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Using Summation Notation to Solve Some Diophantine Equations

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Abstract It is well known that Diophantine equations can be solved by two methods, namely elementary methods, such as decomposition, modular arithmetic, mathematical induction, and Fermat's infinite descent, the second method is an advanced method involving Gaussian integers, quadratic rings, divisors of certain forms, and quadratic reciprocity (see [1], and [2]).

1 Introduction

It is well known that Diophantine equations can be solved by two methods, namely elementary methods , such as decomposition, modular arithmetic, mathematical induction, and Fermat's infinite descent, the second method is an advanced method involving Gaussian integers, quadratic rings, divisors of certain forms, and quadratic reciprocity (see [1]).

In this paper, we will use Summation Notation Method (SNM), which may be not used before.

Consider the equation

$$x^2 - y^2 = z^3$$

where $z = x - y$

This equation solved by Dorin Andrica and Gheorghe M. Tudor[1], by using decomposition method by finding the identity

$$(u(u^2 + 3v^2))^2 - (v(3u^2 + v^2))^2 = (u^2 - v^2)^3$$

Now before solving equation by SNM, I would like to present it geometrically to see the different solution cases

2 The solution of $x^2 - y^2 = z^3$

We begin with the following result.

Theorem 2.1. *The solution of* $x^2 - y^2 = z^3$, *where* $z = x - y$ *is:* $x = \sum\limits_{i=0}^{n} i, and, y = \sum\limits_{i=0}^{n-1} i$

Proof. $\Rightarrow x^2 - y^2 = (\sum\limits_{i=0}^{n} i)^2 - (\sum\limits_{i=0}^{n-1} i)^2 = \sum\limits_{i=0}^{n} i^3 - \sum\limits_{i=0}^{n-1} i^3 = (\frac{n(n+1)}{2})^2$-$(\frac{n(n-1)}{2})^2 = n^3$.

$\Leftarrow z^3 = (x - y)^3 = (\sum\limits_{i=0}^{n} i - \sum\limits_{i=0}^{n-1} i)^3 = (\frac{n(n+1)}{2} - \frac{n(n-1)}{2}))^3 = n^3$.

3 Table

The first ten solutions of $x^2 - y^2 = z^3$ are in the following table

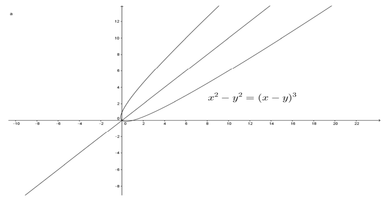

Figure 1. From the above figure we can notice three cases of solutions , case one : when $y > x$(the upper branch of parabola), the second case when $x = y$ (the middle part of parabola), and the third case when $x > y$ (the lower branch of the parabola).So I will consider the third case because of the symmetry property.

x	y	z
1	0	**1**
3	1	**2**
6	3	**3**
10	6	**4**
15	10	**5**
21	15	**6**
28	21	**7**
36	28	**8**
45	36	**9**
55	45	**10**

In fact, the solutions above are not the only one, I have generated more solutions .

Theorem 3.1. *The solution of* $x^2 - y^2 = z^3$ *is*

$$x = 2((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) + 1, y = 2((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) - 1, and; z = 2n$$

Proof. $\Rightarrow x^2 - y^2 = (2((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) + 1)^2 - (2((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) - 1)^2 = (2n)^3.$
$\Leftarrow z^3 = (x-y)^3 = (2((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) + 1 - 2((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) - 1)^3 = (2n)^3.$

4 Table

The first ten solution of $x^2 - y^2 = z^3$ are in the following table

x	y	z
3	1	**2**
17	15	**4**
55	53	**6**
129	127	**8**
251	249	**10**
433	431	**12**
687	685	**14**
1025	1023	**16**
1459	1457	**18**
2001	1999	**20**

Theorem 4.1. *The solution of* $x^2 - y^2 = z^3$ *is*

$$x = 4((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) + 6n^2 + 3n + 1, y = 4((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) + 6n^2 + 3n, and; z = 2n + 1$$

Proof. $\Rightarrow x^2 - y^2 = (4((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) + 6n^2 + 3n + 1)^2 - (4((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) + 6n^2 + 3n)^2$ $= (2n + 1)^3$.

$\Leftarrow z^3 = (x-y)^3 = (4((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) + 6n^2 + 3n + 1 - (4((\sum_{i=0}^{n} i)^2 - (\sum_{i=0}^{n-1} i)^2) + 6n^2 + 3n))^3 =$ $(2n + 1)^3$.

5 Table

The first ten solutions of $x^2 - y^2 = z^3$ are in the following table

x	y	z
14	13	**3**
63	62	**5**
172	171	**7**
365	364	**9**
666	665	**11**
1099	1098	**13**
1688	1687	**15**
2457	2456	**17**
3430	3429	**19**
4631	4630	**21**

Remark 5.1. if we reduce $x^2 - y^2 = z^3$, then we have :

$$x + y = z^2$$

Hence $x + y = z^2$ has infinitely many solutions with $x = \sum_{i=0}^{n} i, and, y = \sum_{i=0}^{n-1} i$.

Remark 5.2. From the above theorems, we can see many relationship between the solutions , which of course generate new chains of identities. For examples we can see the following important notes :

$$6^2 - 3^2 = 14^2 - 13^2$$

$$10^2 - 16^2 = 17^2 - 15^2$$

$$15^2 - 10^2 = 63^2 - 62^2$$
$$21^2 - 15^2 = 55^2 - 53^2, etc$$

Solution system in the above examples , represented the solution of arithmetic progression among four squares of the form $x^2 - y^2 = z^2 - w^2$, which need more studies.

Remark 5.3. with slight modification of $x^2 - y^2 = z^2$, I have founded only two solutions for the equation $x^3 - y^3 = z^2$, namely:

$$10^3 - 6^3 = 28^2$$
$$295296^3 - 294528^3 = 14155780^2$$

Remark 5.4. The equation $x^2 + y^2 - 2xy + x + y = 0$ has infinitely many solutions with $x = \sum_{i=0}^{n} i, and, y = \sum_{i=0}^{n-1} i.$

Conjecture 5.5. Conjecture :

$$x^3 - y^3 = z^4$$

, has no solution in integers (has only trivial solutions) .

References

[1] D. Andrica, and Gh. M.Tudor , Parametric solutions for some Diophantine equations, *General. Mathematics.* **12**, 23–34 (2004).

[2] D. Andrica, and Gh. M.Tudor ,Some Diophantine Equations , *General. Mathematics.* **13**, 121–131 (2005).

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