$$F(y)(F(x)\sigma(x) - \sigma(x)^2) = 0 \tag{3.8}$$

for all $x, y \in I$. Subtracting one from another, we get $F(y)[F(x), \sigma(x)] = 0$ for all $x, y \in I$. Putting $y = yz$, we get $F(y)\sigma(z)[F(x), \sigma(x)] = 0$ for all $x, y, z \in I$. Since $I$ is a left ideal of $R$, it follows that $[F(y), \sigma(y)]\sigma(z)[F(x), \sigma(x)] = 0$ for all $x, y, z \in I$ which yields $[F(y), \sigma(y)]R\sigma(z)[F(x), \sigma(x)] = (0)$ and so $\sigma(z)[F(y), \sigma(y)]R\sigma(z)[F(x), \sigma(x)] = (0)$ for all $x, y, z \in I$. In particular, $\sigma(z)[F(x), \sigma(x)]R\sigma(z)[F(x), \sigma(x)] = (0)$ for all $x, z \in I$. By the semiprimeness of $R$, $\sigma(I)[F(x), \sigma(x)] = (0)$ for all $x \in I$.□

**Theorem 3.2.** *Let $R$ be a semiprime ring, $I$ a nonzero left ideal of $R$ and $\sigma$ any epimorphism of $R$. Suppose that $F : R \to R$ be a multiplicative (generalized)-$(\sigma,\sigma)$-derivation associated with the map $d$. If $F(xy) + F(x)F(y) = 0$ holds for all $x, y \in I$, then $\sigma(I)d(I) = (0)$ and $\sigma(I)[F(x), \sigma(x)] = (0)$ for all $x \in I$.*

**Proof.** If $F$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation associated with the map $d$, then $-F$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation associated with the map $-d$. Thus replacing $F$ with $-F$ and $d$ with $-d$ in Theorem 3.1, we get our conclusions here.□

**Corollary 3.3.** *Let $R$ be a semiprime ring and $\sigma$ be any epimorphism of $R$. Suppose that $F : R \to R$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation associated with the map $d$. If $F(xy) \pm F(x)F(y) = 0$ for all $x, y \in R$, then $d(R) = 0$, $F$ is a multiplicative $\sigma$-centralizer mapping on $R$ and $\sigma$-commuting on $R$.*

*In particular, if $R$ is a prime ring, then $d = 0$ and $F(x) = \mp\sigma(x)$ for all $x \in R$.*

**Proof.** Applying Theorem 3.1 and Theorem 3.2, we may conclude that $d(R) = (0)$ and $[F(x), \sigma(x)] = 0$ for all $x \in R$. Again since $d(R) = (0)$, we have for all $x, y \in R$, $F(xy) = F(x)\sigma(y) + \sigma(x)d(y) = F(x)\sigma(y)$, that is, $F$ is a multiplicative $\sigma$-centralizer mapping on $R$.

In particular, if $R$ is a prime ring, then by our hypothesis, we have $F(x)(F(y) \pm \sigma(y)) = 0$ for all $x, y \in R$. Replacing $x$ with $xr$ for $r \in R$, we have $F(x)\sigma(r)(F(y) \pm \sigma(y)) = 0$ for all $x, y, r \in R$. Since $\sigma$ is an epimorphism of $R$, $F(x)R(F(y) \pm \sigma(y)) = 0$ for all $x, y \in R$. Since $R$ is prime ring, either $F = 0$ or $F(x) = \mp\sigma(x)$ for all $x \in R$. □

**Corollary 3.4.** *Let $R$ be a prime ring, $\sigma$ an epimorphism of $R$ and $J$ a nonzero ideal of $R$ such that $\sigma(J) \neq (0)$. Suppose that $F : R \to R$ is a nonzero generalized $(\sigma, \sigma)$-derivation of $R$. If $F(xy) \pm F(x)F(y) = 0$ for all $x, y \in J$, then $F(x) = \mp\sigma(x)$ for all $x \in R$.*

**Proof.** Let $d$ be the associated $(\sigma, \sigma)$-derivation of $F$. Then by Theorem 3.1 and Theorem 3.2, we have that $\sigma(J)d(J) = (0)$ and $\sigma(J)[F(x), \sigma(x)] = (0)$ for all $x \in J$. Since $R$ is prime ring and $\sigma(J)$ is a nonzero ideal of $R$, $d(J) = (0)$ and $[F(x), \sigma(x)] = 0$ for all $x \in J$. Now $d(J) = (0)$ yields $(0) = d(RJ) = d(R)\sigma(J) + \sigma(R)d(J) = d(R)\sigma(J)$ implying $d(R) = (0)$. Then we have for all $x, y \in R$, $F(xy) = F(x)\sigma(y) + \sigma(x)d(y) = F(x)\sigma(y)$. Then by our hypothesis, we have $F(x)(F(y) \pm \sigma(y)) = 0$ for all $x, y \in J$. Replacing $x$ with $xr$ for $r \in R$, we have $F(x)\sigma(r)(F(y \pm \sigma(y)) = 0$ for all $x, y \in J$ and $r \in R$. Since $\sigma$ is an epimorphism of $R$, it follows that $F(x)R(F(y) \pm \sigma(y)) = (0)$ for all $x, y \in J$. Since $R$ is a prime ring, either $F(J) = (0)$ or $F(x) = \mp\sigma(x)$ for all $x \in J$. If $F(J) = (0)$, then $(0) = F(RJ) = F(R)\sigma(J)$ which implies $F(R) = (0)$, a contradiction. On the other hand if $F(x) = \mp\sigma(x)$ for all $x \in J$, then replace $x$ with $rx$, where $r \in R$, and then we get $0 = F(rx) \pm \sigma(rx) = F(r)\sigma(x) \pm \sigma(r)\sigma(x) = (F(r) \pm \sigma(r))\sigma(x)$. This yields that $F(r) = \mp\sigma(r)$ for all $r \in R$.□

**Theorem 3.5.** *Let $R$ be a semiprime ring, $I$ a nonzero left ideal of $R$ and $\sigma$ any epimorphism of $R$. Suppose that $F : R \to R$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation associated with the map $d$. If $F(xy) - F(y)F(x) = 0$ holds for all $x, y \in I$, then $\sigma(I)[d(x), \sigma(x)] = (0)$ for all $x \in I$.*

**Proof.** We have

$$F(xy) = F(y)F(x) \tag{3.9}$$

for all $x, y \in I$. Now replacing $x$ with $xy$ in (3.9) and then using the fact $F(xy) = F(x)\sigma(y) + \sigma(x)d(y)$ for all $x, y \in R$, we obtain

$$F(xy)\sigma(y) + \sigma(xy)d(y) = F(y)\{F(x)\sigma(y) + \sigma(x)d(y)\}. \tag{3.10}$$

By using (3.9), it reduces to $\sigma(xy)d(y) = F(y)\sigma(x)d(y)$, that gives

$$(\sigma(xy) - F(y)\sigma(x))d(y) = 0 \tag{3.11}$$

for all $x, y \in I$. Now replacing $x$ with $rx$ in the above relation, we get

$$(\sigma(r)\sigma(xy) - F(y)\sigma(r)\sigma(x))d(y) = 0 \tag{3.12}$$

for all $x, y \in I$ and for all $r \in R$. Since $\sigma$ is an epimorphism of $R$, $\sigma(R) = R$ and hence from above we have

$$(r\sigma(xy) - F(y)r\sigma(x))d(y) = 0 \tag{3.13}$$

for all $x, y \in I$. Replacing $r$ with $F(z)$, where $z \in I$, in (3.13), we get

$$(F(z)\sigma(xy) - F(y)F(z)\sigma(x))d(y) = 0. \tag{3.14}$$

Left multiplying (3.11) by $F(z)$ and then subtracting from (3.14), we get

$$\{F(y)F(z) - F(z)F(y)\}\sigma(x)d(y) = 0. \tag{3.15}$$

By (3.9), above relation reduces to

$$\{F(zy) - F(yz)\}\sigma(x)d(y) = 0 \tag{3.16}$$

for all $x, y, z \in I$. Now we put $z = zy$ and then calculate

$$\begin{aligned} 0 &= \{F(zy)\sigma(y) + \sigma(zy)d(y) - F(yz)\sigma(y) - \sigma(yz)d(y)\}\sigma(x)d(y) \\ &= (F(zy) - F(yz))\sigma(yx)d(y) + \sigma([z, y])d(y)\sigma(x)d(y). \end{aligned}$$

By using (3.16), it yields $\sigma([z, y])d(y)\sigma(x)d(y) = 0$ for all $x, y, z \in I$. Since for $r \in R$, $r[z, y] \in I$, we put $x = r[z, y]$ and get

$$\sigma([z, y])d(y)\sigma(r)\sigma([z, y])d(y) = 0 \tag{3.17}$$

for all $y, z \in I$ and for all $r \in R$. Since $\sigma(R) = R$ and $R$ is semiprime,

$$\sigma([z, y])d(y) = 0 \tag{3.18}$$

for all $z, y \in I$. Replacing $z$ with $rz$, where $r \in R$ we get

$$\begin{aligned} 0 = \sigma([rz, y])d(y) &= \sigma(r)\sigma([z, y])d(y) + \sigma([r, y])\sigma(z)d(y) \\ &= \sigma([r, y])\sigma(z)d(y) \\ &= [\sigma(r), \sigma(y)]\sigma(z)d(y) \end{aligned} \tag{3.19}$$

for all $y, z \in I$ and $r \in R$. Right multiplying in (3.19) by $\sigma(y)$ and replacing $z$ with $zy$ in (3.19), we get respectively $[\sigma(r), \sigma(y)]\sigma(z)d(y)\sigma(y) = 0$ and $[\sigma(r), \sigma(y)]\sigma(z)\sigma(y)d(y) = 0$ for all $y, z \in I$ and for all $r \in R$. Subtracting one from another yields $[\sigma(r), \sigma(y)]\sigma(z)[d(y), \sigma(y)] = 0$ for all $y, z \in I$ and for all $r \in R$. Since $\sigma(R) = R$, we have $[R, \sigma(y)]\sigma(z)[d(y), \sigma(y)] = (0)$ for all $y, z \in I$. Replacing $z$ with $rz$, where $r \in R$, we get $[R, \sigma(y)]\sigma(r)\sigma(z)[d(y), \sigma(y)] = (0)$ which gives $[R, \sigma(y)]R\sigma(z)[d(y), \sigma(y)] = (0)$ for all $y, z \in I$. In particular, it follows that $[d(y), \sigma(y)]R\sigma(z)[d(y), \sigma(y)] = (0)$. This implies that $\sigma(z)[d(y), \sigma(y)]R\sigma(z)[d(y), \sigma(y)] = (0)$ for all $y, z \in I$. Since $R$ is semiprime, we conclude that $\sigma(I)[d(x), \sigma(x)] = (0)$ for all $x \in I$, as desired.□

**Theorem 3.6.** *Let $R$ be a semiprime ring, $I$ a nonzero left ideal of $R$ and $\sigma$ any epimorphism of $R$. Suppose that $F : R \to R$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation associated with the map $d$. If $F(xy) + F(y)F(x) = 0$ holds for all $x, y \in I$, then $\sigma(I)[d(x), \sigma(x)] = (0)$ for all $x \in I$.*

**Proof.** If $F$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation associated with the map $d$, then $-F$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation associated with the map $-d$. Thus replacing $F$ with $-F$ and $d$ with $-d$ in Theorem 3.5, we get our conclusions here.□

**Corollary 3.7.** *Let $R$ be a semiprime ring and $\sigma$ be any epimorphism of $R$. Suppose that $F : R \to R$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation associated with the map $d$. If $F(xy) \pm F(y)F(x) = 0$ for all $x, y \in R$, then $d$ is $\sigma$-commuting on $R$. Moreover, if $d$ is a $(\sigma, \sigma)$-derivation of $R$, then $d(R)$ is contained in a nonzero central ideal of $R$, provided $d(R) \neq 0$.*

*In particular, if $R$ is a prime ring and $d$ is a $(\sigma, \sigma)$-derivation of $R$, then $d(R) = 0$, $R$ is commutative and $F(x) = \mp\sigma(x)$ for all $x \in R$.*

**Proof.** By Theorem 3.5 and Theorem 3.6, we have $[d(x), \sigma(x)] = 0$ for all $x \in R$. Then, by Lemma 2.2, if $d(R) \neq (0)$, $d(R)$ is contained in a nonzero central ideal of $R$.

In particular, if $R$ is a prime ring and $d$ is a $(\sigma, \sigma)$-derivation of $R$, then by Lemma 2.1(a) either $d = 0$ or $R$ is commutative. If $d \neq 0$, then $R$ is commutative and so $F(xy) \pm F(x)F(y) = 0$ for all $x, y \in R$. Then by Corollary 3.3, we get $d = 0$, a contradiction. Hence, we conclude that $d = 0$. Therefore, for all $x, y \in R$, $F(xy) = F(x)\sigma(y)$, that is, $F$ is a multiplicative $\sigma$-centralizer mapping of $R$. Then by our assumption, we get

$$F(x)\sigma(y) \pm F(y)F(x) = 0 \tag{3.20}$$

for all $x, y \in R$. Now replacing $x$ with $xr$, we get

$$F(x)\sigma(r)\sigma(y) \pm F(y)F(x)\sigma(r) = 0 \tag{3.21}$$

for all $x, y \in R$. Right multiplying (3.20) by $\sigma(r)$ and then subtracting from (3.21) we have $F(x)[\sigma(r), \sigma(y)] = 0$ for all $x, y, r \in R$, that is $F(x)[\sigma(R), \sigma(R)] = (0)$ for all $x \in R$. Since $\sigma$ is an epimorphism of $R$, it leads to $F(x)[R, R] = (0)$ for all $x \in R$. This gives

$$(0) = F(x)[R^2, R] = F(x)R[R, R] + F(x)[R, R]R = F(x)R[R, R]$$

for all $x \in R$. By the primeness of $R$, we get $F(R) = (0)$ or $R$ is commutative. Since $F \neq 0$, $R$ is commutative and then by Corollary 3.3, we conclude that $F(x) = \mp\sigma(x)$ for all $x \in R$.□

**Corollary 3.8.** *Let $R$ be a prime ring, $\sigma$ an epimorphism of $R$ and $J$ a nonzero ideal of $R$ such that $\sigma(J) \neq (0)$. Suppose that $F : R \to R$ is a nonzero generalized $(\sigma, \sigma)$-derivation of $R$. If $F(xy) \pm F(y)F(x) = 0$ for all $x, y \in J$, then $R$ is commutative and $F(x) = \mp\sigma(x)$ for all $x \in R$.*

**Proof.** Let $d$ be the $(\sigma, \sigma)$-derivation of $R$ associated to $F$. Then by Theorem 3.5 and Theorem 3.6, we find that $\sigma(J)[d(x), \sigma(x)] = (0)$ for all $x \in J$. Since $R$ is prime ring and $\sigma(J)$ is a nonzero ideal of $R$, we have $[d(x), \sigma(x)] = 0$ for all $x \in J$. Then by Lemma 2.4, $R$ is commutative. Thus by Corollary 3.4, we obtain our conclusions. □

## 4 Examples

In this section we construct some examples to show that the semiprimeness condition of the ring in our results are essential.

**Example 4.1.** Consider the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbf{Z} \right\}$, where $\mathbf{Z}$ is the set of all integers. Let $I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbf{Z} \right\}$ be the left ideal of $R$. We define maps $F, d, \sigma : R \to R$, by $F \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 2b \\ 0 & 0 \end{pmatrix}$, $d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 3b \\ 0 & 0 \end{pmatrix}$ and $\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$.

It is easy to verify that $F$ is a multiplicative (generalized)-$(\sigma, \sigma)$-derivation associated with the map $d$. Note that $F$ acts as homomorphism on $I$.

Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0)$, $R$ is not semiprime ring. We see that $\sigma(I)d(I) \neq (0)$ and $\sigma(I)[F(x), x] \neq (0)$ for all $x \in I$, because $\sigma \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} d \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \neq (0)$ and $\sigma \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \left[ F \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \neq (0)$.

Hence, the semiprimeness hypothesis in Theorem 3.1 is crucial.

**Example 4.2.** Consider the ring $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbf{Z} \right\}$, where $\mathbf{Z}$ is the set of all integers. We define maps $F, d, \sigma : R \to R$, by $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ and $\sigma \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. Then $F$ satisfies $F(xy) = F(x)\sigma(y) + \sigma(x)d(y)$ for all $x, y \in R$, where $\sigma$ is an epimorphism of $R$. Then $F$ acts as homomorphism as well as anti-homomorphism on $R$.

Since $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$; $R$ is not semiprime ring. It is easy to verify that $d(R) \neq (0)$, $F$ is neither a multiplicative $\sigma$-centralizer nor a $\sigma$-commuting on $R$ and $d$ is not

$\sigma$-commuting on $R$. Hence, the semiprimeness hypothesis in Corollary 3.3 and Corollary 3.7 can not be omitted.

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