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Some results involving special polynomials and integral transforms

Ahmed Ali Al-Gonah

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Abstract. In this paper, we introduce some special polynomials starting from a suitable operational definition. Some properties of these special polynomials are considered by using the integral representation method.

1. Introduction

Dattoli and his co-workers have shown that by combining the properties of exponential operators and suitable integral representations one can find an efficient way of treating fractional operators. They introduced new families of special polynomials starting from a suitable operational definition, see [2,8]. Also, they have shown that the use of integral representation method is a fairly important tool of analysis to study the properties of old and new families of special polynomials, see [3,5]. The work of this paper is a further attempt in introducing new special polynomials by combining the integral representation method and operational techniques. To this aim, we recall that the Laguerre-Gould Hopper polynomials (LGHP) ${}_LH_n^{(m,s)}(x,y,z)$ are defined by the generating function [11, p.9933]

$$\exp(yt+zt^s)C_0(-xt^m)=\sum_{n=0}^{\infty}{}_LH_n^{(m,s)}(x,y,z)\frac{t^n}{n!}, \tag{1.1}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined as [13]:

$$C_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^r}{r!(n+r)!}. \tag{1.2}$$

The LGHP ${}_LH_n^{(m,s)}(x,y,z)$ are also defined by the following series:

$${}_LH_n^{(m,s)}(x,y,z)=n!\sum_{k=0}^{[\frac{n}{s}]}\frac{z^k\,{}_mL_{n-sk}(x,y)}{k!(n-sk)!}, \tag{1.3}$$

where ${}_mL_n(x,y)$ are the 2-variable generalized Laguerre polynomials (2VgLP) defined by [6; p.213]

$${}_mL_n(x,y)=n!\sum_{r=0}^{[\frac{n}{m}]}\frac{x^r y^{n-mr}}{(r!)^2(n-mr)!} \tag{1.4}$$

and by the operational definition

$${}_mL_n(x,y)=\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\Big\{y^n\Big\}, \tag{1.5}$$

where D_x^{-1} denotes the inverse of the derivative operator $D_x:=\frac{\partial}{\partial x}$ and is defined in such a way that

$$D_x^{-n}\{f(x)\}=\frac{1}{(n-1)!}\int_0^x(x-\xi)^{n-1}f(\xi)d\xi. \tag{1.6}$$

The LGHP ${}_LH_n^{(m,s)}(x,y,z)$ satisfy the differential equation

$$\left(m\frac{\partial^m}{\partial y^m}+sz\frac{\partial^{s+1}}{\partial x\partial y^s}+y\frac{\partial^2}{\partial x\partial y}-n\frac{\partial}{\partial x}\right){}_LH_n^{(m,s)}(x,y,z)=0 \tag{1.7}$$

and the monomiality recurrences

$$\frac{\partial}{\partial y}\ {}_LH_n^{(m,s)}(x,y,z)=n\ {}_LH_{n-1}^{(m,s)}(x,y,z), \tag{1.8}$$

$$\left(mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+sz\frac{\partial^{s-1}}{\partial y^{s-1}}+y\right){}_LH_n^{(m,s)}(x,y,z)={}_LH_{n+1}^{(m,s)}(x,y,z). \tag{1.9}$$

Further, the polynomials ${}_LH_n^{(m,s)}(x,y,z)$ are defined by means of the following operational representations:

$$\exp\left(z\frac{\partial^s}{\partial y^s}\right)\Big\{{}_mL_n(x,y)\Big\}={}_LH_n^{(m,s)}(x,y,z), \tag{1.10}$$

$$\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\Big\{H_n^{(s)}(y,z)\Big\}={}_LH_n^{(m,s)}(x,y,z) \tag{1.11}$$

and

$$\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}+z\frac{\partial^s}{\partial y^s}\right)\Big\{y^n\Big\}={}_LH_n^{(m,s)}(x,y,z), \tag{1.12}$$

where $H_n^{(s)}(x,y)$ are the Gould-Hopper polynomials (GHP), defined by the series [10; p.58] (see also [7])

$$g_n^s(x,y)=H_n^{(s)}(x,y)=n!\sum_{k=0}^{[\frac{n}{s}]}\frac{y^kx^{n-sk}}{k!(n-sk)} \tag{1.13}$$

and by the operational definition

$$H_n^{(s)}(x,y)=\exp\left(y\frac{\partial^s}{\partial x^s}\right)\Big\{x^n\Big\}. \tag{1.14}$$

2. Integral transforms and special polynomials

The integrals offer a direct way to investigate the properties of the functions and polynomials they define. It is well known that one of the starting point of the theory of fractional operators, i.e. operators raised to a fractional power, is the Euler's integral [13; p.218]:

$$a^{-\nu}=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-at}\,t^{\nu-1}dt. \tag{2.1}$$

Now, multiplying both sides of equation (2.1) by $f(x,y)$ and replacing a by $\alpha-z\frac{\partial^s}{\partial y^s}$ in the resultant equation, we find

$$\left(\alpha-z\frac{\partial^s}{\partial y^s}\right)^{-\nu}f(x,y)=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\,t^{\nu-1}\exp\left(zt\frac{\partial^s}{\partial y^s}\right)f(x,y)dt, \tag{2.2}$$

Let us consider the case $f(x,y)={}_mL_n(x,y)$ in equation (2.2), we get

$$\left(\alpha-z\frac{\partial^s}{\partial y^s}\right)^{-\nu}{}_mL_n(x,y)=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\,t^{\nu-1}\exp\left(zt\frac{\partial^s}{\partial y^s}\right){}_mL_n(x,y)dt. \tag{2.3}$$

Using operational formula (1.10) in the r.h.s. of the above equation, we obtain

$$\left(\alpha-z\frac{\partial^s}{\partial y^s}\right)^{-\nu}{}_mL_n(x,y)=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\,t^{\nu-1}{}_LH_n^{(m,s)}(x,y,zt)dt. \tag{2.4}$$

The integral on the r.h.s. of equation (2.4) defines a new special polynomials. Denoting this special polynomials by ${}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$, we find

$$ {}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\;t^{\nu-1}{}_LH_n^{(m,s)}(x,y,zt)dt, \tag{2.5}$$

From equations (2.4) and (2.5), we have

$$\left(\alpha-z\frac{\partial^s}{\partial y^s}\right)^{-\nu}{}_mL_n(x,y)={}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha). \tag{2.6}$$

Using definition (1.3) in the r.h.s. of equation (2.5), we obtain

$$ {}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)=\frac{n!}{\Gamma(\nu)}\sum_{k=0}^{[\frac{n}{s}]}\frac{z^k{}_mL_{n-sk}(x,y)}{k!(n-sk)!}\int_0^\infty e^{-\alpha t}\;t^{\nu+k-1}\,dt, \tag{2.7}$$

which on using equation (2.1) in the r.h.s., becomes

$$ {}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)=\frac{n!}{\alpha^\nu}\sum_{k=0}^{[\frac{n}{s}]}\frac{(\nu)_k\;z^k{}_mL_{n-sk}(x,y)}{\alpha^k k!(n-sk)!}, \tag{2.8}$$

Now, using definition (1.4) in the r.h.s. of the above equation, we get the following series definition for ${}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$:

$$ {}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)=\frac{n!}{\alpha^\nu}\sum_{k=0}^{[\frac{n}{s}]}\sum_{r=0}^{[\frac{n-sr}{m}]}\frac{(\nu)_k\;z^kx^ry^{n-sk-mr}}{\alpha^k k!(r!)^2(n-sk-mr)!}. \tag{2.9}$$

In particular, we note that

$$ {}_LH_{n,1}^{(m,s)}(x,y,D_z^{-1};1)={}_LH_n^{(m,s)}(x,y,z), \tag{2.10}$$

$$ {}_LH_{n,\nu}^{(m,2)}(0,y,z;\alpha)={}_\nu H_n^{(2)}(y,z;\alpha), \tag{2.11}$$

where ${}_\nu H_n^{(2)}(y,z;\alpha)$ denotes the known special polynomials defined by [8; p.731]

$$ {}_\nu H_n^{(2)}(x,y;\alpha)=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\;t^{\nu-1}H_n^{(2)}(x,yt)dt=\frac{n!}{\alpha^\nu}\sum_{k=0}^{[\frac{n}{2}]}\frac{(\nu)_k\;y^kx^{n-2k}}{\alpha^k k!(n-2k)!}. \tag{2.12}$$

As a generalization of the special polynomials ${}_\nu H_n^{(2)}(x,y;\alpha)$ defined by equation (2.12), we introduce the special polynomials ${}_\nu H_n^{(s)}(x,y;\alpha)$ by means of the following integral representation:

$$ {}_\nu H_n^{(s)}(x,y;\alpha)=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\;t^{\nu-1}H_n^{(s)}(x,yt)dt. \tag{2.13}$$

Also, we define the special polynomials ${}_\nu H_n^{(s)}(x,y;\alpha)$ by the series

$$ {}_\nu H_n^{(s)}(x,y;\alpha)=\frac{n!}{\alpha^\nu}\sum_{k=0}^{[\frac{n}{s}]}\frac{(\nu)_k\;y^kx^{n-sk}}{\alpha^k k!(n-sk)!} \tag{2.14}$$

and by the following generating function:

$$\sum_{n=0}^\infty {}_\nu H_n^{(s)}(x,y;\alpha)\frac{t^n}{n!}=\frac{\exp(xt)}{(\alpha-yt^s)^\nu}. \tag{2.15}$$

Now, using definitions (1.5) and (2.14) in the r.h.s. of equation (2.8), we get the following operational definition for the special polynomials ${}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$:

$$\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\left\{{}_\nu H_n^{(s)}(y,z;\alpha)\right\}={}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha). \tag{2.16}$$

We note that

$$ {}_LH_{n,\nu}^{(m,s)}(0,y,z;\alpha) = {}_\nu H_n^{(s)}(y,z;\alpha). \tag{2.17}$$

3. Properties of ${}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$

The special polynomials ${}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$ satisfy the following differential relations:

$$\left(\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right){}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha) = \frac{\partial^m}{\partial y^m}\,{}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$$

$$= n(n-1)\cdots(n-(m-1))\,{}_LH_{n-m,\nu}^{(m,s)}(x,y,z;\alpha), \tag{3.1}$$

$$\frac{\partial}{\partial z}\,{}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha) = \nu\,\frac{\partial^s}{\partial y^s}\,{}_LH_{n,\nu+1}^{(m,s)}(x,y,z;\alpha)$$

$$= \nu\, n(n-1)\cdots(n-(s-1))\,{}_LH_{n-s,\nu+1}^{(m,s)}(x,y,z;\alpha) \tag{3.2}$$

and

$$\frac{\partial}{\partial \alpha}\,{}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha) = -\nu\,{}_LH_{n,\nu+1}^{(m,s)}(x,y,z;\alpha). \tag{3.3}$$

Consequently from equations (3.1) and (3.2), we have

$$\frac{\partial^m}{\partial z^m}\,{}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha) = (\nu)_m\left(\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)^s{}_LH_{n,\nu+m}^{(m,s)}(x,y,z;\alpha) \tag{3.4}$$

and from equations (3.2) and (3.3), we get

$$\frac{\partial}{\partial z}\,{}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha) = -\frac{\partial^{s+1}}{\partial\alpha\partial y^s}\,{}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha), \tag{3.5}$$

which for $s=m$ and using equation (3.1) (for $s=m$), yields

$$\frac{\partial}{\partial z}\,{}_LH_{n,\nu}^{(m,m)}(x,y,z;\alpha) = -\frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right){}_LH_{n,\nu}^{(m,m)}(x,y,z;\alpha). \tag{3.6}$$

The integral representation (2.5) can be used to establish other properties for the special polynomials ${}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$ with the help of the corresponding properties of the LGHP ${}_LH_n^{(m,s)}(x,y,z)$. For instance, multiplying both sides of equation (2.5) by $\frac{\xi^n}{n!}$, summing up over n and using generating function (1.1) in the r.h.s. of the resultant equation, we get

$$\sum_{n=0}^{\infty}{}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)\frac{\xi^n}{n!} = \frac{\exp(y\xi)\;C_0(-x\xi^m)}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp(zt\xi^s)dt, \tag{3.7}$$

which on expanding the second exponential (inside integral) in the r.h.s. gives

$$\sum_{n=0}^{\infty}{}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)\frac{\xi^n}{n!} = \frac{\exp(y\xi)\;C_0(-x\xi^m)}{\Gamma(\nu)}\sum_{k=0}^{\infty}\frac{(z\xi^s)^k}{k!}\int_0^\infty e^{-\alpha t}t^{\nu+k-1}\,dt. \tag{3.8}$$

Using relation (2.1) in the r.h.s. of the above equation, we get the following generating function of the special polynomials ${}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$:

$$\sum_{n=0}^{\infty}{}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)\frac{\xi^n}{n!} = \frac{\exp(y\xi)C_0(-x\xi^m)}{(\alpha-z\xi^s)^\nu}. \tag{3.9}$$

From generating functions (2.15) and (3.9), we get

$${}_LH_{n,\nu+\mu}^{(m,s)}(x,y+w,z;\alpha) = \sum_{k=0}^{n}\binom{n}{k}{}_LH_{n-k,\nu}^{(m,s)}(x,y,z;\alpha)\;{}_\mu H_k^{(s)}(w,z;\alpha). \tag{3.10}$$

Now, replacing z by zt in differential equation (1.7) and multiplying by $\frac{1}{\Gamma(\nu)}\,e^{-\alpha t}\,t^{\nu-1}$ and integrating the resultant equation with respect to t between the limits 0 to ∞, we have

$$\left(m\frac{\partial^m}{\partial y^m}+y\frac{\partial^2}{\partial x\partial y}-n\frac{\partial}{\partial x}\right)\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\,t^{\nu-1}{}_LH_n^{(m,s)}(x,y,zt)dt$$

$$+sz\frac{\partial^{s+1}}{\partial x\partial y^s}\,\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\,t^\nu{}_LH_n^{(m,s)}(x,y,zt)dt=0, \tag{3.11}$$

which on using equation (2.5) in the l.h.s. and then using relation (3.3) in last term of the l.h.s. of the resultant equation, we find that the special polynomials ${}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$ satisfy the following differential equation:

$$\left(m\frac{\partial^m}{\partial y^m}-sz\frac{\partial^{s+2}}{\partial\alpha\partial x\partial y^s}+y\frac{\partial^2}{\partial x\partial y}-n\frac{\partial}{\partial x}\right){}_LH_{n,\nu}^{(m,s)}(x,y,z)=0. \tag{3.12}$$

Also, from the monomiality recurrences relations (1.8) and (1.9) and by proceeding on the same lines as above, we can obtain that the special polynomials ${}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$ satisfy the following monomiality recurrences relations:

$$\frac{\partial}{\partial y}\,{}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)=n\,{}_LH_{n-1,\nu}^{(m,s)}(x,y,z;\alpha) \tag{3.13}$$

and

$$\left(mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}-sz\frac{\partial^s}{\partial\alpha\partial y^{s-1}}+y\right){}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)={}_LH_{n+1,\nu}^{(m,s)}(x,y,z;\alpha), \tag{3.14}$$

respectively.

We remark that the differential equation (3.12) can also be obtained by combining equations (3.13) and (3.14).

Further, replacing z by zt in operational rule (1.12) and multiplying both sides by $\frac{1}{\Gamma(\nu)}\,e^{-\alpha t}\,t^{\nu-1}$ and integrating the resultant equation with respect to t between the limits 0 to ∞, we have

$$\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\,t^{\nu-1}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}+zt\frac{\partial^s}{\partial y^s}\right)\Big\{y^n\Big\}dt=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\,t^{\nu-1}{}_LH_n^{(m,s)}(x,y,zt)dt, \tag{3.15}$$

which on using equation (2.5) in the r.h.s. and breaking the second exponential in the l.h.s. becomes

$$\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\,\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\,t^{\nu-1}\exp\left(zt\frac{\partial^s}{\partial y^s}\right)\Big\{y^n\Big\}dt={}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha). \tag{3.16}$$

Expanding the third exponential operator containing the s-order derivative in the l.h.s. of equation (3.16) and then using equation (2.1) in the resultant equation, we get the following operational rule for the special polynomials ${}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha)$:

$$\left(\alpha-z\frac{\partial^s}{\partial y^s}\right)^{-\nu}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\,\Big\{y^n\Big\}={}_LH_{n,\nu}^{(m,s)}(x,y,z;\alpha), \tag{3.17}$$

which can also be obtained by using operational definition (1.5) in the l.h.s. of equation (2.6).

4. Further new special polynomials ${}_\nu H_n^{(s,m)}(x,y,z;\alpha)$

It is evident that the method we have proposed offers a fairly efficient means to introduce new special polynomials. In order to further stress the importance of this method in introducing new special polynomials, we recall that the 3-variable generalized Hermite polynomials (3VgHP) $H_n^{(s,m)}(x,y,z)$ are defined by the series [9] (see[4])

$$H_n^{(s,m)}(x,y,z)=n!\sum_{k=0}^{[\frac{n}{s}]}\frac{z^kH_{n-sk}^{(m)}(x,y)}{k!(n-sk)!} \tag{4.1}$$

and specified by the generating function

$$\exp(x\xi+y\xi^m+z\xi^s)=\sum_{n=0}^\infty H_n^{(s,m)}(x,y,z)\frac{\xi^n}{n!}. \tag{4.2}$$

Using definitions (1.14) and (1.13) in the r.h.s. of definition (4.1), we get the following operational definition for the 3VgHP $H_n^{(s,m)}(x,y,z)$:

$$H_n^{(s,m)}(x,y,z)=\exp\left(y\frac{\partial^m}{\partial x^m}\right)\Big\{H_n^{(s)}(x,z)\Big\}. \tag{4.3}$$

Now, according to relation (2.1) it is evident that

$$\left(\alpha - y\frac{\partial^m}{\partial x^m}\right)^{-\nu} f(x,z) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\, t^{\nu-1} \exp\left(yt\frac{\partial^m}{\partial x^m}\right) f(x,z)\, dt, \tag{4.4}$$

which for $f(x,z) = H_n^{(s)}(x,z)$ becomes

$$\left(\alpha - y\frac{\partial^m}{\partial x^m}\right)^{-\nu} H_n^{(s)}(x,z) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\, t^{\nu-1} \exp\left(yt\frac{\partial^m}{\partial x^m}\right) H_n^{(s)}(x,z)\, dt. \tag{4.5}$$

Using operational definition (4.3) in the r.h.s. of the above equation, we find

$$\left(\alpha - y\frac{\partial^m}{\partial x^m}\right)^{-\nu} H_n^{(s)}(x,z) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\, t^{\nu-1} H_n^{(s,m)}(x,yt,z)dt. \tag{4.6}$$

The integral on the r.h.s. of equation (4.6) defines a new special polynomials. Denoting this polynomials by ${}_\nu H_n^{(s,m)}(x,y,z;\alpha)$, we find

$${}_\nu H_n^{(s,m)}(x,y,z;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\, t^{\nu-1} H_n^{(s,m)}(x,yt,z)dt. \tag{4.7}$$

From equations (4.6) and (4.7), we have

$$\left(\alpha - y\frac{\partial^m}{\partial x^m}\right)^{-\nu} H_n^{(s)}(x,z) = {}_\nu H_n^{(s,m)}(x,y,z;\alpha). \tag{4.8}$$

Using definition (4.1) in the r.h.s. of equation (4.7), we get

$${}_\nu H_n^{(s,m)}(x,y,z;\alpha) = \frac{n!}{\Gamma(\nu)}\sum_{k=0}^{[\frac{n}{s}]}\frac{z^k}{k!(n-sk)!}\int_0^\infty e^{-\alpha t}\, t^{\nu-1} H_{n-sk}^{(m)}(x,yt)dt, \tag{4.9}$$

which on using equation (2.13), yields

$${}_\nu H_n^{(s,m)}(x,y,z;\alpha) = n!\sum_{k=0}^{[\frac{n}{s}]}\frac{z^k\ {}_\nu H_{n-sk}^{(m)}(x,y;\alpha)}{k!(n-sk)!}. \tag{4.10}$$

Using definition (2.14) in the r.h.s. of the above equation, we get the following series definition for ${}_\nu H_n^{(s,m)}(x,y,z;\alpha)$:

$${}_\nu H_n^{(s,m)}(x,y,z;\alpha) = \frac{n!}{\alpha^\nu}\sum_{k=0}^{[\frac{n}{s}]}\sum_{r=0}^{[\frac{n-sr}{m}]}\frac{(\nu)_r\ z^k y^r x^{n-sk-mr}}{\alpha^r k! r! (n-sk-mr)!}. \tag{4.11}$$

The generating function for the special polynomials ${}_\nu H_n^{(s,m)}(x,y,z;\alpha)$ can be obtained with the help of generating function (4.2) of the 3VgHP $H_n^{(s,m)}(x,y,z)$. Multiplying both side of equation (4.7) by $\frac{\xi}{n!}$ and summing up over n, we have

$$\sum_{n=0}^\infty {}_\nu H_n^{(s,m)}(x,y,z;\alpha)\frac{\xi}{n!} = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\, t^{\nu-1}\sum_{n=0}^\infty H_n^{(s,m)}(x,yt,z)\frac{\xi}{n!}dt, \tag{4.12}$$

which on using generating function (4.2) in the r.h.s. gives

$$\sum_{n=0}^\infty {}_\nu H_n^{(s,m)}(x,y,z;\alpha)\frac{\xi}{n!} = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}\, t^{\nu-1}\ \exp(x\xi + yt\xi^m + z\xi^s)dt. \tag{4.13}$$

Breaking the exponential function in the r.h.s. of the above equation and then expanding the second exponential inside the integral and integrating it, we get the following generating function for ${}_\nu H_n^{(s,m)}(x,y,z;\alpha)$:

$$\sum_{n=0}^\infty {}_\nu H_n^{(s,m)}(x,y,z;\alpha)\frac{\xi}{n!} = \frac{\exp(x\xi + z\xi^s)}{(\alpha - y\xi^m)^\nu}. \tag{4.14}$$

The special polynomials ${}_{\nu}H_n^{(s,m)}(x,y,z;\alpha)$ are linked to the special polynomials ${}_{L}H_{n,\nu}^{(m,s)}(x,y,z;\alpha)$ defined in Section 2 by the following relation:

$$ {}_{\nu}H_n^{(m,s)}(y,z,D_x;\alpha) = {}_{L}H_{n,\nu}^{(m,s)}(x,y,z;\alpha). \tag{4.15}$$

In particular, we note that

$$ {}_{\nu}H_n^{(s,m)}(x,y,0;\alpha) = {}_{\nu}H_n^{(m)}(x,y;\alpha), \tag{4.16}$$

$$ {}_{\nu}H_n^{(2,2)}(x,y,z;\alpha) = {}_{\nu H}H_n^{(2)}(x,y,z;\alpha). \tag{4.17}$$

where ${}_{\nu H}H_n^{(2)}(x,y,z;\alpha)$ denotes the known special polynomials defined by [8; p.732]

$$ {}_{\nu H}H_n^{(2)}(x,y,z;\alpha) = n!\sum_{k=0}^{[\frac{n}{2}]} \frac{z^k \ {}_{\nu}H_{n-2k}^{(2)}(x,y;\alpha)}{k!(n-2k)!}. \tag{4.18}$$

We conclude that further properties of the special polynomials ${}_{\nu}H_n^{(s,m)}(x,y,z;\alpha)$ can be directly inferred from the corresponding ones for the 3VgHP $H_n^{(s,m)}(x,y,z)$ with the help of the integral representation (4.7) and by following the same procedure leading to the results in Section 3.

References

[1] P. Appell, J. Kampé de Fériet , *Fonctions hypergéométriques et hypersphériques: Polynômes d' Hermite*, Gauthier-Villars, Paris 1926.

[2] G. Dattoli , Operational methods, fractional operators and special polynomials, *Appl. Math. Comput.* **141** (2003) 151–159.

[3] G. Dattoli, Integral transforms and Chebyshev-like polynomials, *Appl. Math. Comput.* **148** (2004) 225–234.

[4] G. Dattoli, B. Germano, P.E. Ricci, Hermite polynomials with more than two variables and associated bi-orthogonal function, *Integral Transforms Spec. Funct.* **20** (1) (2009) 17–22.

[5] G. Dattoli, S. Lorenzutta, C. Cesarano, From Hermite to Humbert polynomials, *Rend. Istit. Mat. Univ. Trieste* **XXXV** (2003) 37–48.

[6] G. Dattoli, S. Lorenzutta, A.M. Mancho, A. Torre, Generalized polynomials and associated operational identities, *J. Comput. Appl. Math.* **108** (1999) 209Ű-218.

[7] G. Dattoli, P.L. Ottaviani , A. Torre, L. Vázquez, Evolution operator equations: integration with algebraic and finite-difference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory. *Riv. Nuovo Cimento Soc. Ital. Fis.*(4) **20** (2)(1997) 1–133.

[8] G. Dattoli , P.E. Ricci, C. Cesarano, L. Vázquez, Special polynomials and fractional calculus, *Math. Comput. Modelling* **37** (2003) 729Ű-733.

[9] G. Dattoli, P.E. Ricci, I. Khomasuridze, On the derivation of new families of generating functions involving ordinary Bessel functions and Bessel-Hermite functions, *Math. Comput. Modelling* **46** (2007) 410Ű-414.

[10] H.W. Gould, A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke. Math. J.* **29** (1962) 51–63.

[11] S. Khan, A.A. Al-Gonah, Operational method and Laguerre-Gould Hopper polynomials, *Appl. Math. Comput.* **218**, (2012) 9930Ű-9942.

[12] E.D. Rainville, *Special Functions*, Macmillan, New York, 1960, reprinted by Chelsea Publ. Co., Bronx, New York, 1971.

[13] H.M. Srivastava, H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press, New York, 1984.

Author information

Ahmed Ali Al-Gonah, Department of Mathematics, Aden University, Aden, Yemen.
E-mail: gonah1977@yahoo.com

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