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Fully Idempotent and Multiplication Modules

Nil ORHAN ERTAŞ

This paper is dedicated to Professors P.F. Smith and J. Clark on their 70th birthdays

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Abstract. A submodule N of M is idempotent if $N = N \star N = Hom(M, N)N$. In this paper we give some properties of idempotent submodules. Relations between the multiplication, pure, and idempotent submodules are investigated. We give necessary condition for tensor products of two idempotent submodules to be idempotent.

1 Introduction

All rings are associative with identity element and all modules are unitary right R-modules. Recall that $[N : M] = \{r \in R : Mr \subseteq N\}$. $r(M)$ is the annihilator ideal of M in R, i.e. the ideal consisting of all elements x of R such that $mx = 0$ for all $m \in M$. M is said to be *faithful*, if $r(M) = 0$.

When we generalize notions of ring to module, some difficulties come up with the multiplication in a module. In [5], a product on the lattice of submodules of a module was defined. Let M be an R-module and N and L submodules of M. Set:

$$N \star L := Hom(M, L)N = \sum\{f(N) \mid f : M \to L\}$$

N is called *idempotent* submodule of M if $N \star N = N$. That is, N is idempotent submodule of M, if for each element $n \in N$ there exist a positive integer k, homomorphisms $\varphi_i : M \to N(1 \leq i \leq k)$ and elements $n_i \in N(1 \leq i \leq k)$ such that $n = \varphi_1(n_1) + \cdots + \varphi_k(n_k)$. Idempotent submodules of any modules have been studied in [8] and [9]. Let I be a right ideal of R. Then I is an idempotent submodule of R_R if and only if $I^2 = I$, that is I is an *idempotent* ideal of R. Idempotent submodules of any module generalize the idempotent right ideals of rings. It is well known that every finitely generated idempotent ideal over a commutative ring is generated by idempotent in [1, Exercise 12(3), page 103].

An R-module M is called *fully idempotent* if every submodule of M is idempotent. Clearly every semisimple module is fully idempotent.

Following [4], an R-module M is called a *multiplication* module if every submodule N of M there exists an ideal I of R such that $N = MI$. A right ideal I of a ring R is called a multiplication right ideal if I_R is a multiplication module. It is clear that M_R is a multiplication module $\Leftrightarrow$ $M[N : M] = N$ for each submodule N of $M \Leftrightarrow Mr(M/N) = N$ for each submodule N of M. A submodule K of M is multiplication if and only if $N \cap K = K[N : K]$ for all submodule N of M by [12]. A module M is called *self-generator* if every submodule of M is M-generated. If M is multiplication module over commutative rings, then M is self generator. The simplest example of a multiplication module is an arbitrary simple module over an arbitrary ring. In a commutative ring, the class of multiplication modules contains all projective ideals, all cyclic modules, all finitely generated distributive modules and all ideals generated by idempotents. The fundamental theorem of abelian groups can be expressed that any finitely generated $\mathbb{Z}$-module is a direct of multiplication modules, where $\mathbb{Z}$ is the ring of integers. This motivates us to study multiplication modules.

A submodule N of M is called *pure in M* if the sequence

$$0 \to N \otimes E \to M \otimes E$$

is exact for every R-module E. If N is flat then N is pure in M if and only if $NI = MI \cap N$ for all ideals I of R.

2 Idempotent Submodules

It is well known that every direct summand of a module is idempotent. The following example shows that the converse is not true in general.

Example 2.1. Let K be a field and let $R = \prod_{i=1}^{\infty} K_i$ with $K_i = K$ for all $i \geq 1$. Then the ring R is commutative von Neumann regular. It is easy to check that $Soc(R) = \oplus_{i=1}^{\infty} K_i$ is an idempotent submodule of R_R which is not a direct summand.

Lemma 2.2. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ be a $\mathbb{Z}$-module. Suppose $N = (a,b)\mathbb{Z}$, where $a, b \in \mathbb{Z}, (a,b) = 1$. Then N is a direct summand of M. Hence N is an idempotent submodule of M.

Proof. Since $(a,b) = 1$, $ra + sb = 1$ for some $r, s \in \mathbb{Z}$. Now $(1,0) = (a,b)r + (s,-r)b$ and $(0,1) = (a,b)s - (s,-r)a$. Clearly, $\mathbb{Z} \oplus \mathbb{Z} = (a,b)\mathbb{Z} + (s,-r)\mathbb{Z}$. Now assume that $(a,b)x = (s,-r)y$. Then $ax = sy$ and $bx = -ry$. Therefore $rax + sbx = 0$ and hence $x = 0$. Thus $(a,b)\mathbb{Z} \cap (s,-r)\mathbb{Z} = 0$. Hence N is a direct summand of M. □

Note that any homomorphic image of an idempotent submodule need not be idempotent as we see in the following example.

Example 2.3. Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z} = (1,0)\mathbb{Z} \oplus (0,1)\mathbb{Z}$ and the idempotent submodule $N = (2,3)\mathbb{Z}$. Let $\pi_2 : M \longrightarrow (0,1)\mathbb{Z}$ be the projection map. Then $\pi_2(N) = (0,3)\mathbb{Z}$ is not an idempotent submodule of $(0,1)\mathbb{Z}$.

The next result shows that the homomorphic image of an idempotent submodules is again idempotent in a special case.

Lemma 2.4. Let M be a multiplication module and $K \subseteq N \subseteq M$. If N is an idempotent submodule of M, then N/K is an idempotent submodule of M/K.

Proof. By [15, Lemma 1.2], each submodules of M is fully invariant. Then it is clear by [8, Lemma 2.2]. □

Since $M_{\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}$ is not a multiplication module, multiplication assumption in Lemma 2.4 is not superfluous. The following example shows that the transitivity property of idempotent submodules is not true in general.

Example 2.5. Let $M_{\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}$, $K = (2,3)\mathbb{Z}$ and $L = (1,0)\mathbb{Z}$. Then K and L are direct summands of M and hence they are idempotent submodules of M. $K + L = (1,0)\mathbb{Z} \oplus (0,3)\mathbb{Z}$ is an idempotent submodule of M, but it is not a direct summand of M. Also $(0,3)\mathbb{Z}$ is a direct summand of $K + L$. But $(0,3)\mathbb{Z}$ is not an idempotent submodule of M.

Now we can show that transitivity property of idempotent submodules is inherited in a special case.

Theorem 2.6. Let M be a self generator multiplication R-module with $N \leq K \leq M$ and K be a self generator multiplication submodule of M. If N is idempotent in K and K is idempotent in M, then N is idempotent in M.

Proof. Since K is idempotent in M and M is multiplication, we have

$$\begin{aligned} K \star K &= M[K:M] \star M[K:M] \\ &= M[K:M]^2 = M[K:M][K:M] = K[K:M] = K \end{aligned}$$

by [9, Corollary 3.2]. Since N is idempotent in K and K is multiplication submodule of M, we have

$$\begin{aligned} N \star N &= K[N:K] \star K[N:K] \\ &= K[N:K]^2 = K[N:K][N:K] = N[N:K] = N \end{aligned}$$

by [9, Corollary 3.2]. These imply that $N = K[N:K] = K[K:M][N:K] \subseteq K[K[N:K]:M] = K[N:M] \subseteq K[N:K] = N$. Hence we get $N = K[N:M]$. By [9, Corollary 3.2], we have

$$\begin{aligned} N \star N &= M[N:M] \star M[N:M] \\ &= M[N:M]^2 = M[N:M][N:M] = N[N:M]. \end{aligned}$$

Since K is multiplication submodule of M, we have $N[N:M] \cap K = N[N:M] = K[[N:M]N:K] \supseteq K[N[N:M]:M] \supseteq K[N:M][N:M] \supseteq K[N:M][N:K] \supseteq N[N:K] = N$. Hence N is idempotent in M. □

In Example 2.5, multiplication condition of M is necessary for Theorem 2.6. Since multiplication modules over commutative rings are self generator, we get the following corollary.

Corollary 2.7. Let M be a multiplication module with $N \leq K \leq M$ over a commutative ring R and K be a multiplication submodule of M. If N is idempotent in K, K is idempotent in M, then N is idempotent in M.

Now we construct new idempotent submodule by using idempotent ideals in the following result.

Theorem 2.8. Let M be a module and N idempotent submodule of M and I idempotent ideal of R. Then NI is an idempotent submodule of M.

Proof. Let $r \in I$ and $n \in N$. Since N is an idempotent submodule of M, there exist homomorphisms $f_i : M \to N$ and $n_i \in N$ for every $1 \leq i \leq t$ such that $n = f_1(n_1) + \cdots + f_t(n_t)$. Since I is an idempotent ideal of R, there exist $a_i, b_i \in I$ for every $1 \leq i \leq k$ such that $r = a_1b_1 + \cdots + a_kb_k$. Define homomorphism

$$\varphi_{ij} : M \to NI \quad \text{by} \quad \varphi_{ij}(m) = f_i(m)b_j$$

Clearly φ_{ij} is homomorphism for every $1 \leq i \leq t$ and $1 \leq j \leq k$. Then

$$nr = \sum_{1\leq i\leq t, 1\leq j\leq k} f_i(n_i)a_jb_j = \sum_{1\leq i\leq t, 1\leq j\leq k} f_i(n_ia_j)b_j = \sum_{1\leq i\leq t, 1\leq j\leq k} \varphi_{ij}(n_ia_j).$$

Hence NI is an idempotent submodule of M. □

Clearly every module is an idempotent submodule of itself. Then we have the following as a corollary of [8, Proposition 3.8].

Corollary 2.9. Let M be a module and I idempotent submodule of R. Then MI is an idempotent submodule of M.

Recall that a module M is called *fully idempotent* if every submodule of M is idempotent.

Example 2.10. (i) If R is a von Neumann regular ring, then every right principal ideal of R is a direct summand. So every principal right ideal of R is idempotent. Hence every submodule of R_R is idempotent (as a sum of idempotent submodules). That is R_R is fully idempotent. Note that by [16, 3.15], in the case of commutative rings, they are the same.
(ii) Following [16, 23.5], if R is a right V-ring, then R_R is fully idempotent .

Theorem 2.11. Let M be a multiplication module and $M = M_1 \oplus M_2$ be a direct sum of fully idempotent submodules M_1 and M_2. Then M is also fully idempotent.

Proof. Since M is multiplication every submodule of M is fully invariant. Let A be a submodule of M. Then we have $A = (A \cap M_1) \oplus (A \cap M_2)$. By assumption $A \cap M_1$ and $A \cap M_2$ are idempotent submodules of M_1 and M_2 respectively. Then by [8, Corollary 2.5], A is an idempotent submodule of M. □

Proposition 2.12. Let M be a multiplication module and N submodule of M. If M is fully idempotent, then so is M/N.

Proof. It is clear by Lemma 2.4. □

This lemma is taken from [7, Lemma 2.2]. It is given for the sake of completeness.

Lemma 2.13. Let M be a fully idempotent module. Let $N \leq M$ and I an ideal of R. Then $N \cap MI = NI$.

Proof. Let $n \in N \cap MI$. Then there exist the homomorphisms $\varphi_i : M \longrightarrow N \cap MI$ and the elements $n_i \in N \cap MI$ for some $k \geq 1$ and $1 \leq i \leq k$ such that $n = \varphi_1(n_1) + \cdots + \varphi_k(n_k)$. Let $1 \leq i \leq k$. Then $n_i = x_1u_1 + \cdots + x_tu_t$ for some $t \geq 1$, $x_j \in M$ and $u_j \in I$ $(1 \leq j \leq t)$. Therefore $\varphi_i(n_i) = \varphi_i(x_1)u_1 + \cdots + \varphi_i(x_t)u_t \in NI$. Hence $n \in NI$ and so $NI = N \cap MI$. □

3 Modules over Commutative Rings

In this section all rings are commutative with identity and all modules are unital. In [2, Corollary 2,7], finitely generated faithful multiplication modules are flat. By using Lemma 2.13, idempotent submodule of a flat modules is pure. In this section, we consider when pure submodule is idempotent.

In [3], M.M. Ali and D.J. Smith defined idempotent submodules as follows which are different our idempotent definition: A submodule N of M is called *idempotent* submodule of M, if $N = N[N : M]$. Let $N = N[N : M]$ for any submodule N of M and $n \in N$. Then there exist $r_i \in [N : M]$ and $n_i \in N$ for every $1 \leq i \leq k$, such that $n = n_1 r_1 + \cdots + n_k r_k$. Define homomorphisms,

$$f_i : M \to N \quad (1 \leq \mathrm{i} \leq \mathrm{k})$$

with $f_i(m) = mr_i$. Then $n = f_1(n_1) + \cdots + f_k(n_k)$. Therefore in the sense of our idempotent submodule is a generalization of the M.M. Ali and D.J. Smith's sense. But the converse is not true in general. Consider the example in Lemma 2.2. Clearly $[N : M] = 0$, and hence $N[N : M] = 0 \neq N$. Therefore N is not idempotent submodule in M.M. Ali and D.J. Smith's sense but N is idempotent in our sense. For the converse; let M be a finitely generated faithful multiplication module, N an idempotent submodule of M in our sense. Then by [8, Theorem 3.9], $[N : M]$ is idempotent. Since M is multiplication, $N = M[N : M] = M[N : M]^2 = M[N : M][N : M] = N[N : M]$. This implies that N is idempotent in the sense of M.M Ali and D. J. Smith. In [3, Theorem 1.1], authors characterized pure submodules of multiplication modules by using idempotent submodules in the sense of M.M. Ali and D.J. Smith. In the following theorem we give a relation between pure submodules, multiplication modules and idempotent submodules by motivation of some idea in [3, Theorem 1.1].

Theorem 3.1. Let M be a finitely generated faithful multiplication module and N a submodule of M.Then the following are equivalent:
(i) N is a pure submodule of M.
(ii) N is multiplication and is idempotent in M.
(iii) $I[N : M] = I \cap [N : M]$ for every ideal I of R.

Proof. $(i) \Rightarrow (ii)$ Assume that N is a pure submodule of M. Let K be a submodule of M. We will show that $N \cap K = N[K : N]$. Since M is multiplication $K = M[K : M]$. Since N is a pure submodule of M, we have $N[K : N] = N \cap M[K : N]$.

$$N[K : N] = N \cap M[K : N] \supseteq N \cap M[K : M] = N \cap K \supseteq N[K : N]$$

Hence we get $N[K : N] = K \cap N$. This implies that N is a multiplication submodule of M. Since N is pure, we have $N[N : M] = N \cap M[N : M] = N$.

$$N = N[N : M] = M[N : M][N : M] = M[N : M]^2.$$

Hence we get $M[N : M]^2 = M[N : M]$. By [14, Theorem 9], we have $[N : M]$ is an idempotent ideal of R. Then [8, Theorem 3.9], N is idempotent in M.

$(ii) \Rightarrow (iii)$ Assume that N is multiplication and idempotent in M. By [8, Theorem 3.9], $[N : M]$ is an idempotent submodule, then we have $N = M[N : M] = M[N : M]^2 = M[N : M][N : M] = N[N : M]$. For any submodule K of M we have,

$$N[K : N] = N[K : N][N : M] \subseteq N[K : M] \subseteq N[K : N]$$

implies that $N[K : N] = N[K : M]$. Since N is multiplication submodule of M, for every ideal I of R

$$\begin{aligned} MI \cap N &= N[MI : N] \\ &= N[MI : M] \\ &= NI = MI \cap M[N : M] \end{aligned}$$

Also we have $NI = N[N : M]I = M[N : M]I$ implies that; $MI \cap M[N : M] = M[N : M]I$ for any ideal I of R. By [6, Theorem 1.6], $MI \cap M[N : M] = M(I \cap [N : M])$. By [14, Theorem 9], M is a cancellation module. Then we have $I[N : M] = I \cap [N : M]$.

$(iii) \Rightarrow (i)$ Let I be an ideal of R. By [6, Theorem 1.6], we have $M[N : M] \cap MI = M([N : M] \cap I)$. By assumption, $[N : M] \cap I = I[N : M]$ implies that

$$N \cap MI = M[N : M] \cap MI = M([N : M] \cap I) = M[N : M]I = NI.$$

Then we have desired result. □

In Example 2.5, $(2,3)\mathbb{Z}$ and $(0,1)\mathbb{Z}$ are idempotent submodules of $M_{\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}$. But $(0,1)\mathbb{Z} \cap (2,3)\mathbb{Z} = (0,3)\mathbb{Z}$ is not idempotent in $M_{\mathbb{Z}}$. Hence intersection of two idempotent submodule need not be idempotent again. In below proposition we give a result about intersection of idempotent module is again idempotent in a special case.

Proposition 3.2. Let M be a finitely generated faithful multiplication module. If N and K are finitely generated idempotent submodules of M, then $N \cap K$ is also idempotent in M.

Proof. By [8, Theorem 3.9], $[N : M]$ and $[K : M]$ are idempotent ideals of R. By [10, Lemma 1.4], $[N : M]$ and $[K : M]$ are finitely generated. Clearly there exist idempotent elements $e_1, e_2 \in R$ such that $[N : M] = e_1R$ and $[K : M] = e_2R$. We also have $[N \cap K : M] = [N : M] \cap [K : M] = e_1R \cap e_2R = e_1e_2R$. Hence $[N \cap K : M]$ is also idempotent ideal in R. Then $N \cap K$ is idempotent in M by [8, Theorem 3.9]. □

Lemma 3.3. Let M be a finitely generated faithful multiplication module and N be a finitely generated submodule of M.Then N is an idempotent submodule of M if and only if $nR = n[N : M]$ for each $n \in N$.

Proof. $(\Rightarrow:)$ By [8, Theorem 3.9], $[N : M]$ is an idempotent ideal of R. By [10, Lemma 1.4], $[N : M]$ is finitely generated. Clearly there exists an idempotent element $e \in R$ such that $[N : M] = eR$. Since M is multiplication, we have $M[N : M] = N$. Let $ns \in nR$ for $s \in R$ and $n \in N$. Then there exist $r_i \in [N : M]$ and $m_i \in M$ for every $1 \leq i \leq k$ such that $n = m_1r_1 + \cdots + m_kr_k$. Since $[N : M] = eR$, there exist $s_i \in R$ such that $r_i = es_i$ for every $1 \leq i \leq k$. Then

$$\begin{aligned} ns &= m_1r_1s + \cdots + m_kr_ks \\ &= m_1es_1s \cdots + m_kes_ks \\ &= (m_1s_1 + \cdots + m_ks_k)es \\ &= (m_1s_1 + \cdots + m_ks_k)ees \\ &= (m_1es_1 + \cdots + m_kes_k)es \\ &= (m_1r_1 + \cdots + m_kr_k)es \\ &= nes \in neR = n[N : M]. \end{aligned}$$

$(\Leftarrow:)$ Let N be a submodule of M. We assume that $nR = n[N : M]$ for each $n \in N$. We will show that $N \star N = N$. Take $n \in N$. By assumption, there exist $r \in [N : M]$ such that $n = nr$. Define a homomorphism $\varphi : M \to N$ with $\varphi(m) = mr$. Clearly φ is homomorphism. Then $\varphi(n) = n$. Hence N is an idempotent submodule of M. □

Theorem 3.4. Let M_1 and M_2 are finitely generated faithful multiplication modules and N_1 and N_2 are finitely generated idempotent submodules of M_1 and M_2 respectively. Then $N_1 \otimes N_2$ is an idempotent submodule of $M_1 \otimes M_2$.

Proof. By Lemma 3.3, it is enough to show that

$$(n_1 \otimes n_2)R = (n_1 \otimes n_2)[N_1 \otimes N_2 : M_1 \otimes M_2]$$

for any $n_1 \in N_1$ and $n_2 \in N_2$. By assumption, we have $n_1R = n_1[N_1 : M_1]$ and $n_2R = n_2[N_2 : M_2]$. Then

$$(n_1 \otimes n_2)R = n_1R \otimes n_2R = n_1[N_1 : M_1] \otimes n_2[N_2 : M_2] = (n_1 \otimes n_2)[N_1 : M_1][N_2 : M_2]$$

Clearly we have $[N_1 : M_1][N_2 : M_2] \subseteq [N_1 \otimes N_2 : M_1 \otimes M_2]$. This implies that

$$(n_1 \otimes n_2)R \subseteq (n_1 \otimes n_2)[N_1 \otimes N_2 : M_1 \otimes M_2] \subseteq (n_1 \otimes n_2)R.$$

Then we have desired result. □

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Author information

Nil ORHAN ERTAŞ, Karabuk University, Department of Mathematics, 78050, Karabuk, Turkey.
E-mail: `orhannil@yahoo.com`

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