

MODULES THAT HAVE A WEAK SUPPLEMENT IN EVERY COFINITE EXTENSION

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Abstract We say that over an arbitrary ring a module M has *the property (CWE)* (respectively, *(CWEE)*) if M has a weak supplement (respectively, ample weak supplements) in every cofinite extension. We show that if every submodule of a module M has the property *(CWE)* then M has the property *(CWEE)*. A ring R is semilocal iff every left R -module has the property *(CWE)*. We also prove that over a commutative Von-Neumann regular ring R , an R -module M has the property *(CWE)* iff M is cofinitely injective.

1 Introduction

Throughout this paper, R is an associative ring with identity and all modules are unital left R -modules, unless otherwise stated. Let M be an R -module. The notation $(U \ll M) U \leq M$ means that U is a (*small*) submodule of M . A submodule U of M is called small in M , if $M \neq U + L$ for every proper submodule L of M . By $Rad(M)$ we denote the intersection of all maximal submodules of M , equivalently the sum of all small submodules of M (see [10]). A module M is called *radical* if M has no maximal submodules, that is, $M = Rad(M)$. A submodule U of M is called *cofinite* (in M) if the factor module $\frac{M}{U}$ is finitely generated. Let U and K be submodules of M . K is called *supplement* of U in M if it is minimal with respect to $M = U + K$. A module M is called *supplemented* if every submodule of M has a supplement in M . K is a supplement of U in M if and only if $M = U + K$ and $U \cap K \ll K$ [10]. If $M = U + K$ and $U \cap K \ll M$, then K is called a *weak supplement* of U in M , and clearly in this case, U is a weak supplement of K in M , too. A submodule U of M has *ample (weak) supplements* in M if, whenever $M = U + L$, L contains a (weak) supplement of U in M . M is a *weakly supplemented* module if every submodule of M has a weak supplement in M (see [6], [10], [11]). M is called a *cofinitely supplemented* module if every cofinite submodule of M has a supplement in M (see [1]). In [2], Alizade and Büyükaşık introduced *cofinitely weak supplemented* module (or briefly a *cws-module*), every cofinite submodule of which has a weak supplement in M .

Let R be a ring and let M and N be R -modules. N is called (*cofinite*) *extension* of M in case $M \subseteq N$ (and $\frac{N}{M}$ is finitely generated). A module M is said to be *cofinitely injective* if it is a direct summand in its every cofinite extension (see [4]).

In [12], Zöschinger initiated the study of the modules with the following properties defined as:

- (E) A module M has a supplement in every extension.
- (EE) A module M has ample supplements in every extension.

The author showed in the same paper some properties of these modules. Modifying his concepts, Çalışıcı and Türkmen say that a module M has *the property (CE)* (respectively, *(CEE)*) if M has a supplement (respectively, ample supplements) in every cofinite extension. They give a characterization of semiperfect rings via these modules in [4].

In this paper, we study the modules with the property *(CWE)* (respectively, *(CWEE)*) as a generalization of the modules with the property *(CE)* (respectively, *(CEE)*). We show that if a module M has the property *(CWE)* then every direct summand of M has the property *(CWE)*. We give a characterization of semilocal rings via the modules with the property *(CWE)*. Moreover, we give that for a ring R , every R -module is cofinitely weak supplemented if and only if every R -module has the property *(CWE)*. Finally, we show that there exists a module which has the property *(CWE)* but not *(CE)*.

2 Modules with the Properties (CWE) and (CWEE)

In this section, we define the notion of modules with the properties (CWE) and (CWEE). We obtain various properties of these modules. We prove that every R -module has the property (CWE) if and only if the ring R is semilocal.

Definition 2.1. Let M be an R -module. We say that:

M has the property (CWE) if M has a weak supplement in every cofinite extension, and

M has the property (CWEE) if M has ample weak supplements in every cofinite extension.

It is shown in [12, Lemma 1.3.(a)] that the property (E) is preserved by direct summands. Now we give an analogue of this fact for the modules with the property (CWE).

Proposition 2.2. Let M be a module. If M has the property (CWE), then every direct summand of M has the property (CWE).

Proof. Let M_1 be a direct summand of M . Then there exists a submodule M_2 of M such that $M = M_1 \oplus M_2$. Let N be any cofinite extension of M_1 , K be the external direct sum $N \oplus M_2$ and $\varphi : M \rightarrow K$ be the canonical embedding. Then $M \cong \varphi(M)$ has the property (CWE). We have that

$$\frac{N}{M_1} \cong \frac{N \oplus M_2}{\varphi(M)} = \frac{K}{\varphi(M)}$$

is finitely generated. Since $\varphi(M)$ has the property (CWE), there exists a submodule U of K such that $K = \varphi(M) + U$ and $\varphi(M) \cap U \ll K$. Consider the projection $\pi : K \rightarrow N$. We have $N = M_1 + \pi(U)$. Also since $\text{Ker}(\pi) \subseteq \varphi(M)$, $\pi(\varphi(M) \cap U) = \pi(\varphi(M)) \cap \pi(U) = M_1 \cap \pi(U) \ll N$. Hence $\pi(U)$ is a weak supplement of M_1 in N . \square

Now we give the relation between the modules with the property (CWE) and (CWEE).

Proposition 2.3. If every submodule of a module M has the property (CWE), then M has the property (CWEE).

Proof. Let N be any cofinite extension of M and $N = M + V$ for a submodule V of N . Note that

$$\frac{N}{M} \cong \frac{V}{M \cap V}$$

is finitely generated. By the hypothesis, $M \cap V$ has a weak supplement K in V that is $(M \cap V) + K = V$, $(M \cap V) \cap K = M \cap K \ll V$. Also we have that $N = M + V = M + (M \cap V) + K = M + K$. Consequently, K is an ample weak supplement of M in N . \square

Now we show that every simple module has the property (CWE). Firstly, we give the following well known fact for completeness.

Lemma 2.4. Every simple submodule S of a module M is either a direct summand of M or small in M .

Proof. Suppose that S is not small in M , then there exists a proper submodule K of M such that $S + K = M$. Since S is simple and $K \neq M$, $S \cap K = 0$. Thus $M = S \oplus K$. \square

Proposition 2.5. Every simple module has the property (CWE).

Proof. Let S be a simple module and N be any cofinite extension of S . Then by Lemma 2.4, $S \ll N$ or $S \oplus K = N$ for a submodule $K \leq N$. In the first case N is a weak supplement of S in N . In the second case K is a weak supplement of S in N . Hence, S has the property (CWE). \square

Let M be a module and U be a submodule of M . If the factor module $\frac{M}{U}$ has the property (CWE), M does not need to have the property (CWE). For example, for the ring $R = \mathbb{Z}$, the R -module $M = \frac{3\mathbb{Z}}{9\mathbb{Z}}$ has a weak supplement in every cofinite extension since it is simple. But $3\mathbb{Z}$ does not have a weak supplement in its cofinite extension \mathbb{Z} .

Now we obtain that the statement mentioned at the beginning of the last paragraph is true under a special condition.

Proposition 2.6. Let M be a module and U be a submodule of M . If $U \ll M$ and the factor module $\frac{M}{U}$ has the property (CWE), then M also has the property (CWE).

Proof. Let N be any cofinite extension of M . Then we obtain that

$$\frac{N}{M} \cong \frac{\frac{N}{U}}{\frac{M}{U}}$$

is finitely generated. Since $\frac{M}{U}$ has the property (CWE), there exists a submodule $\frac{V}{U}$ of $\frac{N}{U}$ such that $\frac{M}{U} + \frac{V}{U} = \frac{N}{U}$ and $\frac{M}{U} \cap \frac{V}{U} = \frac{M \cap V}{U} \ll \frac{N}{U}$. Note that $M + V = N$. Suppose that $(M \cap V) + S = N$ for a submodule S of N . Then we obtain $\frac{M \cap V}{U} + \frac{S+U}{U} = \frac{N}{U}$. Since $\frac{M \cap V}{U} \ll \frac{N}{U}$, we have that $\frac{S+U}{U} = \frac{N}{U}$. By hypothesis, it follows that $N = S + U = S$. Hence $M \cap V \ll N$. \square

The following corollary is immediate consequence of the last proposition.

Corollary 2.7. *Let M be a finitely generated module. If $\frac{M}{\text{Rad}(M)}$ has the property (CWE), then M has the property (CWE).*

Proposition 2.8. *Let M be an R -module and N be a cofinite extension of M . If M has a weak supplement V in N , then M has a finitely generated weak supplement W in N such that $W \leq V$.*

Proof. Let V be a weak supplement of M in N . Then we have that $N = M + V$ and $M \cap V \ll N$. Since N is a cofinite extension of M ,

$$\frac{N}{M} = \frac{M+V}{M} \cong \frac{V}{M \cap V}$$

is finitely generated. Suppose that for some positive integer n and elements $x_i \in V (1 \leq i \leq n)$, elements $x_1 + M \cap V, x_2 + M \cap V, \dots, x_n + M \cap V$ generate the factor module $\frac{V}{M \cap V}$. Then for a finitely generated submodule $W = Rx_1 + Rx_2 + \dots + Rx_n$ of V , we have that $W + M = W + (V \cap M) + M = V + M = N$. Consequently, W is a finitely generated weak supplement of M in N . \square

Recall from [2] the ring R is semilocal if and only if every R -module is cofinitely weak supplemented. Now we give a characterization for semilocal rings via the modules with the property (CWE).

Theorem 2.9. *Let R be a ring. R is semilocal if and only if every R -module has the property (CWE).*

Proof. Let R be a semilocal ring, M be an R -module and N be a cofinite extension of M . Since R is semilocal, N is cofinitely weak supplemented module from [2, Corollary 2.22]. Therefore, M has a weak supplement in N .

Conversely, let M be an R -module and U be any cofinite submodule of M . By hypothesis, U has the property (CWE). Then U has a weak supplement in M , so that M is cofinitely weak supplemented. Hence R is semilocal by [2, Corollary 2.22]. \square

The following corollary is immediate result of [2, Corollary 2.22] and Theorem 2.9.

Corollary 2.10. *Let R be a ring. Every R -module is cofinitely weak supplemented if and only if every R -module has the property (CWE).*

In [10] a ring R is said to be *Von-Neumann regular* if every element a of R is regular, i.e., there exists an element $b \in R$ such that $aba = a$. In [4], a module M is called *cofinitely injective* if it is a direct summand in every cofinite extension.

Theorem 2.11. *Let R be a commutative Von-Neumann regular ring. An R -module M has the property (CWE) if and only if M is cofinitely injective.*

Proof. Let M be an R -module with the property (CWE) and N be any cofinite extension of M . Then there is a submodule K of N such that $N = M + K$ and $M \cap K \ll N$. From [10] we have $\text{Rad}(N) = 0$. Hence M is a direct summand of N .

Conversely, let N be any cofinite extension of M . By hypothesis, there is a submodule K of N such that $N = M \oplus K$. Hence K is a weak supplement of M in N . \square

It is clear that every module with the property (CE) has the property (CWE), but the converse can not be true in general. The following example shows that there exists a module that has the property (CWE) but not (CE).

Example 2.12. (see [6, Remark 3.3]) For primes p and q , consider the following ring:

$$R := \mathbb{Z}_{p,q} := \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0, (p, b) = (q, b) = 1 \right\}.$$

R is a semilocal ring that is not semiperfect. Then there exists an R -module M that has not the property (CE) from [4, Theorem 2.12]. Since R is semilocal ring, by Theorem 2.9 M has the property (CWE) .

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