

## Semigroup ideals with semiderivations in 3-prime near-rings

A. Boua and L. Oukhtite and A. Raji

Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

Communicated by Ayman Badawi

MSC 2010 Classifications: 16N60, 16W25, 16Y30.

Keywords and phrases: 3-prime near-rings, semiderivations, commutativity.

**Abstract.** The purpose of this paper is to obtain the structure of certain near-rings satisfying the following conditions:

- (i)  $d(I) \subseteq Z(N)$ ,      (ii)  $d(-I) \subseteq Z(N)$ ,      (iii)  $d([x, y]) = 0$ ,  
 (iv)  $d([x, y]) = [x, y]$ ,      (v)  $d(x \circ y) = 0$ ,      (vi)  $d(x \circ y) = x \circ y$

for all  $x, y \in I$ , with  $I$  is a semigroup ideal and  $d$  is a semiderivation associated with an automorphism. Furthermore; an example is given to illustrate that the 3-primeness hypothesis is not superfluous.

### Definitions and terminology

In this paper  $N$  will denote a zero symmetric left near-ring. For any  $x, y \in N$  the symbol  $[x, y]$  will denote the commutator  $xy - yx$ , while the symbol  $x \circ y$  will stand for the anti-commutator  $xy + yx$ . The symbol  $Z(N)$  will represent the multiplicative center of  $N$ , that is,  $Z(N) = \{x \in N \mid xy = yx \text{ for all } y \in N\}$ . Unless specified, we will use the word near-ring to mean zero symmetric left near-ring. A near-ring  $N$  is said to be 3-prime if  $xNy = \{0\}$  for all  $x, y \in N$  implies  $x = 0$  or  $y = 0$ . A nonempty subset  $I$  of  $N$  is called a semigroup right ideal (resp. semigroup left ideal) if  $IN \subset I$  (resp.  $NI \subset I$ ); and if  $I$  is both a semigroup right and a semigroup left ideal, then  $I$  is said to be a semigroup ideal.  $N$  is said to be 2-torsion free if  $x \in N$  and  $2x = 0$  implies  $x = 0$ . An additive mapping  $\delta : N \rightarrow N$  is called a derivation if  $\delta(xy) = \delta(x)y + x\delta(y)$  holds for all  $x, y \in N$ . Let  $g$  be an additive mapping of  $N$ , an additive mapping  $d : N \rightarrow N$  is called a semiderivation of  $N$  associated with  $g$  if  $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$  and  $d(g(x)) = g(d(x))$  for all  $x, y \in N$ , or equivalently, as noted in [10], that  $d(xy) = xd(y) + d(x)g(y) = g(x)d(y) + d(x)y$  and  $d(g(x)) = g(d(x))$  for all  $x, y \in N$ . In the case of rings, semiderivations have received significant attention in recent years. We prove that some theorems in the setting of a semigroup ideal of a 3-prime near ring admitting a semiderivation, and thereby extend some known results [4, Theorem 2.1], [8, Theorem 2.6] and [8, Theorem 2.9].

### 1 Main Results

In this paper, the semiderivations used are associated with automorphisms. To prove our main theorems, we need the following lemmas.

**Lemma 1.** [4, Lemma 1.4(i)] Let  $N$  be a 3-prime near-ring, and  $I$  a nonzero semigroup ideal of  $N$ . If  $x, y \in N$  and  $xIy = \{0\}$ , then  $x = 0$  or  $y = 0$ .

**Lemma 2.** Let  $N$  be a 3-prime near-ring.

- (i) [4, Lemma 1.2 (iii)] If  $z \in Z(N) \setminus \{0\}$  and  $xz \in Z(N)$ , then  $x \in Z(N)$ .  
 (ii) [2, Lemma 1.5] If  $N \subseteq Z(N)$ , then  $N$  is a commutative ring.

**Lemma 3.** Let  $N$  be a near-ring and  $d$  is a semiderivation of  $N$ . Then  $N$  satisfies the following partial distributive law

- i)  $(d(x)y + g(x)d(y))z = d(x)yz + g(x)d(y)z$  for all  $x, y, z \in N$ .  
 ii)  $(xd(y) + d(x)g(y))z = xd(y)z + d(x)g(y)z$  for all  $x, y, z \in N$ .

**Theorem 1.** Let  $N$  be a 3-prime near-ring and  $I$  be a nonzero semigroup ideal of  $N$ . If  $N$  admits a nonzero semiderivation  $d$ , then the following assertions are equivalent

i)  $d(I) \subseteq Z(N)$

ii)  $N$  is a commutative ring.

**Proof.**  $ii) \Rightarrow i)$  is obvious.

$i) \Rightarrow ii)$  by the hypothesis given, we have

$$d(xy)z = zd(xy) \text{ for all } y \in I, x, z \in N.$$

Taking Lemma 3(i), we get

$$d(x)yz + g(x)d(y)z = zd(x)y + zg(x)d(y) \text{ for all } y \in I, x, z \in N.$$

So that,

$$d(x)yz + d(y)g(x)z = zd(x)y + d(y)zg(x) \text{ for all } y \in I, x, z \in N. \quad (1.1)$$

Substituting  $g(x)$  for  $z$  in (1.1), we obtain

$$d(x)yg(x) = g(x)d(x)y \text{ for all } y \in I, x \in N. \quad (1.2)$$

Replacing  $y$  by  $yt$  in (1.2) and using this, we get

$$\begin{aligned} d(x)ytg(x) &= (g(x)d(x)y)t \\ &= d(x)yg(x)t \text{ for all } y \in I, x, t \in N. \end{aligned}$$

The last equation shows that

$$d(x)y[g(x), t] = 0 \text{ for all } y \in I, x, t \in N$$

this means that

$$d(x)I[g(x), t] = \{0\} \text{ for all } x, t \in N.$$

By Lemma 1, this implies that

$$d(x) = 0 \text{ or } g(x) \in Z(N) \text{ for all } x \in N. \quad (1.3)$$

Taking the fact that  $d \neq 0$ , then (1.3) shows that there is an element  $x_0 \in N$  such as  $g(x_0) \in Z(N)$  and  $d(x_0) \neq 0$ . In this case, equation (1.1) yields

$$d(x_0)yz = zd(x_0)y \text{ for all } y \in I, z \in N.$$

Again replacing  $y$  by  $yt$ , we get

$$\begin{aligned} d(x_0)ytz &= (zd(x_0)y)t \\ &= d(x_0)yzt \text{ for all } y \in I, z, t \in N. \end{aligned}$$

Hence,

$$d(x_0)y[z, t] = 0 \text{ for all } y \in I, z, t \in N$$

this is reduced to

$$d(x_0)I[z, t] = \{0\} \text{ for all } z, t \in N. \quad (1.4)$$

Taking Lemma 1, (1.4) implies that  $d(x_0) = 0$  or  $N \subset Z(N)$ . And since the first of these conditions is impossible, the second must hold  $N$  a commutative ring by Lemma 2(ii).  $\square$

**Corollary 1.** [4, Theorem 2.1] Let  $N$  be a 3-prime near-ring, and let  $I$  be a nonzero semigroup ideal of  $N$ . If  $N$  admits a nonzero derivation  $d$  for which  $d(I) \subseteq Z(N)$ , then  $N$  is a commutative ring.

**Theorem 2.** Let  $N$  be a 2-torsion free 3-prime near-ring and  $I$  be a nonzero semigroup ideal of  $N$ . If  $N$  admits a nonzero semiderivation  $d$ , then the following assertions are equivalent

i)  $d(-I) \subseteq Z(N)$

ii)  $N$  is a commutative ring.

**Proof.** For  $ii) \Rightarrow i)$ , the proof is obvious.

$i) \Rightarrow ii)$ , we have  $d(-x) \in Z(N)$  for all  $x \in I$ , then

$$d(-tx) = d(t(-x)) \in Z(N) \text{ for all } x \in I, t \in N. \quad (1.5)$$

In particular, for all  $t \in Z(N)$  we have

$$d(t(-x)) = td(-x) + d(t)g(-x) \in Z(N) \text{ for all } x \in I.$$

by Lemma 3(ii), we obtain

$$d(t)g(-x) \in Z(N) \text{ for all } x \in I. \quad (1.6)$$

Since  $g$  is an automorphism, then  $d(t) \in Z(N)$ . By the application of Lemma 2(i), (1.6) yields

$$d(t) = 0 \text{ or } g(-x) \in Z(N) \text{ for all } x \in I, t \in Z(N). \quad (1.7)$$

If  $d(Z(N)) = \{0\}$ , taking (1.5) into account, we get

$$d\left(d(t(-x))\right) = 0 \text{ for all } x \in I, t \in N.$$

So that,

$$d^2(t)(-x) + 2g(d(t))d(-x) = 0 \text{ for all } x \in I, t \in N. \quad (1.8)$$

Replacing  $t$  by  $d(t)$  in (1.8), we get

$$d^3(t)(-x) + 2g(d^2(t))d(-x) = 0 \text{ for all } x \in I, t \in N. \quad (1.9)$$

on the other hand, applying  $d$  for (1.8), we find that

$$d^3(t)(-x) + 3g(d^2(t))d(-x) = 0 \text{ for all } x \in I, t \in N. \quad (1.10)$$

From (1.9) and (1.10), we conclude that  $g(d^2(t))d(-x) = 0$  for all  $x \in I, t \in N$ . Taking the fact that  $d(-x) \in Z(N)$ , then

$$d^2(g(t))Nd(-x) = \{0\} \text{ for all } x \in I, t \in N.$$

In the light of the 3-primeness of  $N$ , the last equation implies that

$$d^2 = 0 \text{ or } d = 0. \quad (1.11)$$

If  $d^2 = 0$ , then  $d = 0$  (see prove of Theorem 2 in [10]), and therefore (1.11) shows that  $d = 0$ , a contradiction. Consequently  $d(Z(N)) \neq \{0\}$  and (1.7) prove that  $g(-x) \in Z(N)$  for all  $x \in I$ . Let  $v \in N$  and  $x \in I$ , we have  $g(-vx) = g(v)g(-x) \in Z(N)$ , by Lemma 2(i), we get

$$g(-x) = 0 \text{ or } g(v) \in Z(N) \text{ for all } x \in I, v \in N. \quad (1.12)$$

$i)$  If  $g(-x) = 0$  for all  $x \in I$ , by this hypothesis we have

$$d(-yx) = d(y)g(-x) + yd(-x) \in Z(N) \text{ for all } x \in I, y \in N.$$

So that,

$$yd(-x) \in Z(N) \text{ for all } x \in I, y \in N.$$

Using Lemma 2(i) and taking the fact  $d \neq 0$ , we arrive at  $N \subset Z(N)$ . Applying Lemma 2(ii), we conclude that  $N$  is a commutative ring.

$ii)$  If there is an element  $x_0 \in I$  such that  $g(-x_0) \neq 0$ , then equation (1.12) shows that  $g(v) \in Z(N)$  for all  $v \in N$ . Since  $g$  is an automorphism we conclude that  $N \subset Z(N)$ . Thus  $N$  is a commutative ring. This completes the proof of our theorem.  $\square$

**Corollary 2.** [8, Lemma 2.4] Let  $N$  be a 2-torsion free 3-prime near-ring and  $I$  be a nonzero semigroup ideal of  $N$ . If  $N$  admits a nonzero derivation  $d$  for which  $d(-I) \subseteq Z(N)$ , then  $N$  is a commutative ring.

**Theorem 3.** Let  $N$  be a 3-prime near-ring and  $I$  be a nonzero semigroup ideal of  $N$ . If  $N$  admits a semiderivation  $d$ , then the following assertions are equivalent:

**i)**  $d([x, y]) = 0$  for all  $x, y \in I$ .

ii)  $d([x, y]) = [x, y]$  for all  $x, y \in I$ .

iii)  $N$  is a commutative ring.

**Proof.** iii)  $\Rightarrow$  i) and iii)  $\Rightarrow$  ii) are obvious.

Proving that i)  $\Rightarrow$  iii). Suppose that

$$d([x, y]) = 0 \quad \text{for all } x, y \in I. \quad (1.13)$$

Substituting  $xy$  for  $y$  in (1.13), we have

$$d(x)[x, y] + g(x)d[x, y] = 0 \quad \text{for all } x, y \in I.$$

Hence,

$$d(x)xy = d(x)yx \quad \text{for all } x, y \in I. \quad (1.14)$$

Replacing  $y$  by  $yt$  in (1.14) and using this, we get

$$d(x)I[x, t] = \{0\} \quad \text{for all } x \in I, t \in N.$$

Taking into account the Lemma 1, we get

$$d(x) = 0 \quad \text{or } x \in Z(N) \quad \text{for all } x \in I. \quad (1.15)$$

Since  $d$  is associated with an automorphism, we have  $d(x) \in Z(N)$  for each  $x \in Z(N)$ , then (1.15) illustrated  $d(I) \subseteq Z(N)$ . By the use of Theorem 1, we obtain  $N$  is a commutative ring.

Proving that ii)  $\Rightarrow$  iii). By the hypothesis given, we have

$$d([x, y]) = [x, y] \quad \text{for all } x, y \in I. \quad (1.16)$$

Replacing  $y$  by  $xy$  in (1.16), we get

$$xd([x, y]) + d(x)g([x, y]) = x[x, y] \quad \text{for all } x, y \in I.$$

It follows that

$$d(x)g(x)g(y) = d(x)g(y)g(x) \quad \text{for all } x, y \in I. \quad (1.17)$$

Since  $g$  is an automorphism, (1.17) shows that

$$d(x)g(x)j = d(x)jg(x) \quad \text{for all } x \in I, j \in J \quad (1.18)$$

with  $J = g(I)$ , it is clear that  $J$  is a semigroup ideal of  $N$ . Substituting  $jz$  for  $j$  in (1.18) and using this, we obtain

$$d(x)j[g(x), z] = 0 \quad \text{for all } x \in I, j \in J, z \in N. \quad (1.19)$$

Thus,

$$d(x)J[g(x), z] = \{0\} \quad \text{for all } x \in I, z \in N. \quad (1.20)$$

By the application of Lemma 1, (1.20) yields that

$$d(x) = 0 \quad \text{or } g(x) \in Z(N) \quad \text{for all } x \in I.$$

Which implies that

$$d(g(x)) \in Z(N) \quad \text{for all } x \in I$$

Consequently, we deduce that  $d(J) \subset Z(N)$ . And therefore, Theorem 1 assures that  $N$  is a commutative ring. This completes the proof of our theorem.  $\square$

**Corollary 3.** [4, Theorem 4.1] Let  $N$  be a 3-prime near-ring, and  $U$  a nonzero semigroup ideal. If  $N$  admits a derivation  $d$  such that  $d^2 \neq 0$  and  $d(uv) = d(vu)$  for all  $u, v \in U$ , then  $N$  is a commutative ring.

**Corollary 4.** Let  $N$  be a 3-prime near-ring. If  $N$  admits a nonzero derivation  $d$  such that  $d([x, y]) = 0$  for all  $x, y \in N$ , then  $N$  is a commutative ring.

**Corollary 5.** [8, Theorem 2.6] Let  $N$  be a 3-prime near-ring and  $I$  be a nonzero semigroup ideal of  $N$ . If  $N$  admits a nonzero derivation  $d$  such that  $d([x, y]) = [x, y]$  for all  $x, y \in I$ , then  $N$  is a commutative ring.

**Corollary 6.** [7, Theorem 2.2] Let  $N$  be a 3-prime near-ring. If  $N$  admits a nonzero derivation  $d$  such that  $d([x, y]) = [x, y]$  for all  $x, y \in N$ , then  $N$  is a commutative ring.

Now, replacing the commutator  $[x, y]$  by the anti-commutator  $x \circ y$ , our aim is to study this issue and to see if the results are different.

**Theorem 4.** Let  $N$  be a 2-torsion free 3-prime near-ring and  $I$  be a semigroup ideal of  $N$ , then  $N$  admits no nonzero semiderivation  $d$  satisfying one of the assertions as the following:

- i)  $d(x \circ y) = 0$  for all  $x, y \in I$ .
- ii)  $d(x \circ y) = x \circ y$  for all  $x, y \in I$ .

**Proof.** i) Suppose that there is  $d$  which indicates the following

$$d(x \circ y) = 0 \text{ for all } x, y \in I. \quad (1.21)$$

Replacing  $y$  by  $xy$  in (1.21) and taking the fact that  $x \circ xy = x(x \circ y)$ , we get

$$d(x)(x \circ y) = 0 \text{ for all } x, y \in I.$$

So that,

$$d(x)xy = -d(x)yx \text{ for all } x, y \in I. \quad (1.22)$$

Substituting  $yt$  for  $y$  in (1.22), we obtain

$$d(x)y(-x)t = d(x)yt(-x) \text{ for all } x, y \in I, t \in N$$

which can be rewritten as

$$d(x)I[-x, t] = \{0\} \text{ for all } x \in I, t \in N.$$

By using the lemma 1, we have

$$d(x) = 0 \text{ or } -x \in Z(N) \text{ for all } x \in I.$$

Hence,  $d(-x) \in Z(N)$  for all  $x \in I$ , it means that  $d(-I) \subset Z(N)$ . According to theorem 2, we get  $N$  is a commutative ring. In this case, returning to the hypothesis given, we have

$$d(xy) = 0 \text{ for all } x, y \in I.$$

It follows that

$$d(x)y + g(x)d(y) = 0 \text{ for all } x, y \in I \quad (1.23)$$

Taking  $yz$  instead of  $y$  in (1.23), we get

$$d(x)yz = 0 \text{ for all } x, y, z \in I.$$

Therefore,

$$d(x)Iz = \{0\} \text{ for all } x, z \in I.$$

By Lemma 1, the last expression shows that  $d = 0$ , a contradiction.

ii) Suppose there is  $d$  such that

$$d(x \circ y) = x \circ y \text{ for all } x, y \in I. \quad (1.24)$$

Putting  $xy$  instead of  $y$  in (1.24), we arrive at

$$d(x)g(x)g(y) = -d(x)g(y)g(x) \text{ for all } x, y \in I.$$

Which implies that

$$d(x)g(x)n = -d(x)ng(x) \text{ for all } x \in I, n \in J = g(I). \quad (1.25)$$

Writing  $nm$  instead of  $n$  in (1.25), we find that

$$d(x)n[g(-x), m] = 0 \text{ for all } x \in I, n \in J, m \in N$$

implying

$$d(x)J[g(-x), m] = \{0\} \text{ for all } x \in I, m \in N. \quad (1.26)$$

Applying Lemma 1, (1.26) shows that

$$d(x) = 0 \text{ or } g(-x) \in Z(N) \text{ for all } x \in I.$$

And therefore  $d(g(-x)) \in Z(N)$  for all  $x \in I$ , then  $d(-J) \subseteq Z(N)$ . According to Theorem 2, we conclude that  $N$  is a commutative ring. In this case, returning to the hypothesis given, we have

$$d(xy) = xy \text{ for all } x, y \in I$$

it follows that

$$d(x)y + g(x)d(y) = xy \text{ for all } x, y \in I. \quad (1.27)$$

Substituting  $xz$  for  $x$  in (1.27), we obtain

$$g(x)g(z)d(y) = 0 \text{ for all } x, y, z \in I.$$

Which can be rewritten as

$$g(x)Jd(y) = \{0\} \text{ for all } x, y \in I. \quad (1.28)$$

By Lemma 1, (1.28) demonstrates  $g(I) = \{0\}$  or  $d = 0$ , but each of these conditions yields a contradiction.  $\square$

**Corollary 7.** [8, Theorem 2.9] Let  $N$  be a 2-torsion free 3-prime near-ring, and  $I$  be a nonzero semigroup ideal of  $N$ . Then there is no derivation  $d$  such that  $d(x \circ y) = x \circ y$  for all  $x, y \in I$ .

The following example shows that the primeness is necessary in the hypotheses of the above theorems.

**Example** Let  $S$  be a 2-torsion free noncommutative near-ring. Let us define  $N$  and  $d, g : N \rightarrow N$  by:

$$N = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\}.$$

$$d \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, it is straightforward to check that  $N$  is not 3-prime left near-ring admitting a nonzero semiderivation  $d$  associated with  $g$ . Moreover; it is easy to verify that  $d$  satisfies the properties:

- i)**  $d(N) \subseteq Z(N)$       **ii)**  $d(-N) \subseteq Z(N)$       **iii)**  $d([A, B]) = 0$   
**iv)**  $d([A, B]) = [A, B]$       **v)**  $d(A \circ B) = 0$       **vi)**  $d(A \circ B) = A \circ B$

for all  $A, B \in N$ . However,  $N$  is not commutative.

## References

- [1] M. Ashraf and A. Shakir, *On  $(\sigma, \tau)$ -derivations of prime near-rings II*, *Sarajevo J. Math.*, 4 (2008), no.16, 23-30.
- [2] H. E. Bell and G. Mason, *On derivations in near-rings*, *North-Holland Mathematics Studies*, 137 (1987), 31-35.
- [3] H. E. Bell and G. Mason, *On derivations in near-rings and rings*, *Math. J. Okayama Univ.*, 34 (1992), 135-144.
- [4] H. E. Bell, *On derivations in near-rings II*, *Kluwer Academic Publishers Netherlands* (1997), 191-197.
- [5] H. E. Bell and N. Argac, *Derivations, products of derivations, and commutativity in near-rings*, *Algebra Colloq.*, 8 (2001), no.4, 399-407.
- [6] H. E. Bell, A. Boua, and L. Oukhtite, *Semigroup ideals and commutativity in 3-prime near rings*, *Communications in Algebra*, (to appear).
- [7] A. Boua and L. Oukhtite, *Derivations on prime near-rings*, *Int. J. Open Probl. Comput. Sci. Math.*, 4 (2011), no.2, 162-167.
- [8] A. Boua, *Some conditions under which prime near-rings are commutative rings*, *Int. J. Open Probl. Comput. Sci. Math.*, 5(2012), no.2, 7-15.

- [9] A. Boua, H. E. Bell and L. Oukhtite, *On derivations of prime near-rings*, *Afr. Diaspora J. Math.*, 14 (2012), no.1, 65-72.
- [10] A. Boua and L. Oukhtite, *Semiderivations satisfying certain algebraic identities on prime near-rings*, *Asian-Eur. J. Math.*, 6 (3)(2013) 1350043 (8 pages).
- [11] A. Boua, L. Oukhtite and H. E. Bell, *Differential identities on semigroup ideals of right near-rings*, *Asian-Eur. J. Math.*, 6 (4) (2013) 1350050 (9 pages).
- [12] M. N. Daif and H. E. Bell, *Remarks on derivations on semiprime rings*, *Int. J. Math. Math. Sci.*, 15 (1992), no.1, 205-206.
- [13] X. K. Wang, *Derivations in prime near-rings*, *Proc. Amer. Math. Soc.*, 121 (1994), 361-366.

### Author information

A. Boua, Ibn Zohr University, Faculty of Sciences of Agadir, Department of Mathematics, P. O. Box 8106, Agadir, Morocco.

E-mail: karimoun2006@yahoo.fr

L. Oukhtite and A. Raji, Université Moulay Ismaïl, Faculté des Sciences et Techniques Département de Mathématiques, Groupe d'Algèbre et Applications B.P. 509 Boutalamine, Errachidia, Morocco.

E-mail: oukhtitel@hotmail.com; rajiabd2@gmail.com

Received: February 10, 2014.

Accepted: May 8, 2014.