

ON GENERALIZATION OF SISTER CELINE’S POLYNOMIALS

Khursheed Ahmad, M. Kamarujjama and M. Ghayasuddin

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Abstract. This paper deals with a generalization of Sister Celine’s polynomials, their generating functions and integral representations. A set of relations with other polynomials are given in the last section.

1 Introduction

Sister Celine [1] has introduced the polynomial $f_n(x)$

$$f_n \left[\begin{matrix} a_1, & \dots, & a_p; \\ b_1, & \dots, & b_q; \end{matrix} x \right] = {}_{p+2}F_{q+2} \left[\begin{matrix} -n, & n+1, & a_1, & \dots, & \dots, & a_p; \\ & 1, & \frac{1}{2}, & b_1, & \dots, & b_q; \end{matrix} x \right], \quad (1.1)$$

which is defined by the following generating function (see [4, p.290])

$$(1-t)^{-1} {}_pF_q \left[\begin{matrix} a_1, & \dots, & a_p; \\ b_1, & \dots, & b_q; \end{matrix} \frac{-4xt}{(1-t)^2} \right] = \sum_{n=0}^{\infty} f_n \left[\begin{matrix} a_1, & \dots, & a_p; \\ b_1, & \dots, & b_q; \end{matrix} x \right] t^n, \quad (1.2)$$

where ${}_pF_q$ denotes the generalized hypergeometric function [4].

For $p = 1, q = 1, a_1 = \frac{1}{2}, b_1 = 1$ the following integral representation of Sister Celine polynomials is given

$$f_n\left(\frac{1}{2}; 1; x\right) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} y^{-\frac{1}{2}} e^{-y} f_n(-; 1; xy) dy. \quad (1.3)$$

Equation (1.1) with no a’s and no b’s denotes simply

$$f_n(x) = {}_2F_2 \left[-n, n+1; 1, \frac{1}{2}; x \right] = \sum_{r=0}^n \frac{(-1)^r (n)! x^r}{(r!)^2 (\frac{1}{2})_r (n-r)!}. \quad (1.4)$$

For the $f_n(x)$ the generating function (1.2) becomes

$$(1-t)^{-1} \exp\left(\frac{-4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} f_n(x) t^n. \quad (1.5)$$

2 Generalization of Sister Celine’s Polynomials

In the view of above results, we define the generalized Sister Celine polynomial in following manner

$$f_n^{(\alpha, \beta)} \left[\begin{matrix} a_1, & \dots, & a_p; \\ b_1, & \dots, & b_q; \end{matrix} x \right] = \frac{(1 + \alpha + \beta)_n}{n!} {}_{p+2}F_{q+2} \left[\begin{matrix} -n, & n + \alpha + \beta + 1, & a_1, & \dots, & a_p; \\ 1 + \alpha, & \frac{1}{2}, & b_1, & \dots, & b_q; \end{matrix} x \right]. \quad (2.1)$$

Equation (2.1) with no a's and no b's denotes simply

$$f_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha + \beta)_n}{n!} {}_2F_2 \left[\begin{matrix} -n, & n + \alpha + \beta + 1; \\ 1 + \alpha, & \frac{1}{2}; \end{matrix} x \right] \quad (2.2)$$

$$= \frac{(1 + \alpha + \beta)_n}{n!} \sum_{r=0}^n \frac{(-n)_r (n + \alpha + \beta + 1)_r x^r}{(1 + \alpha)_r (\frac{1}{2})_r r!}. \quad (2.3)$$

$$\text{Indeed} \quad f_n^{(0,0)}(x) = f_n(x). \quad (2.4)$$

3 Generating functions

The following generating function can be easily obtained

$$\sum_{n=0}^{\infty} f_n^{(\alpha, \beta)}(x) t^n = (1 - t)^{-1 - \alpha - \beta} {}_2F_2 \left[\begin{matrix} \frac{1 + \alpha + \beta}{2}, & \frac{2 + \alpha + \beta}{2}; \\ 1 + \alpha, & \frac{1}{2}; \end{matrix} \frac{-4xt}{(1 - t)^2} \right], \quad (3.1)$$

$$\sum_{n=0}^{\infty} (C)_n f_n^{(\alpha, \beta)}(x) t^n = (1 - t)^{-C - \alpha - \beta} {}_3F_3 \left[\begin{matrix} C, & \frac{1 + \alpha + \beta}{2}, & \frac{2 + \alpha + \beta}{2}; \\ 1 + \alpha, & 1 + \alpha + \beta, & \frac{1}{2}; \end{matrix} \frac{-4xt}{(1 - t)^2} \right] \quad (3.2)$$

(Obviously for $C = 1, \alpha = \beta = 0$, equation (3.2) reduces to the generating function (1.5)), and

$$\begin{aligned} (1 - t)^{-1 - \alpha - \beta} {}_{p+2}F_{q+2} \left[\begin{matrix} a_1, \dots, a_p, & \frac{1 + \alpha + \beta}{2}, & \frac{2 + \alpha + \beta}{2}; \\ b_1, \dots, b_q, & 1 + \alpha, & \frac{1}{2}; \end{matrix} \frac{-4xt}{(1 - t)^2} \right] \\ = \sum_{n=0}^{\infty} f_n^{(\alpha, \beta)} \left[\begin{matrix} a_1, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} x \right] t^n. \end{aligned} \quad (3.3)$$

Proof. Expanding left hand member of (3.1) with the help of (2.2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} f_n^{(\alpha, \beta)}(x) t^n &= \frac{(1 + \alpha + \beta)_n}{n!} \sum_{n=0}^{\infty} \sum_{r=0}^n \frac{(-n)_r (1 + \alpha + \beta + n)_r x^r}{(1 + \alpha)_r (\frac{1}{2})_r r!} t^n \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^n \frac{(-1)^r}{(n - r)!} \frac{(1 + \alpha + \beta)_{n+r} x^r}{(1 + \alpha)_r (\frac{1}{2})_r r!} t^n \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r}{(n)!} \frac{(1 + \alpha + \beta)_{n+2r} x^r}{(1 + \alpha)_r (\frac{1}{2})_r r!} t^{n+r} \\ &= \sum_{r=0}^{\infty} \frac{(1 + \alpha + \beta)_{2r} (-xt)^r}{(1 + \alpha)_r (\frac{1}{2})_r r!} \sum_{n=0}^{\infty} \frac{(1 + \alpha + \beta + 2r)_n}{n!} t^n \\ &= \sum_{r=0}^{\infty} \frac{(1 + \alpha + \beta)_{2r} (-xt)^r}{(1 + \alpha)_r (\frac{1}{2})_r r! (1 - t)^{1 + \alpha + \beta + 2r}} \\ &= (1 - t)^{-1 - \alpha - \beta} \sum_{r=0}^{\infty} \frac{(1 + \alpha + \beta)_{2r} (-xt)^r}{(1 + \alpha)_r (\frac{1}{2})_r r! (1 - t)^{2r}} \\ &= (1 - t)^{-1 - \alpha - \beta} \sum_{r=0}^{\infty} \frac{(\frac{1 + \alpha + \beta}{2})_r (\frac{2 + \alpha + \beta}{2})_r (\frac{-4xt}{(1 - t)^2})^r}{(1 + \alpha)_r (\frac{1}{2})_r r!} \end{aligned}$$

Thus we arrive at the result (3.1). Similarly we can easily prove equations (3.2) and (3.3). \square

4 Integral representations

The following integral representations involving Sister Celine's polynomials are obtained as follows:

$$(a). \quad f_n^{(\alpha,\beta)}(x) = \frac{(1 + \alpha + \beta)_n}{(1 + \alpha)_n \sqrt{\pi}} \int_0^1 t^{-1}(1-t)^{\frac{1}{2}-1} P_n^{(\alpha,\beta)}(1-2xt) dt. \tag{4.1}$$

For $\alpha = \beta = 0$, equation (4.1) reduces to known result [1]

$$f_n(x) = \frac{1}{\sqrt{\pi}} \int_0^1 t^{-1}(1-t)^{\frac{1}{2}-1} P_n(1-2xt) dt, \tag{4.2}$$

where

$$P_n^{(\alpha,\beta)}(x) = \frac{(1 + \alpha)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, & n + \alpha + \beta + 1; & \frac{1-x}{2} \\ & 1 + \alpha; & \end{matrix} \right].$$

$$(b). \quad \frac{\Gamma(\beta + \gamma)}{\Gamma(\beta)\Gamma(\gamma)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-1} f_n^{(\alpha,\beta)}(x(1-t)) dt = \frac{(1 + \alpha + \beta)_n}{n!} {}_3F_3 \left[\begin{matrix} -n, & \gamma, 1 + \alpha + \beta + n; \\ & 1 + \alpha, \beta + \gamma, \frac{1}{2}; & x \end{matrix} \right]. \tag{4.3}$$

$$(c). \quad P_n^{(\alpha,\beta)}(1-2x) = \frac{(1 + \alpha)_n}{(1 + \alpha + \beta)_n \sqrt{\pi}} \int_0^\infty e^{-y} y^{-\frac{1}{2}} f_n^{(\alpha,\beta)}(xy) dy, \tag{4.4}$$

for $\alpha = \beta = 0$, equation (4.4) reduces to known result [1, p.291]

$$P_n(1-2x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{-\frac{1}{2}} f_n(xy) dy \tag{4.5}$$

$$\text{and } f_n^{(\alpha,\beta)}\left(\frac{1}{2}; 1; x\right) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{-\frac{1}{2}} f_n^{(\alpha,\beta)}(-; 1; xy) dy. \tag{4.6}$$

Proof. (a)

$$\begin{aligned} & \frac{(1 + \alpha + \beta)_n}{\sqrt{\pi}(1 + \alpha)_n} \int_0^1 t^{-1}(1-t)^{\frac{1}{2}-1} P_n^{(\alpha,\beta)}(1-2xt) dt \\ &= \frac{(1 + \alpha + \beta)_n}{\sqrt{\pi}n!} \sum_{r=0}^n \frac{(-n)_r (1 + \alpha + \beta + n)_r}{(1 + \alpha)_r} \frac{x^r}{r!} \int_0^1 t^{r-1}(1-t)^{\frac{1}{2}-1} dt \\ &= \frac{(1 + \alpha + \beta)_n}{n!} \sum_{r=0}^n \frac{(-n)_r (1 + \alpha + \beta + n)_r}{(1 + \alpha)_r (\frac{1}{2})_r} \frac{x^r}{r!} \\ &= f_n^{(\alpha,\beta)}(x). \end{aligned}$$

(b)

$$\begin{aligned} & \frac{\Gamma(\beta + \gamma)}{\Gamma(\beta)\Gamma(\gamma)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-1} f_n^{(\alpha,\beta)}(x(1-t)) dt \\ &= \frac{(1 + \alpha + \beta)_n}{n!} \sum_{r=0}^n \frac{(-n)_r (1 + \alpha + \beta + n)_r}{(1 + \alpha)_r (\frac{1}{2})_r} \frac{x^r}{r!} \int_0^1 t^{\beta-1}(1-t)^{\gamma+r-1} \frac{\Gamma(\beta + \gamma)}{\Gamma(\beta)\Gamma(\gamma)} dt \\ &= \frac{(1 + \alpha + \beta)_n}{n!} \sum_{r=0}^n \frac{(-n)_r (1 + \alpha + \beta + n)_r (\gamma)_r}{(1 + \alpha)_r (\frac{1}{2})_r (\beta + \gamma)_r} \frac{x^r}{r!} \\ &= \frac{(1 + \alpha + \beta)_n}{n!} {}_3F_3 \left[\begin{matrix} -n, & 1 + \alpha + \beta + n, & \gamma; \\ & 1 + \alpha, & \beta + \gamma, & \frac{1}{2}; & x \end{matrix} \right]. \end{aligned}$$

(c)

$$\begin{aligned}
& \frac{(1+\alpha)_n}{(1+\alpha+\beta)_n \sqrt{\pi}} \int_0^\infty e^{-y} y^{-\frac{1}{2}} f_n^{(\alpha, \beta)}(xy) dy \\
&= \frac{(1+\alpha)_n}{\sqrt{\pi} n!} \sum_{r=0}^n \frac{(-n)_r (1+\alpha+\beta+n)_r x^r \Gamma(\frac{1}{2}) \Gamma(r+\frac{1}{2})}{(1+\alpha)_r \Gamma(\frac{1}{2}+r)} \\
&= \frac{(1+\alpha)_n}{n!} \sum_{r=0}^n \frac{(-n)_r (1+\alpha+\beta+n)_r x^r}{(1+\alpha)_r r!} \\
&= P_n^{\alpha, \beta}(1-2x).
\end{aligned}$$

This completes the proof of (4.1), (4.3) and (4.4). \square

5 Relation with other polynomials

Bateman's polynomial

$$f_n^{(\alpha, \beta)}\left(\frac{1}{2}; 1; x\right) = \frac{n!}{(1+\alpha+\beta)_n} Z_n^{(\alpha, \beta)}(x), \quad (5.1)$$

where $Z_n^{(\alpha, \beta)}(x)$ Generalized Bateman's polynomials [3].

For $\alpha = \beta = 0$, we get

$$f_n^{(0, 0)}\left(\frac{1}{2}; 1; x\right) = Z_n(x), \quad (5.2)$$

and also we get

$$\frac{(1+\alpha)_n}{(1+\alpha+\beta)_n} Z_n^{(\alpha, \beta)}(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty y^{\frac{1}{2}} e^{-y} f_n^{(\alpha, \beta)}(-; 1; xy) dy, \quad (5.3)$$

for $\alpha = \beta = 0$, equation (5.3) reduces to known result [4, pp.291]

$$Z_n(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty y^{\frac{1}{2}} e^{-y} f_n(-; 1; xy) dy. \quad (5.4)$$

Bessel polynomials

$$y_n^{(\alpha, \beta)}\left(1; \frac{1}{2}; -2x\right) = \frac{1}{(1+\alpha)_n} f_n^{(\alpha, \beta)}(x), \quad (5.5)$$

where $y_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha+\beta)_n}{n!} {}_2F_0(-n, 1+\alpha+\beta+n; -; -\frac{x}{2})$, is the Bessel's polynomials [5, p.75].

Obviously,

$$y_n(0, 0) \equiv y_n(x) = {}_2F_0(-n, 1+n; -; -\frac{x}{2}). \quad (5.6)$$

Laguerre polynomials

$$\left\{ \frac{(1+\alpha)_n}{n!} \right\}^2 f_n^{(\alpha, \alpha)}(-; 1; \frac{1}{4}x^2) = L_n^\alpha(x) L_n^\alpha(-x), \quad (5.7)$$

obviously for $\alpha = 0$, the result (5.7) reduces to the known result [4, p.291]

$$f_n(-; 1; \frac{x^2}{4}) = L_n(x) \cdot L_n(-x). \quad (5.8)$$

Generalized Rice polynomials

$$\frac{(1+\alpha)_n}{(1+\alpha+\beta)_n} \int_0^\infty e^{-t} t^{\xi-1} f_n^{(\alpha, \beta)}(xt) dt = \Gamma(\xi) H_n^{(\alpha, \beta)}(\xi, p, v), \quad (5.9)$$

where $H_n^{(\alpha, \beta)}(\xi, p, v) = \frac{(1+\alpha+\beta)_n}{n!} \sum_{r=0}^n \frac{(-n)_r (n+\alpha+\beta+1)_r (\xi)_r}{(1+\alpha)_r (p)_r} \frac{v^r}{r!}$ is the generalized Rices polynomial due to Khandekar [2].

$$\begin{aligned} f_n^{(\alpha, \beta)}(-, -, \xi; -p; v) &= \frac{n!}{(1+\alpha+\beta)_n} H_n^{(\alpha, \beta)}(\xi, p, v) \\ &= {}_3F_2 \left[\begin{matrix} -n, & n+\alpha+\beta+1, & \xi; \\ & 1+\alpha, & p; \end{matrix} v \right]. \end{aligned} \quad (5.10)$$

Generalized Bateman's polynomials

$$\begin{aligned} f_n^{(\alpha, \beta)}(-, -, \frac{1+z}{2}; -, p; 1) &= \sum_{r=0}^n \frac{(-n)_r (n+\alpha+\beta+1)_r (\frac{1+z}{2})_r}{(1+\alpha)_r (p)_r} \frac{1}{r!} \\ &= {}_3F_2 \left[\begin{matrix} -n, & n+\alpha+\beta+1, & \frac{(1+z)}{2}; \\ & 1+\alpha, & p; \end{matrix} 1 \right] \\ &= F_n^{(\alpha, \beta)}(p, z), \end{aligned} \quad (5.11)$$

which is a Jacobi type generalized Bateman's polynomial $F_n(Z)$ and

$$\begin{aligned} f_n^{(\alpha, \beta)}(-, -, \frac{1+Z}{2}; -, -, 1) &= {}_3F_2 \left[\begin{matrix} -n, & n+\alpha+\beta+1, & \frac{1+Z}{2}; \\ & 1+\alpha, & \frac{1}{2}; \end{matrix} 1 \right] \\ &= F_n^{(\alpha, \beta)}(\frac{1}{2}, Z). \end{aligned} \quad (5.12)$$

For $\beta = \alpha$, equation (5.12) reduces to

$$\begin{aligned} f_n^{(\alpha, \alpha)}(-, -, \frac{1+Z}{2}; -, -, 1) &= {}_3F_2 \left[\begin{matrix} -n, & n+2\alpha+1; \\ & 1+\alpha, & \frac{1}{2}; \end{matrix} 1 \right] \\ &= F_n^{(\alpha, \alpha)}(\frac{1}{2}, Z). \end{aligned} \quad (5.13)$$

Pasternak's polynomials

$$f_n^{(\alpha, \beta)}(-, -, \frac{1+Z+m}{2}; m+1; 1) = {}_3F_2 \left[\begin{matrix} -n, & n+\alpha+\beta+1, & \frac{1+Z+m}{2}; \\ & 1+\alpha, & m+1; \end{matrix} 1 \right], \quad (5.14)$$

which is generalized Pasternak's polynomial denoted by $F_n^m(Z)$, For $\alpha = \beta$, the above equation (5.14) reduces to

$$f_n^{(\alpha, \alpha)}(-, -, \frac{1+Z+m}{2}; -, m+1; 1) = {}_3F_2 \left[\begin{matrix} -n, & n+2\alpha+1, & \frac{1}{2}(1+Z+m); \\ & 1+\alpha, & m+1; \end{matrix} 1 \right], \quad (5.15)$$

which is an ultra spherical type generalized Pasternak's polynomial $F_{n,m}^{(\alpha, \alpha)}(Z)$.

For $\alpha = 0$ it reduces to Pasternak's polynomials $F_n^m(Z)$.

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Author information

Khursheed Ahmad, Department of Mathematics, Vivekanand College of Technology and Management, Kher-Road, Aligarh., India.

E-mail: khursheedahmad08@rediffmail.com

M. Kamarujjama, Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India.

E-mail: mdkamarujjama@rediffmail.com

M. Ghayasuddin, Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India.

E-mail: ghayas.maths@gmail.com

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