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ON GENERALIZATION OF SISTER CELINE'S POLYNOMIALS

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Abstract. This paper deals with a generalization of Sister Celine's polynomials, their generating functions and integral representations. A set of relations with other polynomials are given in the last section.

1 Introduction

Sister Celine [1] has introduced the polynomial $f_n(x)$

$$f_n\left[\begin{array}{ccc} a_1, & \cdots, & a_p; \\ b_1, & \cdots, & b_q; \end{array} x\right] = {}_{p+2}F_{q+2}\left[\begin{array}{cccccc} -n, & n+1, & a_1, & \cdots & \cdots & a_p; \\ & 1, & \frac{1}{2}, & b_1, & \cdots, & b_q; \end{array} x\right], \tag{1.1}$$

which is defined by the following generating function (see [4, p.290])

$$(1-t)^{-1}\ {}_pF_q\left[\begin{array}{cccc} a_1, & \cdots, & a_p; & \frac{-4xt}{(1-t)^2} \\ b_1, & \cdots, & b_q; & \end{array}\right] = \sum_{n=0}^{\infty} f_n\left[\begin{array}{ccc} a_1, & \cdots, & a_p; \\ b_1, & \cdots, & b_q; \end{array} x\right] t^n, \tag{1.2}$$

where ${}_pF_q$ denotes the generalized hypergeometric function [4].

For $p = 1, q = 1, a_1 = \frac{1}{2}, b_1 = 1$ the following integral representation of Sister Celine polynomials is given

$$f_n(\frac{1}{2};1;x) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} y^{-\frac{1}{2}}e^{-y}f_n(-;1;xy)dy. \tag{1.3}$$

Equation (1.1) with no a's and no b's denotes simply

$$f_n(x) = {}_2F_2\left[-n, n+1; 1, \frac{1}{2}; x\right] = \sum_{r=0}^{n} \frac{(-1)^r (n)! x^r}{(r!)^2 (\frac{1}{2})_r (n-r)!}. \tag{1.4}$$

For the $f_n(x)$ the generating function (1.2) becomes

$$(1-t)^{-1}exp(\frac{-4xt}{(1-t)^2}) = \sum_{n=0}^{\infty} f_n(x)t^n. \tag{1.5}$$

2 Generalization of Sister Celine's Polynomials

In the view of above results, we define the generalized Sister Celine polynomial in following manner

$$f_n^{(\alpha,\beta)}\left[\begin{array}{ccc} a_1, & \cdots, & a_p; \\ b_1, & \cdots, & b_q; \end{array} x\right] = \frac{(1+\alpha+\beta)_n}{n!}$$

$${}_{p+2}F_{q+2}\left[\begin{array}{ccccc} -n, & n+\alpha+\beta+1, & a_1, & \cdots, & a_p; \\ 1+\alpha, & \frac{1}{2}, & b_1, & \cdots, & b_q; \end{array} x\right]. \tag{2.1}$$

Equation (2.1) with no a's and no b's denotes simply

$$f_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_n}{n!} {}_2F_2 \left[\begin{matrix} -n, & & n+\alpha+\beta+1; \\ & 1+\alpha, & \frac{1}{2}; \end{matrix} \; x \right] \tag{2.2}$$

$$= \frac{(1+\alpha+\beta)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r (n+\alpha+\beta+1)_r x^r}{(1+\alpha)_r (\frac{1}{2})_r r!}. \tag{2.3}$$

Indeed $$f_n^{(0,0)}(x) = f_n(x). \tag{2.4}$$

3 Generating functions

The following generating function can be easily obtained

$$\sum_{n=0}^{\infty} f_n^{(\alpha,\beta)}(x) t^n = (1-t)^{-1-\alpha-\beta} {}_2F_2 \left[\begin{matrix} \frac{1+\alpha+\beta}{2}, & \frac{2+\alpha+\beta}{2}; \\ 1+\alpha, & \frac{1}{2}; \end{matrix} \; \frac{-4xt}{(1-t)^2} \right], \tag{3.1}$$

$$\sum_{n=0}^{\infty} (C)_n f_n^{(\alpha,\beta)}(x) t^n = (1-t)^{-C-\alpha-\beta} {}_3F_3 \left[\begin{matrix} C, & \frac{1+\alpha+\beta}{2}, & \frac{2+\alpha+\beta}{2}; \\ 1+\alpha, & 1+\alpha+\beta & \frac{1}{2}; \end{matrix} \; \frac{-4xt}{(1-t)^2} \right] \tag{3.2}$$

(Obviously for $C = 1, \alpha = \beta = 0$, equation (3.2) reduces to the generating function (1.5)), and

$$(1-t)^{-1-\alpha-\beta} {}_{p+2}F_{q+2} \left[\begin{matrix} a_1, \cdots a_p, & \frac{1+\alpha+\beta}{2}, & \frac{2+\alpha+\beta}{2}; \\ b_1, \cdots b_q, & 1+\alpha, & \frac{1}{2}; \end{matrix} \; \frac{-4xt}{(1-t)^2} \right]$$

$$= \sum_{n=0}^{\infty} f_n^{(\alpha,\beta)} \left[\begin{matrix} a_1, \cdots a_p; \\ b_1, \cdots b_q; \end{matrix} \; x \right] t^n. \tag{3.3}$$

Proof. Expanding left hand member of (3.1) with the help of (2.2), we get

$$\sum_{n=0}^{\infty} f_n^{(\alpha,\beta)}(x) t^n = \frac{(1+\alpha+\beta)_n}{n!} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-n)_r (1+\alpha+\beta+n)_r}{(1+\alpha)_r (\frac{1}{2})_r} \frac{x^r}{r!} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^r}{(n-r)!} \frac{(1+\alpha+\beta)_{n+r}}{(1+\alpha)_r (\frac{1}{2})_r} \frac{x^r}{r!} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r}{(n)!} \frac{(1+\alpha+\beta)_{n+2r}}{(1+\alpha)_r (\frac{1}{2})_r} \frac{x^r}{r!} t^{n+r}$$

$$= \sum_{r=0}^{\infty} \frac{(1+\alpha+\beta)_{2r} (-xt)^r}{(1+\alpha)_r (\frac{1}{2})_r r!} \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta+2r)_n}{n!} t^n$$

$$= \sum_{r=0}^{\infty} \frac{(1+\alpha+\beta)_{2r} (-xt)^r}{(1+\alpha)_r (\frac{1}{2})_r r! (1-t)^{1+\alpha+\beta+2r}}$$

$$= (1-t)^{-1-\alpha-\beta} \sum_{r=0}^{\infty} \frac{(1+\alpha+\beta)_{2r} (-xt)^r}{(1+\alpha)_r (\frac{1}{2})_r \; r! (1-t)^{2r}}$$

$$= (1-t)^{-1-\alpha-\beta} \sum_{r=0}^{\infty} \frac{(\frac{1+\alpha+\beta}{2})_r (\frac{2+\alpha+\beta}{2})_r}{(1+\alpha)_r (\frac{1}{2})_r} \frac{(\frac{-4xt}{(1-t)^2})^r}{r!}$$

Thus we arrive at the result (3.1). Similarly we can easily prove equations (3.2) and (3.3). □

4 Integral representations

The following integral representations involving Sister Celine's polynomials are obtained as follows:

$$(a). \qquad f_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_n}{(1+\alpha)_n\sqrt{\pi}} \int_0^1 t^{-1}(1-t)^{\frac{1}{2}-1} P_n^{(\alpha,\beta)}(1-2xt)dt. \tag{4.1}$$

For $\alpha = \beta = 0$, equation (4.1) reduces to known result [1]

$$f_n(x) = \frac{1}{\sqrt{\pi}} \int_0^1 t^{-1}(1-t)^{\frac{1}{2}-1} P_n(1-2xt)dt, \tag{4.2}$$

where

$$\mathrm{P}_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left[\begin{array}{cc} -n, & n+\alpha+\beta+1; \\ & 1+\alpha; \end{array} \frac{1-x}{2}\right].$$

$$(b). \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-1} f_n^{(\alpha,\beta)}(x(1-t)) = \frac{(1+\alpha+\beta)_n}{n!} {}_3F_3\left[\begin{array}{cc} -n, & \gamma, 1+\alpha+\beta+n; \\ & 1+\alpha, \beta+\gamma, \frac{1}{2}; \end{array} x\right]. \tag{4.3}$$

$$(c). \qquad P_n^{(\alpha,\beta)}(1-2x) = \frac{(1+\alpha)_n}{(1+\alpha+\beta)_n\sqrt{\pi}} \int_0^\infty e^{-y} y^{-\frac{1}{2}} f_n^{(\alpha,\beta)}(xy)dy, \tag{4.4}$$

for $\alpha = \beta = 0$, equation (4.4) reduces to known result [1, p.291]

$$P_n(1-2x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{-\frac{1}{2}} f_n(xy)dy \tag{4.5}$$

$$\text{and } f_n^{(\alpha,\beta)}(\frac{1}{2};1;x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{-\frac{1}{2}} f_n^{(\alpha,\beta)}(-;1;xy)dy. \tag{4.6}$$

Proof. (a)

$$\frac{(1+\alpha+\beta)_n}{\sqrt{\pi}(1+\alpha)_n} \int_0^1 t^{-1}(1-t)^{\frac{1}{2}-1} P_n^{(\alpha,\beta)}(1-2xt)dt$$

$$= \frac{(1+\alpha+\beta)_n}{\sqrt{\pi}n!} \sum_{r=0}^{n} \frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r} \frac{x^r}{r!} \int_0^1 t^{r-1}(1-t)^{\frac{1}{2}-1}dt$$

$$= \frac{(1+\alpha+\beta)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r(\frac{1}{2})_r} \frac{x^r}{r!}$$

$$= f_n^{(\alpha,\beta)}(x).$$

(b)

$$\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-1} f_n^{(\alpha,\beta)}(x(1-t))dt$$

$$= \frac{(1+\alpha+\beta)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r(\frac{1}{2})_r} \frac{x^r}{r!} \int_0^1 t^{\beta-1}(1-t)^{\gamma+r-1} \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}dt$$

$$= \frac{(1+\alpha+\beta)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r(1+\alpha+\beta+n)_r(\gamma)_r}{(1+\alpha)_r(\frac{1}{2})_r(\beta+\gamma)_r} \frac{x^r}{r!}$$

$$= \frac{(1+\alpha+\beta)_n}{n!} {}_3F_3\left[\begin{array}{ccc} -n, & 1+\alpha+\beta+n, & \gamma; \\ 1+\alpha, & \beta+\gamma, & \frac{1}{2}; \end{array} x\right].$$

(c)

$$\frac{(1+\alpha)_n}{(1+\alpha+\beta)_n\sqrt{\pi}}\int_0^\infty e^{-y}y^{-\frac{1}{2}}f_n^{(\alpha,\beta)}(xy)dy$$
$$=\frac{(1+\alpha)_n}{\sqrt{\pi}n!}\sum_{r=0}^{n}\frac{(-n)_r(1+\alpha+\beta+n)_r x^r\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}{(1+\alpha)_r\Gamma((\frac{1}{2}+r)}$$
$$=\frac{(1+\alpha)_n}{n!}\sum_{r=0}^{n}\frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r}\frac{x^r}{r!}$$
$$=P_n^{\alpha,\beta)}(1-2x).$$

This completes the proof of (4.1), (4.3) and (4.4). □

5 Relation with other polynomials

Bateman's polynomial

$$f_n^{(\alpha,\beta)}(\frac{1}{2};1;x)=\frac{n!}{(1+\alpha+\beta)_n}Z_n^{(\alpha,\beta)}(x), \tag{5.1}$$

where $Z_n^{(\alpha,\beta)}(x)$ Generalized Bateman's polynomials [3].

For $\alpha=\beta=0$, we get

$$f_n^{(0,0)}(\frac{1}{2};1;x)=Z_n(x), \tag{5.2}$$

and also we get

$$\frac{(1+\alpha)_n}{(1+\alpha+\beta)_n}Z_n^{(\alpha,\beta)}(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty y^{\frac{1}{2}}e^{-y}f_n^{(\alpha,\beta)}(-;1;xy)dy, \tag{5.3}$$

for $\alpha=\beta=0$, equation (5.3) reduces to known result [4, pp.291]

$$Z_n(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty y^{\frac{1}{2}}e^{-y}f_n(-;1;xy)dy. \tag{5.4}$$

Bessel polynomials

$$y_n^{(\alpha,\beta)}(1;\frac{1}{2};-2x)=\frac{1}{(1+\alpha)_n}f_n^{(\alpha,\beta)}(x), \tag{5.5}$$

where $y_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha+\beta)_n}{n!}{}_2F_o(-n,1+\alpha+\beta+n;-;-\frac{x}{2})$, is the Bessel's polynomials [5, p.75].

Obviously,

$$y_n(0,0)\equiv y_n(x)={}_2F_o(-n,1+n;-;-\frac{x}{2}). \tag{5.6}$$

Laguerre polynomials

$$\left\{\frac{(1+\alpha)_n}{n!}\right\}^2 f_n^{(\alpha,\alpha)}(-;1-;\frac{1}{4}x^2)=L_n^\alpha(x)L_n^\alpha(-x), \tag{5.7}$$

obviously for $\alpha=0$, the result (5.7) reduces to the known result [4, p.291]

$$f_n(-;1;\frac{x^2}{4})=L_n(x).L_n(-x). \tag{5.8}$$

Generalized Rice polynomials

$$\frac{(1+\alpha)_n}{(1+\alpha+\beta)_n}\int_0^\infty e^{-t}t^{\xi-1}f_n^{(\alpha,\beta)}(xt)dt=\Gamma(\xi)H_n^{(\alpha,\beta)}(\xi,p,v), \tag{5.9}$$

where $H_n^{(\alpha,\beta)}(\xi,p,v) = \frac{(1+\alpha+\beta)_n}{n!}\sum_{r=0}^{n}\frac{(-n)_r(n+\alpha+\beta+1)_r(\xi)_r}{(1+\alpha)_r(p)_r}\frac{v^r}{r!}$ is the genralized Rices polynomial due to Khandekar [2].

$$f_n^{(\alpha,\beta)}(-,-,\xi;-p;v) = \frac{n!}{(1+\alpha+\beta)_n}H_n^{(\alpha,\beta)}(\xi,p,v)$$

$$= {}_3F_2\left[\begin{array}{ccc} -n, & n+\alpha+\beta+1, & \xi; \\ & 1+\alpha, & p; \end{array} v\right]. \tag{5.10}$$

Generalized Bateman's polynomials

$$f_n^{(\alpha,\beta)}(-,-,\frac{1+z}{2};-,p;1) = \sum_{r=0}^{n}\frac{(-n)_r(n+\alpha+\beta+1)_r}{(1+\alpha)_r(p)_r}\frac{(\frac{1+z}{2})_r}{r!}$$

$$= {}_3F_2\left[\begin{array}{ccc} -n, & n+\alpha+\beta+1, & \frac{(1+z)}{2}; \\ & 1+\alpha, & p; \end{array} 1\right]$$

$$= F_n^{(\alpha,\beta)}(p,z), \tag{5.11}$$

which is a Jacobi type generalized Bateman's polynomial $F_n(Z)$
and

$$f_n^{(\alpha,\beta)}(-,-,\frac{1+Z}{2};-,-;1) = {}_3F_2\left[\begin{array}{ccc} -n, & n+\alpha+\beta+1, & \frac{1+Z}{2}; \\ & 1+\alpha, & \frac{1}{2}; \end{array} 1\right]$$

$$= F_n^{(\alpha,\beta)}(\frac{1}{2},Z). \tag{5.12}$$

For $\beta=\alpha$, equation (5.12) reduces to

$$f_n^{(\alpha,\alpha)}(-,-,\frac{1+Z}{2};-,-;1) = {}_3F_2\left[\begin{array}{ccc} -n, & n+2\alpha+1; & \\ & 1+\alpha, & \frac{1}{2}; \end{array} 1\right]$$

$$= F_n^{(\alpha,\alpha)}(\frac{1}{2},Z). \tag{5.13}$$

Pasternak's polynomials

$$f_n^{(\alpha,\beta)}(-,-,\frac{1+Z+m}{2};m+1;1) = {}_3F_2\left[\begin{array}{ccc} -n, & n+\alpha+\beta+1, & \frac{1+Z+m}{2}; \\ & 1+\alpha, & m+1; \end{array} 1\right], \tag{5.14}$$

which is generalized Pasternak's polynomial denoted by $F_n^m(Z)$, For $\alpha=\beta$, the above equation (5.14) reduces to

$$f_n^{(\alpha,\alpha)}(-,-,\frac{1+Z+m}{2};-,m+1;1) = {}_3F_2\left[\begin{array}{ccc} -n, & n+2\alpha+1, & \frac{1}{2}(1+Z+m); \\ & 1+\alpha, & m+1; \end{array} 1\right], \tag{5.15}$$

which is an ultra spherical type generalized Pasternak's polynomial $F_{n,m}^{(\alpha,\alpha)}(Z)$.

For $\alpha=0$ it reduces to Pasternak's polynomials $F_n^m(Z)$.

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