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On some new analytic function spaces in polyball

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Abstract. In this note we extend in two directions known classical results for Bergman spaces in their unit ball to polyballs and mixed norm spaces simultaneously.

1 Introduction

The intention of this note is to extend some results from [4] for analytic spaces on polydisk to analytic function spaces in polyballs.

Note the first results of this type were obtained by authors in [5]. The intention of this paper to continue that investigation. For formulation of main result of this note we need several basic definitions taken from [5].

Let $\mathbb{C}$ denote the set of complex numbers and let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n. The open unit ball in $\mathbb{C}^n$ is the set $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$. We denote by $H(B_n)$ the space of holomorphic functions on the open unit ball in $\mathbb{C}^n$.

For every function $f \in H(B_n)$ having a series expansion $f(z) = \sum_{|k|\geq 0} a_k z^k$, we define the operator of fractional differentiation by

$$D^\alpha f(z) = \sum_{|k|\geq 0} (|k|+1)^\alpha a_k z^k,$$

where α is any real number. It is obvious that for any α, D^α operator is acting from $H(B_n)$ to $H(B_n)$.

We will apply this derivative to analytic functions in polyballs by each variable separately and here α will be also as vector $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

Moreover, let $d\nu$ denote the Lebesgue measure on B_n normalized such that $\nu(B_n) = 1$ and for any $\alpha \in R$, let $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z)$ for $z \in B_n$. Here, if $\alpha \leq -1$, $c_\alpha = 1$ and if $\alpha > -1$, $c_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$ is the normalizing constant so that ν_α has unit total mass. The Bergman metric on B_n is

$$\beta(z, w) = \frac{1}{2} \log \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|},$$

where φ_z is the Möbius transformation of B_n that interchanges 0 and z. Let $\mathcal{D}(a, r) = \{z \in B_n : \beta(z, a) < r\}$ denote the Bergman metric ball centered at $a \in B_n$ with radius $r > 0$.

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The following lemmas is the key result, a base for all our proofs.

Lemma 1. [7] (a) There exists a positive number $N \geq 1$ such that for any $0 < r \leq 1$ we can find a sequence $\{v_k\}_{k=1}^\infty$ in B_n to be r-lattice in the Bergman metric of B_n. This means that $B_n = \cup_{k=1}^\infty \mathcal{D}(v_k, r)$, $\mathcal{D}(v_l, r/4) \cap \mathcal{D}(v_k, r/4) = \emptyset$ if $k \neq l$ and each $z \in B_n$ belongs to at most N of the sets $\mathcal{D}(v_k, 2r)$.
(b) For any $r > 0$ there is a constant $C > 0$ so that $\frac{1}{C} \leq |\frac{1-\langle z,w\rangle}{1-\langle z,v\rangle}| \leq C$ for all $z \in B_n$ and all w, v with $\beta(w, v) < r$.
(c) For any $\alpha > -1$ and $r > 0$, $\int_{\mathcal{D}(z,r)} (1-|w|^2)^\alpha d\nu(w)$ is comparable with $(1-|z|^2)^{n+1+\alpha}$ for all $z \in B_n$.
(d) Suppose $r > 0$ and $p > 0$ and $\alpha > -1$. Then there is a constant $C > 0$ such that

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{\mathcal{D}(z,r)} |f(w)|^p d\nu_\alpha(w),$$

for all $f \in H(B_n)$ and all $z \in B_n$.

Lemma 2. [7] Suppose that $c > 0$ and $t > -1$. Then there are positive constants $C_1,\ C_2$ such that
$$C_1 \frac{\Gamma(t+1)\Gamma(c)}{(1-|z|^2)^c} \le \int_{B_n} \frac{1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}} d\nu(w) \le C_2 \frac{\Gamma(t+1)\Gamma(c)}{(1-|z|^2)^c},$$
for all $z \in B_n$. The constants C_1 and C_2 depend on n, c and t and they are bounded as $t \to -1$ and $s \to 0$.

Let B_n^m denote the polyball $B_n^m = B_n \times \ldots \times B_n$. Let also $S_n^m = S_n \times \ldots \times S_n$, where $S_n = \{z \in \mathbb{C}^n : |z| = 1\}$. As usual, we denote by $H(B_n^m)$ the space of all analytic functions in B_n^m by each variable separately.

Let $dm(\xi)$ be Lebesque measure on B_n^m and $d\xi$ be Lebesque measure on S_n^m.

In this paper from one hand we extend known classical results on Bergman spaces in unit ball to mixed norm spaces in polyballs. From the other hand our results for $n = 1$ coincide with results taken from [4]. The properties of r-lattice are the base of all mentioned results and results of this paper. Arguments for our proofs we take from [4] where the case of polydisk was considered.

2 Main results

We now introduce the mixed norm classes in polyballs
$$A^{p_1,\cdots,p_m}_{\alpha_1,\ldots,\alpha_m}(B_n^m) = \{f \in H(B_n^m) : \|f\|_{A^{p_1,\cdots,p_m}_{\alpha_1,\ldots,\alpha_m}} := (\int_{B_n} (1-|z_m|)^{\alpha_m} (\int_{B_n} (1-|z_{m-1}|)^{\alpha_{m-1}} \cdots$$
$$\cdots \int_{B_n} |f(z_1,\ldots,z_m)|^{p_1} (1-|z_1|)^{\alpha_1} d\nu(z_1))^{\frac{p_2}{p_1}} \cdots d\nu(z_{m-1}))^{\frac{p_m}{p_{m-1}}} d\nu(z_m))^{\frac{1}{p_m}} < \infty\},$$
where $0 < p_i < \infty$, $\alpha_i > -1$, $i = 1,\ldots,m$. Note that for $n = 1$ these classes were studied in [4]. For $m = 1$ we have the classical Bergman spaces on the unit ball. Formally replacing B_n by $\mathbb{R}^n$ we arrive at well studied function classes in $\mathbb{R}^n$ (see [1], [3]).

Let $L^{p_1,\cdots,p_m}_{\alpha_1,\ldots,\alpha_m}(B_n^m)$ denote the space of all measurable functions $f : B_n^m \to \mathbb{C}$ such that $\|f\|_{A^{p_1,\cdots,p_m}_{\alpha_1,\ldots,\alpha_m}} < \infty$. It is not difficult to show that $A^{p_1,\cdots,p_m}_{\alpha_1,\ldots,\alpha_m}(B_n^m)$ is a Banach space for $1 \le p_i < \infty$, $i = 1,\ldots,m$. Moreover, it can be shown that $A^{p_1,\cdots,p_m}_{\alpha_1,\ldots,\alpha_m}(B_n^m)$ is a complete metric space for $0 < p_i < 1$, $i = 1,\ldots,m$.

For $f \in A^{p_1,\cdots,p_m}_{\alpha_1,\ldots,\alpha_m}(B_n^m)$, we have the following estimate
$$|f(z_1,\ldots,z_m)| \le C \frac{||f||_{A^{p_1,\cdots,p_m}_{\alpha_1,\ldots,\alpha_m}}}{\prod_{k=1}^m (1-|z_k|)^{\frac{\alpha_k}{p_k}+\frac{n+1}{p_k}}}, \tag{2.1}$$
where $z_j \in B_n$, $j = 1,\ldots,m$. The proof of (2.1) can be obtained by modification of standard arguments from [7].

Our intention is to prove projection theorems and theorems on representation of functionals on these spaces. For $n = 1$ or $m = 1$ (polydisk case) all our results are known (see [4], [7]).

Theorem 1. Let $p_j \in (1,\infty),\ \alpha_j > -1,\ j = 1,\ldots,m$. Then the Bergman type $T_{\overrightarrow{\alpha}}$ operator
$$T_{\overrightarrow{\alpha}} f(z) = \int_{B_n^m} \frac{f(\xi)\prod_{j=1}^m (1-|\xi_j|)^{\alpha_j}}{\prod_{j=1}^m (1-z_j\overline{\xi_j})^{\alpha_j+n+1}} dm(\xi),\ z \in B_n^m,$$
maps $L^{\overrightarrow{p}}(\overrightarrow{\alpha})$ to $A^{\overrightarrow{p}}(\overrightarrow{\alpha})$ and $\|T_{\overrightarrow{\alpha}} f\|_{A^{\overrightarrow{p}}_{\overrightarrow{\alpha}}} \le C\|f\|_{L^{\overrightarrow{p}}_{\overrightarrow{\alpha}}}$.

Theorem 2. Let Φ be bounded linear functional on $A^{\overrightarrow{p}}_{\overrightarrow{\alpha}}$, $1 < p_j < \infty,\ \alpha_j > -1,\ j = 1,\ldots,m$. Let $g(z) = \Phi(l_z) = \Phi\left(\prod_{j=1}^m \frac{1}{1-z_j\overline{\xi_j}}\right),\ z \in B_n^m$. Then

(A) $g \in H(B_n^m)$, $D^{\alpha+1}_{z_1\ldots z_m} g \in A^{\overrightarrow{q}}_{\overrightarrow{\alpha}}$, $\frac{1}{p_j} + \frac{1}{q_j} = 1,\ j = 1,\ldots,m$ and
$$\Phi(f) = \lim_{\rho\to 1-0} \int_{S_n^m} f(\rho\xi)\, g(\rho\xi) d\xi \ \ and$$

(B) $\|\Phi\| \asymp \|D^{\alpha+1}_{z_1,\ldots,z_m} g\|_{A^{\overrightarrow{q}}_{\alpha}}$,

and the reverse is also true: each g function so that $D^{\alpha+1} g \in A^{\overrightarrow{q}}_{\overrightarrow{\alpha}}$ by (A) produce a bounded linear functional on $A^{\overrightarrow{p}}_{\overrightarrow{\alpha}}$, $\alpha_j > -1,\ 1 < p_j < \infty,\ j = 1,\ldots,m$ for which estimate (B) holds.

Let $0 < p_j \leqslant 1,\ j = 1, \ldots, m$. Let us denote

$$\lambda_{\overrightarrow{\alpha}}^{\overrightarrow{p}} = \left\{ f \in H(B_n^m) : |D_{z_1,\ldots,z_m}^{\widetilde{\alpha}+1} g(z_1, \ldots, z_m)| \le C \prod_{j=1}^{m} (1 - |z_j|)^{(\alpha_j+n+1)/p_j - \widetilde{\alpha} - (n+1)},\ z_j \in B_n \right\},$$

where $\widetilde{\alpha} > \frac{\alpha_j+n+1}{p_j} - (n+1),\ j = 1, \ldots, m$. It can be shown as in case of polydisk (see [4]) these spaces are independent from $\widetilde{\alpha}$.

Theorem 3. Let Φ be bounded linear functional on $A_{\overrightarrow{\alpha}}^{\overrightarrow{p}}$, $0 < p_j \le 1,\ \alpha_j > -1,\ j = 1, \ldots, m$. Let $g(z) = \Phi(l_z) = \Phi\left(\prod_{j=1}^{m} \frac{1}{1 - z_j \overline{\xi_j}}\right),\ z \in B_n^m$. Then

(A) $g \in \lambda_{\overrightarrow{\alpha}}^{\overrightarrow{p}}$ and $\Phi(f) = \lim_{\rho \to 1-0} \int_{S_n^m} f(\rho\xi) g(\rho\xi) d\xi$ and

(B) $\|\Phi\| \asymp \|g\|_{\lambda_{\overrightarrow{\alpha}}^{\overrightarrow{p}}}$,

and the reverse is also true: each g function $g \in \lambda_{\overrightarrow{\alpha}}^{\overrightarrow{p}}$ by (A) produce a bounded linear functional on $A_{\overrightarrow{\alpha}}^{\overrightarrow{p}},\ \alpha_j > -1,\ 0 < p_j < 1,\ j = 1, \ldots, m$ for which estimate (B) holds.

Let us denote by S the class of all slowly varying functions, i.e. the class of all positive measurable functions $\omega(t)$ on $(0, 1]$ such that there are constants $m = m_\omega,\ M = M_\omega$ and $= q_\omega$ satisfying: $0 < m, q < 1$ and

$$m \le \frac{\omega(\lambda r)}{\omega(r)} \le M, \quad 0 < r < 1, \quad q \le \lambda \le 1,$$

see [2], [6] for detailed study of such functions. The constants m, M, q are the structural constants of the slowly varying function ω. We note that functions $\omega(r) = r^\alpha, \alpha \in \mathbb{R}$ are in class S. In fact, for any $\omega \in S$ there is an $\beta \ge 0$ depending on the structural constants of ω such that $\omega(r) \ge Cr^\beta, 0 < r \le 1$.

Let $A_\omega^{\overrightarrow{p}},\ 1 < p_j < \infty,\ j = 1, \ldots, m$ denote mixed norm space:

$$A_\omega^{\overrightarrow{p}}(B_n^m) = \{f \in H(B_n^m) : \|f\|_{A_\omega^{\overrightarrow{p}}} := (\int_{B_n} \omega(1 - |z_m|)(\int_{B_n} \omega(1 - |z_{m-1}|) \cdots$$

$$\cdots \int_{B_n} |f(z_1, \ldots, z_m)|^{p_1} \omega(1 - |z_1|) d\nu(z_1))^{\frac{p_2}{p_1}} \cdots d\nu(z_{m-1}))^{\frac{p_m}{p_{m-1}}} d\nu(z_m))^{\frac{1}{p_m}} < \infty\}.$$

Some results of this paper can be extended to more general spaces $A_\omega^{\overrightarrow{p}}$.

Remarks. Using nice properties of recently invented r-lattices of pseudoconvex domains with smooth boundary projection results of this note can be partially extended to bounded strongly pseudoconvex domains with smooth boundary by similar arguments.

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