

# Arithmetic Progression of more Squares

Muneer Jebreel Karama

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**Abstract.** The aim of this paper is to present new arithmetic progressions among squares, i.e., more than three squares. Moreover we will introduce arithmetic progression of higher power.

## 1 Introduction

Referring to my research [3] it is possible to produce triplets of the type  $(a, b, c)$  of positive integers so that  $a^2, b^2, c^2$  are in Arithmetic Progression (AP) such that :  $b^2 - a^2 = c^2 - b^2$ .

In 2008 [2] Peth'o and Ziegler have found an arithmetic progression of length 4 that lies on some curve  $X^2 - dY^2 = m$  and they have found an arithmetic progression such that there does not exist such a curve.

In 2010 [1] Enrique Gonz and J. ÑDorn Steuding gave a partial answer to this question: Let  $d$  be a squarefree integer. Does there exist four squares in arithmetic progression over  $\mathbb{Q}(\sqrt{d})$  depending on the value of  $d$ ? In the affirmative case, they construct explicit arithmetic progressions consisting of four squares over  $\mathbb{Q}(\sqrt{d})$ .

It is well known that the right triangles whose sides are integers  $X, Y, Z$ (a Pythagorean triple) formed Arithmetical Progressions (AP) among three squares such that the following equation holds:

$$(X + Y)^2 + (X - Y)^2 = 2Z^2$$

For example the Pythagorean triple  $(3, 4, 5)$  represents the AP  $(1, 5, 7)$ , i.e.  $7^2 + 1^2 = 2(5)^2$ , or more familiarly  $5^2 - 1^2 = 7^2 - 5^2$ , in fact equation  $(X + Y)^2 + (X - Y)^2 = 2Z^2$  has infinitely many solutions, and chains with common difference 24.

## 2 The Extension of $(X + Y)^2 + (X - Y)^2 = 2Z^2$

We begin with the following important extension of  $(X + Y)^2 + (X - Y)^2 = 2Z^2$ .

**Theorem 2.1.** *With slight modification of  $(X + Y)^2 + (X - Y)^2 = 2Z^2$ , we got the following new equation:*

$$(X + Y)^2 + (X - Y)^2 + (-X + Y)^2 + (-X - Y)^2 = 4Z^2$$

**Proof.** Assume that  $X = 2mn, Y = m^2 - n^2$ , and  $Z = m^2 + n^2$ , then by direct substitutions we got  $(2mn + m^2 - n^2)^2 + (2mn - (m^2 - n^2))^2 + (-2mn + m^2 - n^2)^2 + (-2mn - (m^2 - n^2))^2 = 4(m^2 + n^2)^2 = 4Z^2$ .

We show the first sixteen consecutive AP of this equation in the following table , for examples: i)  $7^2 + 1^2 + (-7)^2 + (-1)^2 = 100 = 4(25) = 4(5)^2$ , i.e.  $7^2 + (-7)^2 - (5)^2 - (5)^2 = (5)^2 + (5)^2 - (-1)^2 - 1^2$ , so we have the AP  $(-7, -1, 1, 5, 7)$ . ii)  $137^2 + 7^2 + (-7)^2 + (-137)^2 = 37636 = 4(9409) = 4(97)^2$ , i.e.  $137^2 + (-137)^2 - (97)^2 - (97)^2 = (97)^2 + (97)^2 - (-7)^2 - 7^2$ , so we have the AP  $(-137, -7, 7, 97, 137)$ .

X	Y	Z	(X+Y)	(X-Y)	(-X+Y)	(-X-Y)	4(Z)
3	4	5	7	1	-1	-7	4(5)
5	12	13	17	7	-7	-17	4(13)
7	24	25	31	17	-17	-31	4(25)
8	15	17	23	7	-7	-23	4(17)
9	40	41	49	31	-31	-49	4(41)
11	60	61	71	49	-49	-71	4(61)
12	35	37	47	23	-23	-47	4(37)
13	84	85	97	71	-71	-97	4(85)
16	63	65	79	47	-47	-79	4(65)
20	21	29	41	1	-1	-41	4(29)
28	45	53	73	17	-17	-73	4(53)
33	56	65	89	23	-23	-89	4(65)
36	77	85	113	41	-41	-113	4(85)
39	80	89	119	41	-41	-119	4(89)
48	55	73	103	7	-7	-103	4(73)
65	72	97	137	7	-7	-137	4(97)

**Theorem 2.2.** *If X, Y, Z are the sides of right triangle then:*

$$(X + Y + Z)^2 + (X - Y + Z)^2 + (X + Y - Z)^2 + (-X + Y + Z)^2 = 8Z^2$$

**Proof.** Assume that  $X = 2mn, Y = m^2 - n^2$ , and  $Z = m^2 + n^2$ . Then by direct substitutions we got:  $(2mn + m^2 - n^2 + m^2 + n^2)^2 + (2mn - m^2 + n^2 + m^2 + n^2)^2 + (2mn + m^2 - n^2 - m^2 - n^2)^2 + (-2mn + m^2 - n^2 + m^2 + n^2)^2 = 8(m^2 + n^2)^2 = 8Z^2$ .

Table below shows the first sixteen AP of  $(X + Y + Z)^2 + (X - Y + Z)^2 + (X + Y - Z)^2 + (-X + Y + Z)^2 = 8Z^2$ .

X	Y	Z	(X+Y+Z)	(X-Y+Z)	(X+Y-Z)	(-X+Y+Z)	8(Z)
3	4	5	12	4	2	6	8(5)
5	12	13	30	6	4	20	8(13)
7	24	25	56	8	6	42	8(25)
8	15	17	40	10	6	24	8(17)
9	40	41	90	10	8	72	8(41)
11	60	61	132	12	10	110	8(61)
12	35	37	84	14	10	60	8(37)
13	84	85	182	14	12	156	8(85)
16	63	65	144	18	14	112	8(65)
20	21	29	70	28	12	30	8(29)
28	45	53	126	36	20	70	8(53)
33	56	65	154	42	24	88	8(65)
36	77	85	198	44	28	126	8(85)
39	80	89	208	48	30	130	8(89)
48	55	73	176	66	30	80	8(73)
65	72	97	234	90	40	104	8(97)

For example the Pythagorean triple (3, 4, 5) represents the AP (2, 4, 6, 12), i.e.  $2^2 + 4^2 + 6^2 + 12^2 = 8(5)^2$ , hence:  $12^2 + 6^2 - 5^2 - 5^2 - 5^2 - 5^2 = 5^2 + 5^2 + 5^2 + 5^2 - 2^2 - 4^2$ , if we cancel 2

from the AP (2, 4, 6, 12) , then we get (1, 2, 3, 6) , which is of course AP ( $1^2 + 2^2 + 3^2 + 6^2 = 2(5)^2$ ) , this example has new ideas that maybe provoke for more new results which is open for Mathematicians . Table above shows the first sixteen AP of equation:

$$(X + Y + Z)^2 + (X - Y + Z)^2 + (X + Y - Z)^2 + (-X + Y + Z)^2 = 8Z^2$$

**Theorem 2.3.** *If X, Y, Z are the sides of right triangle then;*  $(X+Y+Z)^2+(X-Y+Z)^2+(X+Y-Z)^2+(-X+Y+Z)^2+(-X-Y-Z)^2+(-X+Y-Z)^2+(X-Y-Z)^2+(-X-Y+Z)^2 = 16Z^2$

**Proof.** Assume that  $X = 2mn, Y = m^2 - n^2$ , and  $Z = m^2 + n^2$ , then by direct substitutions we got:  $(2mn + m^2 - n^2 + m^2 + n^2)^2 + (2mn - m^2 + n^2 + m^2 + n^2)^2 + (2mn + m^2 - n^2 - m^2 - n^2)^2 + (-2mn + m^2 - n^2 + m^2 + n^2)^2 + (-2mn - m^2 + n^2 - m^2 - n^2)^2 + (-2mn + m^2 - n^2 - m^2 - n^2)^2 + (2mn - m^2 + n^2 - m^2 - n^2)^2 + (-2mn - m^2 + n^2 + m^2 + n^2)^2 = 16(m^2 + n^2)^2 = 16Z^2$ .

Table below shows selected solutions of equation :  $(X + Y + Z)^2 + (X - Y + Z)^2 + (X + Y - Z)^2 + (-X + Y + Z)^2 + (-X - Y - Z)^2 + (-X + Y - Z)^2 + (X - Y - Z)^2 + (-X - Y + Z)^2 = 16Z^2$

For the following tabular:  $t_1 = (X+Y+Z), t_2 = (X-Y+Z), t_3 = (X+Y-z), t_4 = (-X+Y+Z), t_5 = (-X-Y-Z), t_6 = (-X+Y-Z), t_7 = (X-Y-Z), t_8 = (-X-Y+Z), t_9 = 16Z$

X	Y	Z	t <sub>1</sub>	t <sub>2</sub>	t <sub>3</sub>	t <sub>4</sub>	t <sub>5</sub>	t <sub>6</sub>	t <sub>7</sub>	t <sub>8</sub>	t <sub>9</sub>
3	4	5	12	4	2	6	-12	-4	-6	-2	16(5)
5	12	13	30	6	4	20	-30	-6	-20	-4	16(13)
7	24	25	56	8	6	42	-56	-8	-42	-6	16(25)
8	15	17	40	10	6	24	-40	-10	-24	-6	16(17)
9	40	41	90	10	8	72	-90	-10	-72	-8	16(41)
11	60	61	132	12	10	110	-132	-12	-110	-10	16(61)
12	35	37	84	14	10	60	-84	-14	-60	-10	16(37)
13	84	85	182	14	12	156	-182	-14	-156	-12	16(85)
16	63	65	144	18	14	112	-144	-18	-112	-14	16(65)
20	21	29	70	28	12	30	-70	-28	-30	-12	16(29)
28	45	53	126	36	20	70	-126	-36	-70	-20	16(53)
33	56	65	154	42	24	88	-154	-42	-88	-24	16(65)
36	77	85	198	44	28	126	-198	-44	-126	-28	16(85)
39	80	89	208	48	30	130	-208	-48	-130	-30	16(89)
48	55	73	176	66	30	80	-176	-66	-80	-30	16(73)
65	72	97	234	90	40	104	-234	-90	-104	-40	16(97)

It seems that equation 4 is the general form of the previous equations, so we may plug some variables by zero to get them.

**Theorem 2.4.** *If X, Y, Z are the sides of right triangle then;*

$$X^8 + Y^8 + Z^8 = 2W^2$$

**Proof.** Let  $m, n$  integers, with  $(m, n) = 1$ , then the solution of equation  $X^8 + Y^8 + Z^8 = 2W^2$ :  $X = 2mn, Y = m^2 - n^2, Z = m^2 + n^2$ , and  $W = (m^4 + 2m^3n + 2m^2n^2 - 2mn^3 + n^4)(m^4 - 2m^3n + 2m^2n^2 + 2mn^3 + n^4)$  It can be proved by direct substitution .

Table below shows selected solutions of equation  $X^8 + Y^8 + Z^8 = 2W^2$ :

X	Y	Z	(X) <sup>8</sup>	(Y) <sup>8</sup>	(Z) <sup>8</sup>	2W <sup>2</sup>
3	4	5	6,561	65,536	390,625	2(481) <sup>2</sup>
5	12	13	390,625	429,981,696	815,730,721	2(24961) <sup>2</sup>
7	24	25	5,764,801	110,075,314,176	152,587,890,625	2(362401) <sup>2</sup>
8	15	17	16,777,216	2,562,890,625	6,975,757,441	2(69121) <sup>2</sup>
9	40	41	43,046,721	6,553,600,000,000	7,984,925,229,121	2(2696161) <sup>2</sup>
11	60	61	214,358,881	167,961,600,000,000	191,707,312,997,281	2(13410241) <sup>2</sup>
12	35	37	429,981,696	2,251,875,390,625	3,512,479,453,921	2(1697761) <sup>2</sup>
13	84	85	815,730,721	2,478,758,911,082,500	2,724,905,250,390,620	2(51008161) <sup>2</sup>
16	63	65	4,294,967,296	248,155,780,267,521	318,644,812,890,625	2(16834561) <sup>2</sup>

Conclusion: From the equations above we can conclude such a sequence of patterns that began from equation  $(X + Y)^2 + (X - Y)^2 = 2Z^2$  with  $2Z^2$ , continuo in equation  $(X + Y)^2 + (X - Y)^2 + (-X + Y)^2 + (-X - Y)^2 = 4Z^2$  with  $4Z^2$ , same of equation  $(X + Y + Z)^2 + (X - Y + Z)^2 + (X + Y - Z)^2 + (-X + Y + Z)^2 = 8Z^2$  with  $8Z^2$ , and  $16Z^2$  with equation  $(X + Y + Z)^2 + (X - Y + Z)^2 + (X + Y - Z)^2 + (-X + Y + Z)^2 + (-X - Y - Z)^2 + (-X + Y - Z)^2 + (X - Y - Z)^2 + (-X - Y + Z)^2 = 16Z^2$ , so what about  $32Z^2$ ,  $64Z^2$ , and so on. Are there common pattern connected AP with  $2nZ^2$ ?

## References

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## Author information

Muneer Jebreel Karama, Department of Mathematics, Palestine Polytechnic University ,College of Applied Science, Palestine.  
E-mail: muneerk@ppu.edu

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