On some new analytic function spaces in polyball

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Abstract. In this note we extend in two directions known classical results for Bergman spaces in their unit ball to polyballs and mixed norm spaces simultaneously.

1 Introduction

The intention of this note is to extend some results from [4] for analytic spaces on polydisk to analytic function spaces in polyballs.

Note the first results of this type were obtained by authors in [5]. The intention of this paper to continue that investigation. For formulation of main result of this note we need several basic definitions taken from [5].

Let \( \mathbb{C} \) denote the set of complex numbers and let \( \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C} \) denote the Euclidean space of complex dimension \( n \). The open unit ball in \( \mathbb{C}^n \) is the set \( B_n = \{ z \in \mathbb{C}^n : |z| < 1 \} \). We denote by \( H(B_n) \) the space of holomorphic functions on the open unit ball in \( \mathbb{C}^n \).

For every function \( f \in H(B_n) \) having a series expansion \( f(z) = \sum_{|k| \geq 0} a_k z^k \), we define the operator of fractional differentiation by

\[
D^\alpha f(z) = \sum_{|k| \geq 0} (|k| + 1)^\alpha a_k z^k,
\]

where \( \alpha \) is any real number. It is obvious that for any \( \alpha, D^\alpha \) operator is acting from \( H(B_n) \) to \( H(B_n) \).

We will apply this derivative to analytic functions in polyballs by each variable separately and here \( \alpha \) will be also as vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \).

Moreover, let \( dv \) denote the Lebesgue measure on \( B_n \) normalized such that \( \nu(B_n) = 1 \) and for any \( \alpha \in \mathbb{R} \), let \( dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha dv(z) \) for \( z \in B_n \). Here, if \( \alpha < -1 \), \( c_\alpha = 1 \) and if \( \alpha > -1 \), \( c_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \) is the normalizing constant so that \( \nu_\alpha \) has unit total mass. The Bergman metric on \( B_n \) is

\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|},
\]

where \( \varphi_z \) is the Möbius transformation of \( B_n \) that interchanges 0 and \( z \). Let \( D(a, r) = \{ z \in B_n : \beta(z, a) < r \} \) denote the Bergman metric ball centered at \( a \in B_n \) with radius \( r > 0 \).

Throughout the paper, we write \( C \) (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The following lemmas is the key result, a base for all our proofs.

Lemma 1. [7] (a) There exists a positive number \( N \geq 1 \) such that for any \( 0 < r \leq 1 \) we can find a sequence \( \{v_k\}_{k=1}^\infty \) in \( B_n \) to be r-lattice in the Bergman metric of \( B_n \). This means that \( B_n = \bigcup_{k=1}^\infty D(v_k, r) \cap D(v_k, r/4) = \emptyset \) if \( k \neq \ell \) and each \( z \in B_n \) belongs to at most \( N \) of the sets \( D(v_k, 2r) \).

(b) For any \( r > 0 \) there is a constant \( C > 0 \) so that \( \frac{1}{C} \leq |\frac{1 - |(z, w)|}{1 - |(z, v)|}| \leq C \) for all \( z \in B_n \) and all \( w, v \) with \( \beta(w, v) < r \).

(c) For any \( \alpha > -1 \) and \( r > 0 \), \( \int_{D(z, r)} (1 - |w|^2)^\alpha dv(w) \) is comparable with \( (1 - |z|^2)^{n+1+\alpha} \) for all \( z \in B_n \).

(d) Suppose \( r > 0 \) and \( p > 0 \) and \( \alpha > -1 \). Then there is a constant \( C > 0 \) such that

\[
|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z, r)} |f(w)|^p dv_\alpha(w),
\]
for all $f \in H(B_n)$ and all $z \in B_n$.

**Lemma 2.** [7] Suppose that $c > 0$ and $t > -1$. Then there are positive constants $C_1$, $C_2$ such that

$$C_1 \frac{\Gamma(t+1)\Gamma(c)}{(1-|z|^2)^c} \leq \int_{B_n} \frac{1-|w|^2}{{(1-\langle z, w \rangle)^{n+1+t+c}} \ dv(w) \leq C_2 \frac{\Gamma(t+1)\Gamma(c)}{(1-|z|^2)^c},$$

for all $z \in B_n$. The constants $C_1$ and $C_2$ depend on $n, c$ and $t$ and they are bounded as $t \to -1$ and $s \to 0$.

Let $B_n^m$ denote the polyball $B_n^m = B_n \times \ldots \times B_n$. Let also $S_n^m = S_n \times \ldots \times S_n$, where $S_n = \{ z \in \mathbb{C}^n : |z| = 1 \}$. As usual, we denote by $H(B_n^m)$ the space of all analytic functions in $B_n^m$ by each variable separately.

Let $dm(\xi)$ be Lebesque measure on $B_n^m$ and $d\xi$ be Lebesque measure on $S_n^m$.

In this paper from one hand we extend known classical results on Bergman spaces in unit ball to mixed norm spaces in polyballs. From the other hand our results for $n = 1$ coincide with results taken from [4]. The properties of $r$-lattice are the base of all mentioned results and results of this paper. Arguments for our proofs we take from [4] where the case of polydisk was considered.

## 2 Main results

We now introduce the mixed norm classes in polyballs

$$A_{\alpha_1, \ldots, \alpha_m}^{p_1, \ldots, p_m}(B_n^m) = \{ f \in H(B_n^m) : ||f||_{A_{\alpha_1, \ldots, \alpha_m}^{p_1, \ldots, p_m}} := (\int_{B_n^m} (1-|z_m|)^{\alpha_m} (\int_{B_n^m} (1-|z_{m-1}|)^{\alpha_{m-1}} \ldots 

\cdots (\int_{B_n^m} |f(z_1, \ldots, z_m)|^{p_1} (1-|z_1|)^{\alpha_1} dv(z_1) \int_{B_n^m} \cdots \int_{B_n^m} dv(z_{m-1}))^{p_{m-1}} dm(z_m))^{1/p_m} < \infty \},$$

where $0 < p_i < \infty$, $\alpha_i > -1$, $i = 1, \ldots, m$. Note that for $n = 1$ these classes were studied in [4]. For $m = 1$ we have the classical Bergman spaces on the unit ball. Formally replacing $B_n$ by $\mathbb{R}^n$ we arrive at well studied function classes in $\mathbb{R}^n$ (see [1], [3]).

Let $L_{p_1, \ldots, p_m}(B_n^m)$ denote the space of all measurable functions $f : B_n^m \to C$ such that $||f||_{L_{p_1, \ldots, p_m}(B_n^m)} < \infty$. It is not difficult to show that $A_{\alpha_1, \ldots, \alpha_m}^{p_1, \ldots, p_m}(B_n^m)$ is a Banach space for $1 \leq p_i < \infty$, $i = 1, \ldots, m$. Moreover, it can be shown that $A_{\alpha_1, \ldots, \alpha_m}^{p_1, \ldots, p_m}(B_n^m)$ is a complete metric space for $0 < p_i < 1$, $i = 1, \ldots, m$.

For $f \in A_{\alpha_1, \ldots, \alpha_m}^{p_1, \ldots, p_m}(B_n^m)$, we have the following estimate

$$|f(z_1, \ldots, z_m)| \leq C \frac{||f||_{A_{\alpha_1, \ldots, \alpha_m}^{p_1, \ldots, p_m}}}{\prod_{k=1}^{m} (1-|z_k|)^{\frac{\alpha_k}{p_k} + \frac{n+1}{p_k}}}, \quad (2.1)$$

where $z_j \in B_n$, $j = 1, \ldots, m$. The proof of (2.1) can be obtained by modification of standard arguments from [7].

Our intention is to prove projection theorems and theorems on representation of functionals on these spaces. For $n = 1$ or $m = 1$ (polydisk case) all our results are known (see [4], [7]).

**Theorem 1.** Let $p_j \in (1, \infty)$, $\alpha_j > -1$, $j = 1, \ldots, m$. Then the Bergman type $T_{\alpha\overrightarrow{\alpha}}$ operator

$$T_{\alpha\overrightarrow{\alpha}} f(z) = \int_{B_n^m} f(\xi) \prod_{j=1}^{m} (1-|\xi_j|)^{\alpha_j} \prod_{j=1}^{m} (1-|z_j\xi_j|)^{\alpha_j + n+1} d\mu(\xi), \quad z \in B_n^m,$$

maps $L^2_{\alpha\overrightarrow{\alpha}}(\mathbb{R}^n)$ to $A^2_{\alpha\overrightarrow{\alpha}}(\mathbb{R}^n)$ and $||T_{\alpha\overrightarrow{\alpha}} f||_{A^2_{\alpha\overrightarrow{\alpha}}} \leq C ||f||_{L^2_{\alpha\overrightarrow{\alpha}}}.$

**Theorem 2.** Let $\Phi$ be bounded linear functional on $A^2_{\alpha\overrightarrow{\alpha}}$, $1 < p_j < \infty$, $\alpha_j > -1$, $j = 1, \ldots, m$.

Let $g(z) = \Phi(\xi_2) = \Phi \left( \prod_{j=1}^{m} \frac{1}{1-\xi_j z_j} \right)$, $z \in B_n^m$. Then

(A) $g \in H(B_n^m)$, $D_{\alpha_j}^{\alpha_j + 1} g \in A^2_{\alpha\overrightarrow{\alpha}}$, $1 \leq \frac{1}{p_j} + \frac{1}{\alpha_j} = 1$, $j = 1, \ldots, m$ and

$$\Phi(f) = \lim_{\rho \to 1^-} \int_{S_n^m} f(\rho \xi) \ g(\rho \xi) d\xi$$

and

(B) $||\Phi|| \geq ||D_{\alpha_j}^{\alpha_j + 1} g||_{A^2_{\alpha\overrightarrow{\alpha}}},$

and the reverse is also true: each $g$ function so that $D_{\alpha_j}^{\alpha_j + 1} g \in A^2_{\alpha\overrightarrow{\alpha}}$ by (A) produce a bounded linear functional on $A^2_{\alpha\overrightarrow{\alpha}}$, $\alpha_j > -1$, $1 < p_j < \infty$, $j = 1, \ldots, m$ for which estimate (B) holds.
Let $0 < p_j \leq 1$, $j = 1, \ldots, m$. Let us denote
\[
\lambda_{\alpha}^p \left\{ f \in H(B_n^m) : |D^{\alpha+1}_{z_1, \ldots, z_m} g(z_1, \ldots, z_m)| \leq C \prod_{j=1}^m (1 - |z_j|)^{(\alpha_j+n+1)/p_j - \alpha_j -(n+1)}, z_j \in B_n \right\},
\]
where $\alpha > \frac{\alpha_j+n+1}{p_j} - (n+1)$, $j = 1, \ldots, m$. It can be shown as in case of polydisk (see [4]) these spaces are independent from $\alpha$.

**Theorem 3.** Let $\Phi$ be bounded linear functional on $A^p_{\alpha}$, $0 < p_j \leq 1$, $\alpha_j > -1$, $j = 1, \ldots, m$.

Let $g(z) = \Phi(l_z) = \Phi \left( \prod_{j=1}^m \frac{1}{1-z_j^2} \right)$, $z \in B_n^m$. Then

(A) $g \in \lambda_{\alpha}^p$ and $\Phi(f) = \lim_{\rho \to 0} \int_{S^{n-1}} f(\rho \xi)g(\rho \xi)d\xi$ and

(B) $\|\Phi\| \approx \|g\|_{L_{\alpha}^p}$.

and the reverse is also true: each $g$ function $g \in \lambda_{\alpha}^p$ by (A) produce a bounded linear functional on $A^p_{\alpha}$, $\alpha_j > -1$, $0 < p_j < 1$, $j = 1, \ldots, m$ for which estimate (B) holds.

Let us denote by $S$ the class of all slowly varying functions, i.e. the class of all positive measurable functions $\omega(t)$ on $(0, 1]$ such that there are constants $m = m_\omega$, $M = M_\omega$ and $q_\omega$ satisfying: $0 < m_\omega < q_\omega$ for which
\[
m \omega(x\lambda) \leq M, \quad 0 < r < 1, \quad q \leq \lambda \leq 1,
\]
see [2], [6] for detailed study of such functions. The constants $m_\omega$, $M_\omega$, $q$ are the structural constants of the slowly varying function $\omega$. We note that functions $\omega(r) = r^\alpha$, $\alpha \in \mathbb{R}$ in class $S$. In fact, for any $\omega \in S$ there is an $\beta \geq 0$ depending on the structural constants of $\omega$ such that $\omega(r) \geq Cr^\beta$, $0 < r \leq 1$.

Let $A^p_{\alpha}$, $1 < p_j < \infty$, $j = 1, \ldots, m$ denote mixed norm space:
\[
A^p_{\alpha}(B_n^m) = \{ f \in H(B_n^m) : \|f\|_{A^p_{\alpha}} := (\int_{B_n} \omega(1 - |z_m|)(\int_{B_n} \omega(1 - |z_{m-1}|)) \cdots \\
\cdots (\int_{B_n} |f(z_1, \ldots, z_m)|^{p_m}\omega(1 - |z_1|)\omega(z_1) \frac{dm}{\omega(z_1)} \cdots \omega(z_{m-1}) \frac{dm}{\omega(z_{m-1})})^{\frac{1}{p_m}} < \infty \}.
\]

Some results of this paper can be extended to more general spaces $A^p_{\alpha}$. 

**Remarks.** Using nice properties of recently invented r-lattices of pseudoconvex domains with smooth boundary projection results of this note can be partially extended to bounded strongly pseudoconvex domains with smooth boundary by similar arguments.

**References**


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