PSEUDO-DIFFERENTIAL OPERATORS ON $W_{\Omega M}^{\Omega}(\mathbb{C}^n)$-SPACE

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Abstract. Pseudo-differential operators associated with symbol $\theta(z, \xi)$, $z = x + iy$ and $\xi = u + it$ on $W_{\Omega M}^{\Omega}(\mathbb{C}^n)$-space is defined and using the theory of Fourier transformation its various properties are studied. $L^p(\mathbb{R}^n)$-boundedness is investigated for $1 < p \leq \infty$. Sobolev space associated with distributional space $[W_{\Omega M}^{\Omega}(\mathbb{C}^n)]'$ is defined and its properties are obtained.

1 Introduction

The spaces $W_{M}(\mathbb{R}^n)$, $W^{\Omega}(\mathbb{C}^n)$ and $W_{\Omega M}^{\Omega}(\mathbb{C}^n)$ were introduced and analyzed by Gurevich [4], Gel’fand and Shilov [3] and Friedman [2]. They applied these $W$-type spaces for investigating uniqueness and corrected class of Cauchy problem and other problems of partial differential equations by using Fourier transformation tool. Recently, continuous Wavelet transformation on $W$-spaces are studied by [6, 9] and many interesting properties are obtained.

The theory of pseudo-differential operators is developed by Wong [10], Rodino [1], Pathak [5] and others. They exploited this theory on Schwartz space $S(\mathbb{R}^n)$, Gel’fand and Shilov space of type $\mathcal{S}$ and Gevery space by using the technique of Fourier transformation. They also studied pseudo-differential operators on their respective Sobolev space and got many important results. Sobolev spaces are used to study the minimal-maximal properties, global regularities and spectral properties of pseudo-differential operators on Schwartz space $S(\mathbb{R}^n)$. The pseudo differential operators on $W_{M}(\mathbb{R}^n)$ and $W^{\Omega}(\mathbb{C}^n)$ are studied by author and others and obtained many important results, see [7, 8].

Our main aim in this paper is to introduce more general symbol and precise study of pseudo-differential operators on $W_{\Omega M}^{\Omega}(\mathbb{C}^n)$-space and to study many properties because its dual space $[W_{\Omega M}^{\Omega}(\mathbb{C}^n)]'$ is more general than $[W_{M}(\mathbb{R}^n)]'$ and Schwartz distributional space $S'(\mathbb{R}^n)$.

The present article is divided into three sections. Section 2 gives the various definitions of pseudo-differential operators, symbol, $W$-type spaces, Fourier transformation and Sobolev space. Section 3 contains a study of properties of pseudo-differential operators on $W_{\Omega M}^{\Omega}(\mathbb{C}^n)$ and $L^p(\mathbb{R}^n)$-boundedness result of pseudo differential operators. In the last section, using $L^p(\mathbb{R}^n)$-boundedness result, the Sobolev space $G^{s-p}(\mathbb{C}^n)$, $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ on $[W_{\Omega M}^{\Omega}(\mathbb{C}^n)]'$ space is defined and it is proved that the pseudo-differential operator $A_\theta : G^{s-p}(\mathbb{C}^n) \rightarrow G^{s-p}(\mathbb{C}^n)$ and $A_\theta : G^{s-p}(\mathbb{C}^n) \rightarrow G^{s-m,p}(\mathbb{C}^n)$ are bounded linear operator for $s, m \in \mathbb{R}$.

2 Preliminary

Now in this section we recall the definitions of $W_{M}(\mathbb{R}^n)$, $W^{\Omega}(\mathbb{C}^n)$ and $W_{\Omega M}^{\Omega}(\mathbb{C}^n)$ from [2] and [3].

Let $M_j$ and $\Omega_j$ be the convex functions such that

$$M_j(x_j) = \int_0^{x_j} \mu_j(\xi_j) \, d\xi_j \quad (x_j \geq 0) \tag{2.1}$$

and

$$\Omega_j(y_j) = \int_0^{y_j} w_j(\eta_j) \, d\eta_j \quad (y_j \geq 0) \tag{2.2}$$

for $j = 1, 2, 3, \cdots n$.

We set
\[ \begin{align*}
\mu(\xi) & = (\mu_1(\xi_1), \cdots, \mu_n(\xi_n)) \\
w(n) & = (w_1(\eta_1), \cdots, w_n(\eta_n))
\end{align*} \]

and
\[ \begin{align*}
M_j(-x_j) & = M_j(x_j), \\
M_j(x_j) + M_j(x'_j) & \leq M_j(x_j + x'_j) \quad (2.3) \\
\Omega_j(-y_j) & = \Omega_j(y_j), \\
\Omega_j(y_j) + \Omega_j(y'_j) & \leq \Omega_j(y_j + y'_j). \quad (2.4)
\end{align*} \]

The space \( W_M(\mathbb{R}^n) \) consists of all \( C^\infty \)-functions which satisfy the inequalities
\[ |D_x^{(k)} \phi(x)| \leq C_k \exp[-M(ax)], \quad (2.5) \]
where \( D_x^{(k)} = D_x^{(k_1)} D_x^{(k_2)} \cdots D_x^{(k_n)} \),
\[ \exp[-M(ax)] = \exp[-M_1(a_1 x_1) \cdots - M_n(a_n x_n)] \quad (2.6) \]
and constants \( C_k, a > 0 \) depending on the function \( \phi \). A function \( \phi(z) \in W^\Omega(\mathbb{C}^n) \) if and only if for \( b > 0 \) there exists a constant \( C_k > 0 \) such that
\[ |z^b \phi(z)| \leq C_k \exp\Omega(by)], \quad z = x + iy \quad (2.7) \]
where
\[ z^b = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n} \]
and
\[ \exp\Omega(by)] = \exp[\Omega_1(b_1 y_1) + \cdots + \Omega_j(b_j y_j) + \cdots + \Omega_n(b_n y_n)], \quad (2.8) \]
the constants \( C_k \) and \( b > 0 \) depend on the function \( \phi \).

The space \( W^\Omega_M(\mathbb{C}^n) \) consists of all entire analytic functions \( \phi(z) \) which satisfy
\[ |\phi(z)| \leq C \exp[-M([ax]) + \Omega([by])], \quad (2.9) \]
where \( z = x + iy \) and \( \exp[-M(ax)] \) and \( \exp[\Omega(by)] \) have similar meaning like (2.5) and (2.7) and constants \( C, a \) and \( b \) depend on the function \( \phi \).

Now, we define the duality of functions \( M(x) \) and \( \Omega(y) \) in the following way:

Let \( M_j(x_j) \) and \( \Omega_j(y_j) \) be defined by (2.1) and (2.2) respectively and let \( \mu_j(\xi) \) and \( w_j(\eta_j) \) be mutually inverse, that is \( \mu_j(w_j(\eta_j)) = \eta_j, \quad \eta_j(\mu_j(\xi)) = \xi_j \), then the corresponding functions \( M_j(x_j) \) and \( \Omega_j(y_j) \) are called dual in sense of Young. In this case the Young inequality is
\[ x_j y_j \leq M_j(x_j) + \Omega_j(y_j). \quad (2.10) \]
This inequality holds for any \( x_j \geq 0, y_j \geq 0 \) and equality holds if and only if \( y_j = \mu_j(x_j) \), where \( x_j \) varies in the interval \( x_j^0 < x_j < \infty \) and \( y_j \) varies in the interval \( y_j^0 < y_j < \infty \). That equality will be
\[ \begin{align*}
x_j y_j & = M_j^0(x_j) + \Omega_j^0(y_j) \quad (2.11) \\
x_j y_j & = M_j(x_j) + \Omega_j^0(y_j) \quad (2.12)
\end{align*} \]
for \( M_j(x_j) < M_j^0(x_j) \) and \( \Omega_j^0(y_j) < \Omega_j(y_j) \).

From [2, pp.132-133,Theorem 12] and [2, p.134,Theorem 13 and Theorem 15] the Fourier transformation of a function \( \phi \in W^\Omega_M(\mathbb{C}^n) \) is defined by
\[ \hat{\phi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{iz \cdot \xi} \phi(z) \, dx \quad (2.13) \]
for \( z = x + iy \) and \( \xi = u + it \).

From [1] and [2] the Fourier duality relation is given by
\[ \begin{align*}
F[W^\Omega_M(\mathbb{C}^n)] & = W_M(\mathbb{R}^n), \\
F[W_M(\mathbb{R}^n)] & = W^\Omega(\mathbb{C}^n) \quad \text{and} \\
F[W^\Omega_M(\mathbb{C}^n)] & = W^\Omega_M(\mathbb{C}^n).
\end{align*} \]
Now, we recall the definitions of translation $T$ and modulation $M$ of function $f$ on $\mathbb{R}^n$ from [10, p. 14]:

Let $f$ be a measurable function defined on $\mathbb{R}^n$. For any fixed $y \in \mathbb{R}^n$, we define $T_yf$ and $M_yf$ by

$$
(T_yf)(x) = f(x + y)
$$

and

$$
(M_yf)(x) = e^{i\pi y}f(x).
$$

Next, we define a symbol associated with pseudo-differential operator by the following way:

$$
M_z = \text{Theorem 2.1.}
$$

Let $\phi \in W^{m}_{\mathcal{D}}(\mathbb{C}^n)$ and $\sigma(z, \xi) \in \mathbb{C}^{\infty}(\mathbb{C}^n \times \mathbb{C}^n)$ which is the set of all entire analytic functions of $z = x + iy$ and $\xi = u + it$ is said to be class $V^m$ iff for any two multiindices $\alpha$ and $\beta$, there is a positive constant $C_{\alpha, \beta}$ depending on $\alpha$ and $\beta$ only such that

$$
|\hat{f}(\alpha D_{x}^{\alpha})\hat{\sigma}(z, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{m-|\beta|},
$$

where $m \in \mathbb{R}$ and $z, \xi \in \mathbb{C}^n$. If we take $y = 0, t = 0$ then the symbol will to the well known class of $S^m$.

**Theorem 2.1.** Let $\phi \in W^{m}_{\mathcal{D}}(\mathbb{C}^n)$ and let $\sigma(z, \xi)$ be an entire function in $(z, \xi)$ and satisfy

$$
|\sigma(z, \xi)| \leq C(1 + |\xi|)^m,
$$

then $\phi(\xi)\sigma(z, \xi) \in W^{m}_{\mathcal{D}}(\mathbb{C}^n)$.

**Proof.** Let $\phi \in W^{m}_{\mathcal{D}}(\mathbb{C}^n)$ and $|\sigma(z, \xi)| \leq C(1 + |\xi|)^m$. Then

$$
|\sigma(z, \xi)\phi(\xi)| = |\sigma(z, \xi)||\phi(\xi)| \leq C(1 + |\xi|)^m \exp[-M(au) + \Omega(bt)].
$$

Since $(1 + |\xi|)^m \leq \exp[-M(a_0u) + \Omega(b_0t)], m \in \mathbb{R}$ then using the definition of $W^{m}_{\mathcal{D}}(\mathbb{C}^n)$ space we find that

$$
|\sigma(z, \xi)\phi(\xi)| \leq C \exp[-M(a_0u) + \Omega(b_0t)] \exp[M(au) + \Omega(bt)].
$$

By the definition of convex function (2.1) and (2.2), we get

$$
|\sigma(z, \xi)\phi(\xi)| \leq C \exp[-M(a - a_0)u + \Omega(b + b_0)t]] .
$$

This implies that

$$
\phi(\xi)\sigma(z, \xi) \in W^{m}_{\mathcal{D}}(\mathbb{C}^n).
$$

Using this argument and argument of Fourier transform in $W^{m}_{\mathcal{D}}(\mathbb{C}^n)$, we can define the partial differential operator (2.17) and pseudo differential operator (2.18).

A linear partial differential operator $P(z, D)$ as $z = x + iy$ on $\mathbb{C}^n$ is given by

$$
P(z, D) = \sum_{|\alpha| \leq m} a_{\alpha}(z) D^{(\alpha)}.
$$

If we replace $D^{(\alpha)}$ by a monomial $\xi^\alpha \in \mathbb{R}^n$ then we get a symbol

$$
P(z, \xi) = \sum_{|\alpha| \leq m} a_{\alpha}(z) \xi^\alpha.
$$

We take $\phi \in W^{m}_{\mathcal{D}}(\mathbb{C}^n)$ then we get

$$
(P(z, D) \phi)(z) = \sum_{|\alpha| \leq m} a_{\alpha}(z) (D^{(\alpha)} \phi)(z)
$$

By the property of Fourier transformation and using the technique of [3] we get

$$
(P(z, D) \phi)(z) = \sum_{|\alpha| \leq m} a_{\alpha}(z) (D^{(\alpha)} \phi)(z) = \sum_{|\alpha| \leq m} a_{\alpha}(z) (\xi^\alpha \phi)(z)
$$
\[(P(z, D) \phi)(z) = \sum_{|\alpha| \leq m} a_\alpha(z) (2\pi)^{-n/2} \int_{\mathbb{R}^n} \xi^\alpha e^{i(z, \xi)} \hat{\phi}(\xi) \, du \]

\[= \int_{\mathbb{R}^n} e^{i(z, \xi)} \left( \sum_{|\alpha| \leq m} a_\alpha(z) \xi^\alpha \right) \hat{\phi}(\xi) \, du \]

\[= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(z, \xi)} P(z, \xi) \hat{\phi}(\xi) \, du. \tag{2.17} \]

Hence,

\[(P(z, D) \phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(z, \xi)} P(z, \xi) \hat{\phi}(\xi) \, du. \tag{2.17} \]

In (2.17) if we replace \(P(z, \xi)\) by more general symbol \(\theta(z, \xi)\) which are no longer polynomial in \(\xi\). The operator is so called pseudo differential operator.

The pseudo-differential operators associated with symbol \(\theta(z, \xi) \in V^m\) is defined by

\[(A_\theta \phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(z, \xi)} \theta(z, \xi) \hat{\phi}(\xi) \, du \tag{2.18} \]

as \(\xi = u + it \in \mathbb{C}^n\) and \(\xi \in \mathbb{C}^n\) and \(\phi \in W^0_{2\pi}(\mathbb{C}^n)\).

For \(s \in \mathbb{R}\), the pseudo-differential operators \(V_s\) associated with symbol \(\theta(\xi) = (1 + |\xi|^2)^{-s/2}\) as \(\xi = u + it\) is defined by

\[(V_s f)(z) = F^{-1}((1 + |\xi|^2)^{-s/2} \hat{f})(z), \quad \text{for } f \in W^0_{2\pi}(\mathbb{C}^n). \tag{2.19} \]

Now, the Sobolev space \(G^{s,p}(\mathbb{C}^n)\) of \(L^p(\mathbb{R}^n)\)-type is defined to be the set of all \(f \in W^0_{2\pi}(\mathbb{C}^n)\) such that

\[\|f\|_{s,p} = \|(V_s f)(z)\|_p \quad \text{for } 1 \leq p < \infty. \tag{2.20} \]

The notations and terminologies of this paper are taken from Wong [10, pp 1-4] and Friedman [2].

3 Properties of Pseudo-differential Operators

In this section we study the various properties of pseudo-differential operators \(A_\theta\) associated with symbol \(\theta(z, \xi)\) on \(W^0_{2\pi}(\mathbb{C}^n)\)-space.

**Theorem 3.1.** Let \(\theta(z, \xi)\) be the symbol belong to \(V^m\). Then \(A_\theta\) maps \(W^0_{2\pi}(\mathbb{C}^n)\) into itself.

**Proof.** Let \(\phi \in W^0_{2\pi}(\mathbb{C}^n)\). Then, for any multi-indices \(\alpha\) and \(\beta\), we have to show that

\[
\sup_{z \in \mathbb{C}^n} |\exp[M(|a_x|) - \Omega(|b_y|)](A_\theta \phi)(z)| < \infty
\]

Now from (2.18) the pseudo-differential operator can be written as

\[z^\beta(A_\theta \phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} D^{(\beta)}_{\xi} e^{i(z, \xi)} \theta(z, \xi) \hat{\phi}(\xi) \, du, \quad z, \xi \in \mathbb{C}^n.\]

Using integration by parts we have

\[z^\beta(A_\theta \phi)(z) = (2\pi)^{-n/2} (-1)^{|eta|} \int_{\mathbb{R}^n} e^{i(z, \xi)} D^{(\beta)}_{\xi} [\theta(z, \xi) \hat{\phi}(\xi)] \, du \]

\[= (2\pi)^{-n/2} (-1)^{|eta|} \int_{\mathbb{R}^n} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D^{(\beta-\gamma)}_{\xi} \theta(z, \xi) D^{(\gamma)}_{\xi} \hat{\phi}(\xi) \, du \]

\[= (2\pi)^{-n/2} (-1)^{|eta|} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \int_{\mathbb{R}^n} e^{i(z, \xi)} (D^{(\beta-\gamma)}_{\xi} \theta)(z, \xi) D^{(\gamma)}_{\xi} \hat{\phi}(\xi) \, du \]

\[= (2\pi)^{-n/2} (-1)^{|eta|} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \int_{\mathbb{R}^n} D^{(\alpha)}_{\xi} \left( e^{i(z, \xi+1)} e^{-i(z, 1)} \prod_{j=1}^n [(1 + \xi_j)]^{-\alpha_j} \right) \]

\[= (D^{(\beta-\gamma)}_{\xi} \theta)(z, \xi) D^{(\gamma)}_{\xi} \hat{\phi}(\xi) \, du.\]
Again using integration by parts we have

\[
z^\beta(A\phi)(z) = (2\pi)^{-n/2}(-1)^{|\alpha|+|\beta|} \sum_{|\gamma| \leq |\beta|} \left( \frac{\beta}{\gamma} \right) e^{i(z,\xi)+1} D_z^{(\alpha)}(e^{-i(z,1)} D_{\xi}^{(\beta-\gamma)}(z,\xi)) \int_{\mathbb{R}^n} (1+\xi_j)^{-\alpha_j} D_{\xi}^{(\gamma)} \phi(\xi) \, du
\]

\[
= (2\pi)^{-n/2}(-1)^{|\alpha|+|\beta|} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) e^{i(z,\xi)+1} D_z^{(\alpha-\delta)}(D_{\xi}^{(\beta-\gamma)}(z,\xi)) \int_{\mathbb{R}^n} (1+\xi_j)^{-\alpha_j} e^{-i(z,1)} D_{\xi}^{(\gamma)} \phi(\xi) \, du
\]

\[
= (2\pi)^{-n/2}(-1)^{|\alpha|+|\beta|} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) e^{i(z,\xi)+1} (D_{\xi}^{(\alpha-\delta)} D_z^{(\beta-\gamma)}(z,\xi) \prod_{j=1}^n (1+\xi_j)^{-\alpha_j}) D_{\xi}^{(\gamma)} \phi(\xi) \, du
\]

Then

\[
|z^\beta(A\phi)(z)| \leq (2\pi)^{-n/2} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) \int_{\mathbb{R}^n} |e^{i(z,\xi)}(D_z^{(\alpha-\delta)} D_{\xi}^{(\beta-\gamma)}(z,\xi))| (1+|\xi|)^{-|\alpha|} |D_{\xi}^{(\gamma)} \phi(\xi)| \, du
\]

\[
\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) \int_{\mathbb{R}^n} |e^{i(z+i\xi,1)}| |D_z^{(\alpha-\delta)} D_{\xi}^{(\beta-\gamma)}(z,\xi)| (1+|\xi|)^{-|\alpha|} |D_{\xi}^{(\gamma)} \phi(\xi)| \, du
\]

\[
\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) \int_{\mathbb{R}^n} |\exp(-\langle y, u \rangle - \langle x, t \rangle)| |D_z^{(\alpha-\delta)} D_{\xi}^{(\beta-\gamma)}(z,\xi)| (1+|\xi|)^{-|\alpha|} |D_{\xi}^{(\gamma)} \phi(\xi)| \, du.
\]

Now,

\[
|z^\beta(A\phi)(z)| \leq (2\pi)^{-n/2} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) \int_{\mathbb{R}^n} |\exp([y, u] - [x, t])| |D_z^{(\alpha-\delta)} D_{\xi}^{(\beta-\gamma)}(z,\xi)| (1+|\xi|)^{-|\alpha|} |D_{\xi}^{(\gamma)} \phi(\xi)| \, du
\]

\[
\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) \int_{\mathbb{R}^n} |\exp([y, u] - [x, t])| C_{\alpha-\delta,\beta-\gamma} (1+|\xi|)^{|\beta|-|\gamma|-|\alpha|} |D_{\xi}^{(\gamma)} \phi(\xi)| \, du
\]

Using inequality \((1+|\xi|)^{|\beta|-|\gamma|-|\alpha|} \leq \exp[M^0(a_1 u) + \Omega^0(b_1 t)] \) for \(|\beta| + |\gamma| - |\alpha| > 0\).
and (2.9) we have
\[ |z^\beta (A_\theta \phi)(z)| \leq (2\pi)^{-n/2} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) C_{\alpha-\delta, \beta-\gamma} \int_{\mathbb{R}^n} \left| \exp[(y, u) - (x, t)] \right| \exp[M^0(a_1 u) + \Omega^0(b_1 t)]\]
\[ \exp[-M^0(a'' u) + \Omega^0(b'' t)] \, du. \]

Using (2.3) and (2.4) we have
\[ |z^\beta (A_\theta \phi)(z)| \leq (2\pi)^{-n/2} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) C_{\alpha-\delta, \beta-\gamma} \int_{\mathbb{R}^n} \left| \exp[(y, u) - M^0((a'' - a_1) u)] \exp[-(x, t) + \Omega^0(b_1 + b'') t] \right| \]
\[ \leq (2\pi)^{-n/2} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) C_{\alpha-\delta, \beta-\gamma} \int_{\mathbb{R}^n} \left| \exp[(y, u) - M^0((a'' - a_1) u)] \right| \]
\[ \leq (2\pi)^{-n/2} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) C_{\alpha-\delta, \beta-\gamma} \int_{\mathbb{R}^n} \left| \exp[(y, u) - M^0((a'' - a_1) u)] \right| du. \]

Using (2.11) and (2.12) and the arguments of [2, p.134]
\[ |z^\beta (A_\theta \phi)(z)| \leq (2\pi)^{-n/2} \sum_{|\gamma| \leq |\beta|} \sum_{|\delta| \leq |\alpha|} \left( \frac{\beta}{\gamma} \right) \left( \frac{\alpha}{\delta} \right) C_{\alpha-\delta, \beta-\gamma} \exp[-M[[b'' + b_1]^{-1} x] + \Omega[[a'' - 2a_1]^{-1} y]] \]
\[ \int_{\mathbb{R}^n} \left| \exp[-M^0(a_1 u)] \right| du \]
\[ \leq C_{\alpha, \beta} \exp[-M[[b'' + b_1]^{-1} x] + \Omega[[a'' - 2a_1]^{-1} y]]. \]

Hence
\[ |\exp[M((b_1 + b'')^{-1} x) - \Omega[[a'' - 2a_1]^{-1} y]](A_\theta \phi)(z)| \leq C_{\alpha, \beta}(1 + |z|^\beta)^{-1}. \]

Thus
\[ \sup_{z \in \mathbb{C}^n} |\exp[M((b_1 + b'')^{-1} x) - \Omega[[a'' - 2a_1]^{-1} y]](A_\theta \phi)(z)| \leq C_{\alpha, \beta} \]
\[ < \infty. \]

This implies that
\[ (A_\theta \phi)(z) \in W_{\gamma}^M(\mathbb{C}^n). \]

**Theorem 3.2.** $A_\theta$ is continuous linear mapping $W_{\gamma}^M(\mathbb{C}^n)$ into itself.

**Proof.** If the functions $\phi(z)$ converge uniformly to zero as $\nu \to \infty$ in any bounded domain of the $z$-plane and in addition satisfy the inequalities
\[ |\phi_\nu (z)| \leq C \exp[-M[[a x]] + \Omega[[b y]]], \]
then the sequence $\phi_\nu(z) \in W_{\gamma}^M(\mathbb{C}^n)$ is said to converge to zero as $\nu \to \infty$, where the constants $C$, $a$ and $b$ do not depend on the index $\nu$.

Since from Theorem 3.1 $A_\theta \phi$ is a mapping from $W_{\gamma}^M(\mathbb{C}^n)$ into itself. Using above results, $A_\theta \phi_\nu \in W_{\gamma}^M(\mathbb{C}^n)$ converge to zero uniformly in any bounded domain of the $z$-plane as $\nu \to \infty$ and satisfies the above inequality. Therefore, the sequence $A_\theta \phi \in W_{\gamma}^M(\mathbb{C}^n)$ is converges to zero as $\nu \to \infty$. This shows that $A_\theta$ maps continuously into itself.

Now, we define the pseudo-differential operator $A_\theta$ on $W_{\gamma}^M(\mathbb{C}^n)'$-space by
\[ (A_\theta f, \phi) = (f, A_\theta \phi), \quad \phi \in W_{\gamma}^M(\mathbb{C}^n). \]
Theorem 3.3. \( A_\theta \) is a linear mapping from \([W^\Omega_{M} (C^n)]'\) into itself.

**Proof.** Let \( f \in [W^\Omega_{M} (C^n)]' \). Then, for any sequence \( \{ \phi_\nu \} \) of functions in \( W^\Omega_{M} (C^n) \) converging to zero in \( W^\Omega_{M} (C^n) \), as \( \nu \to \infty \). From (2.20) we have

\[
\langle A_\theta f, \phi_\nu \rangle = \langle f, A_\theta^* \phi_\nu \rangle, \quad \nu = 1, 2, 3, \ldots
\]  

(3.2)

By the arguments of Theorem 3.2, we conclude that \( \langle A_\theta f, \phi_\nu \rangle \to 0 \) as \( \nu \to \infty \). Hence \( A_\theta f \in [W^\Omega_{M} (C^n)]' \).

**Definition 3.4.** A sequence of distributions \( \{ f_\nu \} \) in \([W^\Omega_{M} (C^n)]'\) is said to converge to zero in \([W^\Omega_{M} (C^n)]'\) if \( \langle f_\nu, \phi \rangle \to 0 \) as \( \nu \to \infty \) for all \( \phi \in W^\Omega_{M} (C^n) \).

**Theorem 3.5.** \( A_\theta \) maps continuously \([W^\Omega_{M} (C^n)]'\) into itself.

**Proof.** Let \( \phi \in [W^\Omega_{M} (C^n)]' \). Then, using (3.2) and the fact that \( f_\nu \to 0 \) in \([W^\Omega_{M} (C^n)]'\) as \( \nu \to \infty \),

\[
\langle A_\theta f_\nu, \phi \rangle = \langle f_\nu, A_\theta^* \phi \rangle \to 0
\]

as \( \nu \to \infty \). Hence \( A_\theta f_\nu \to 0 \) in \([W^\Omega_{M} (C^n)]'\) as \( \nu \to \infty \), and the proof is complete.

**Theorem 3.6.** Let \( \theta \in C^k(C^n), \ k \geq n/2 \), be such that there exists a positive constant \( B \) such that

\[
|\langle \partial^\alpha \theta \rangle| \leq C_{\alpha, n} (1 + |\xi|)^{-|\alpha|}, \quad \xi \neq 0
\]

(3.3)

for multi-indices \( \alpha \) with \( |\alpha| \leq k \). Then, for \( 1 \leq p < \infty \), there exists a positive constant \( B \), depending on \( \alpha \) and \( N \), such that

\[
\| (A\phi) (z) \|_p \leq M_{\alpha, n} \| \phi \|_p, \quad \phi \in W^\Omega_{M} (C^n),
\]

(3.4)

where

\[
(A\phi)(z) = (2\pi)^{-n/2} \int_{R^n} e^{i(z,\xi)} \hat{\theta}(\xi) \hat{\phi}(\xi) \, d\xi,
\]

(3.5)

\( \xi = u + it \), and \( \hat{\phi} \) denotes the Fourier transformation of \( \phi \).

**Proof.** (3.5) can be written as

\[
(A\phi)(z) = (2\pi)^{-n/2} F^{-1}[\theta(\xi) \hat{\phi}(\xi)](z)
\]

(3.6)

where \( F^{-1} \) denotes the inverse Fourier transformation of a function \( z \) as \( z = x + iy \).

Now, we assume that

\[
F^{-1}[\theta(\xi) \hat{\phi}(\xi)](z) = (f * g)(z).
\]

(3.7)

Then by convolution property of Fourier transformation, we have

\[
\theta(\xi) \hat{\phi}(\xi) = F[(f * g)](\xi)
\]

\[
= \hat{f}(\xi) \cdot \hat{g}(\xi).
\]

This implies that

\[
f(z) = F^{-1}[\theta(\xi)](z), \quad g(z) = \phi(z).
\]

Thus, the expression (3.7) yields

\[
(A\phi)(z) = (2\pi)^{-n/2} (F^{-1}[\theta(\xi)] * \phi)(z).
\]

Using convolution property \( \| f * \phi \|_p \leq \| f \|_1 \| \phi \|_p \) for \( f \in L^1(R^n) \) and \( \phi \in L^p(R^n) \) we have

\[
\| (A\phi)(z) \|_p = (2\pi)^{-n/2} \| (F^{-1}[\theta(\xi)] * \phi)(x) \|_p
\]

\[
\leq (2\pi)^{-n/2} \| F^{-1}[\theta(\xi)] \|_1 \| \phi \|_p.
\]

(3.8)

Next, we have to prove that

\[
F^{-1}[\theta(\xi)] \in L^1(R^n).
\]

Thus, from [3, p. 24] we have

\[
F^{-1}[\theta(\xi)](z) = (2\pi)^{-n/2} \int_{R^n} e^{i(z,\xi)} \theta(\xi) \, d\xi.
\]
By property of Fourier transformation the above expression gives
\[(z)^\alpha F^{-1}[\theta(\xi)](z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} D_\xi^{(\alpha)}(e^{-i(z,\xi)})\theta(\xi) \, du.\]

Integration by parts, above expression can be obtained
\[\frac{\partial}{\partial z^j} e^{-i(z,\xi)} \theta(\xi) = (i\xi_j - \partial_j) e^{-i(z,\xi)} \theta(\xi)\]
by property of Fourier transformation the above expression gives
\[\frac{\partial}{\partial z^j} e^{-i(z,\xi)} \theta(\xi) = (i\xi_j - \partial_j) e^{-i(z,\xi)} \theta(\xi)\]
for \(\xi\) and \(\partial\) as differential operators.

Therefore,
\[\lambda, \xi = (\alpha,\beta), \text{ depending on } \alpha \text{ and } \beta \text{ and positive integers } N\]
then, by using the property of Fourier transformation we have
\[\lambda, \xi = (\alpha,\beta), \text{ depending on } \alpha \text{ and } \beta \text{ and positive integers } N\]

This implies that
\[\|F^{-1}[\theta(\xi)]\| \leq B_{\alpha,n}(1 + |z|^n)^{-1}.\]  

From (3.8) - (3.9), we find the required result (3.4)

**Theorem 3.7.** Let \(\phi \in W^B_M(\mathbb{C}^n)\) and symbol \(\theta_m(z,\xi)\) has compact support in \(z\). Then, pseudo-differential operators \(A_{\theta_m,\phi}\) can be expressed as
\[\langle A_{\theta_m,\phi} \rangle(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(z,\xi)} \theta_m(z,\xi) \theta(\xi) \, du\]
where
\[\langle A_{\lambda,\phi} \rangle(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(z,\lambda)} \theta_m(z,\xi) \theta(\xi) \, du\] 
(3.10)
as \(z = x + iy, \lambda = v + iv'\)
\[\hat{\theta}_m(z,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(z,\xi)} \theta_m(z,\xi) \, dx, \quad \lambda, \xi \in \mathbb{C}^n.\]

**Proof.** Since
\[\langle A_{\lambda,\phi} \rangle(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(z,\xi)} \theta_m(z,\xi) \theta(\xi) \, du,\]
then, by using the property of Fourier transformation we have
\[\langle A_{\theta_m,\phi} \rangle(z) = (2\pi)^{-n/2} \left( \int_{\mathbb{R}^n} e^{-i(z,\lambda)} \int_{\mathbb{R}^n} e^{-i(z,\xi)} \theta_m(z,\xi) \, dx \, du \right) \theta(\xi) \, du\]
as \(\lambda = v + iv'\). By Fubini’s theorem and (3.10) we get
\[\langle A_{\theta_m,\phi} \rangle(z) = (2\pi)^{-n} \left( \int_{\mathbb{R}^n} e^{-i(z,\lambda)} \int_{\mathbb{R}^n} e^{-i(z,\xi)} \theta_m(z,\xi) \theta(\xi) \, du \right) \, dx.\] 
(3.11)

**Lemma 3.8.** For all multi-indices \(\alpha\) and \(\beta\) and positive integers \(N\), there is a positive constant \(C_{\alpha,N}\), depending on \(\alpha\) and \(N\) such that
\[\|D_\xi^{(\alpha)}\theta_m(\lambda,\xi)\| \leq C_{\alpha,N}(1 + |\lambda|^{|\beta|})(1 + |\xi|)^{-|\alpha|}\]
for \(\xi = u + it\) and \(\lambda = v + iv'\).

**Proof.** The Fourier transformation of \(\theta_m\) with respect to \(\lambda = v + iv'\) is given by
\[\hat{\theta}_m(\lambda,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(z,\lambda)} \theta_m(z,\xi) \, dx.\]
Proof. From Wong [10, p. 80], we can write

\[ (i\lambda)^{n/2} D^{(n)}_{\xi} \hat{\theta}_m(\lambda, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \partial_{\xi}^{(\beta)} [e^{-i(z,\lambda)}] D^{(\alpha)}_{\xi} \theta_m(z, \xi) \, dx. \]

Integration by parts we have

\[ (i\lambda)^{n/2} D^{(n)}_{\xi} \hat{\theta}_m(\lambda, \xi) = (2\pi)^{-n/2} (-1)^{n/2} \int_{\mathbb{R}^n} e^{i(z,\lambda)} \partial_{\xi}^{(\beta)} D^{(\alpha)}_{\xi} \theta_m(z, \xi) \, dx \]

as \( z = x + iy \).

Hence,

\[
(i\lambda)^{n/2} D^{(n)}_{\xi} \hat{\theta}_m(\lambda, \xi) = (2\pi)^{-n/2} (-1)^{n/2} \int_{\mathbb{R}^n} e^{i(z,\lambda)} \partial_{\xi}^{(\beta)} D^{(\alpha)}_{\xi} \theta_m(z, \xi) \, dx
\]

\[
= (2\pi)^{-n/2} (-1)^{n/2} \int_{\mathbb{R}^n} e^{i(z,\lambda)} \sum_{|\gamma| \leq \beta} \left( \frac{\beta}{\gamma} \right) D^{(\gamma)}_{\xi} \eta(z - m) \, dx.
\]

Now

\[
|\lambda^{n/2} D^{(n)}_{\xi} \hat{\theta}_m(\lambda, \xi)|
\]

\[
\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \left( \frac{\beta}{\gamma} \right) \int_{\mathbb{R}^n} |\exp(-(x, v) - (y, v'))| \, |D^{(\gamma)}_{\xi} \eta(z - m)| \, |D^{(\beta - n)}_{\xi} D^{(\alpha)}_{\xi} \theta_m(z, \xi)| \, dx
\]

\[
\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \left( \frac{\beta}{\gamma} \right) \int_{\mathbb{R}^n} |\exp(-(x, v) - (y, v'))| \, |\partial_{\xi}^{(\gamma)} \eta(z - m)| \, C_{\beta - \gamma, \alpha} (1 + |\xi|)^{-|\alpha|} \, dx
\]

\[
\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \left( \frac{\beta}{\gamma} \right) C_{\beta - \gamma, \alpha} (1 + |\xi|)^{-|\alpha|} \int_{\mathbb{R}^n} |\exp(-(x, v))| \, |\partial_{\xi}^{(\gamma)} \eta(z - m)| \, dx.
\]

Then

\[
|\lambda^{n/2} D^{(n)}_{\xi} \hat{\theta}_m(\lambda, \xi)|
\]

\[
\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \left( \frac{\beta}{\gamma} \right) C_{\beta - \gamma, \alpha} (1 + |\xi|)^{-|\alpha|} \int_{\mathbb{R}^n} |\partial_{\xi}^{(\gamma)} \eta(z - m)| \, dx
\]

\[
\leq (2\pi)^{-n/2} (1 + |\xi|)^{-|\alpha|} \sum_{|\gamma| \leq \beta} \left( \frac{\beta}{\gamma} \right) C_{\gamma} C_{\beta - \gamma, \alpha}
\]

\[
\leq (2\pi)^{-n/2} (1 + |\xi|)^{-|\alpha|} C_{\beta}
\]

\[
\leq C_{\beta, \alpha}(1 + |\xi|)^{-|\alpha|}.
\]

Hence, for large arbitrary positive integers \( N \), we have

\[
\left| D^{(\alpha)}_{\xi} \hat{\theta}_m(\lambda, \xi) \right| \leq C_{n, \beta}(1 + |\lambda|^N - (1 + |\xi|)^{-|\alpha|}.
\]

as \( \xi = u + i t \).

**Theorem 3.9.** Let \( \theta \in V^0 \). Then we get the following relation

\[
\int_{Q_m} |(A_{\theta} \phi)(z)|^p \, dx \leq C_N^p \|\phi\|^p_p \quad \forall \phi \in W_M^n(C^0).
\]

**Proof.** From Wong [10, p. 80], we can write

\[
\left( \int_{Q_m} |(A_{\theta} \phi)(z)|^p \, dx \right) \leq \left( \int_{\mathbb{R}^n} |(A_{\theta_m} \phi)(z)|^p \, dx \right). \tag{3.12}
\]
Using Lemma 3.8 and Theorem 3.6, we find that
\[ \|A_\lambda \phi\|_p \leq C_N (1 + |\lambda|)^{-N} \|\phi\|_p \quad \forall \phi \in W^Q_M(\mathbb{C}^n). \quad (3.13) \]

Using (3.11), (3.13) and Minkowski’s inequality in the integral form we obtain
\[
\|A_{\theta_m} \phi\|_p = (2\pi)^{-n/2} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(z: \lambda)} (A_\lambda \phi)(z) dv |z|^{p} dx \right)^{1/p} \\
= (2\pi)^{-n/2} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp[-\langle x, v \rangle - \langle y, v \rangle] |(A_\lambda \phi)(z)| dv |z|^{p} dx \right)^{1/p} \\
\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \exp[-\langle x, v \rangle] |(A_\lambda \phi)(z)| dv \right)^{1/p} \|z\|_p dx \\
\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|A_\phi(z)\|_p dv.
\]

Using (3.13) we get
\[
\|A_{\theta_m} \phi\|_p \leq (2\pi)^{-n/2} C_N \left( \int_{\mathbb{R}^n} (1 + |\lambda|)^{-N} dv \right) \|\phi\|_p \\
\leq (2\pi)^{-n/2} C_N \|\phi\|_p, \quad \phi \in W^Q_M(\mathbb{C}^n).
\]

Hence from (3.12) and (3.13) we have
\[
\int_{Q_m} |(A_\phi(z)|^p dx \leq C_{N,q,n} \|\phi\|_p, \quad \phi \in W^Q_M(\mathbb{C}^n). \quad (3.14)
\]

Now, we represent $A_\theta$ as a singular integral operator.

**Lemma 3.10.** Let $K(z, w) = \int_{\mathbb{R}^n} e^{i(z: \omega)} \theta(z, w) ds$, $z = x + iy \in \mathbb{C}^n$ $w = s + iv \in \mathbb{C}^n$ in the distributional sense. Then

(i) for each $z \in \mathbb{C}^n$, $K(z, w)$ is a function defined on $\mathbb{R}^n$,

(ii) for each sufficiently large positive integer $N$, there is a positive constant $C_N$ such that
\[
|K(z - w, w)| \leq C_N (1 + |z - w|^N)^{-1}, \quad (3.15)
\]

(iii) for each fixed $z = x + iy$ and $\phi \in W^Q_M(\mathbb{C}^n)$ vanishing in the neighbourhood of $\mathbb{C}^n$, we find that
\[
(A_\theta \phi)(z) = \int_{\mathbb{R}^n} K(z, w) \phi(w) ds. \quad (3.16)
\]

**Proof.** (i) can be defined by using the arguments of [10, p. 26] and [1, pp. 23-24].

To prove (ii), let $\alpha$ be a multi-index with length greater than $w$. Then by the property of Fourier transformation $(D^{(\alpha)} u) = \xi^{(\alpha)} \hat{u}$ we have
\[
(iw)^\alpha K(z, w) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} e^{i(x, w)} D^{(\alpha)} \theta(z, \xi) du.
\]
Therefore, using (2.16) and tools of theorem (3.6) we have
\[
|K(z, w)| \leq C'_\alpha (1 + |w|^N)^{-1}.
\]

For large positive integer $N$ we can obtain
\[
|K(z, z - w)| \leq C'_\alpha (1 + |z - w|^N)^{-1}.
\]
To prove (iii), we define the distribution $L_z$ on $W^0_{M}(\mathbb{C}^n)$ by

$$\langle L_z, \psi \rangle = \int_{\mathbb{R}^n} \theta(z, \xi) \psi(\xi) \, du,$$

where $z = x + iy, \xi = u + i\tau$ and $w = s + iv$. By the definition of pseudo-differential operator (2.18)

$$(A_{\theta}\phi)(z) = \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \theta(z, \xi) \hat{\phi}(\xi) \, du = L_z(M_{z}\hat{\phi})$$

(3.17)

Using Gelfand and Shilov [3] technique of integration we get

$$(A_{\theta}\phi)(z) = \hat{L}_z(T_z\phi) = \hat{L}_z(T_z\phi)$$

(3.18)

From (i) we have

$$\hat{L}_z(\psi) = \int_{\mathbb{R}^n} \theta(z, -w) \psi(w) \, ds.$$

Hence

$$(A_{\theta}\phi)(z) = \int_{\mathbb{R}^n} \theta(z, -w)(T_z\phi)(w) \, ds = \int_{\mathbb{R}^n} \theta(z, -w)\phi(z + w) \, ds = \int_{\mathbb{R}^n} \theta(z, z - w)\phi(w) \, ds.$$

This completes the proof of the theorem.

**Theorem 3.11.** Let $\theta(z, \xi)$ be a symbol in $V^0$. Then $A_{\theta} : L^n(\mathbb{R}^n) \rightarrow L^n(\mathbb{R}^n)$ is a bounded linear operator for $1 < p < \infty$.

**Proof.** From Theorem 3.6, Theorem 3.7, Theorem 3.9 and Lemma 3.10 we can show that the pseudo-differential operator $A_{\theta}$ is a bounded linear operator from $L^n(\mathbb{R}^n)$ into $L^n(\mathbb{R}^n)$ for $1 < p < \infty$. \qed

4 The Sobolev Space

In this section, we study the pseudo-differential operators on Sobolev type space $G^{s,p}(\mathbb{C}^n)$ which is defined in Section 2.

For $s \in \mathbb{R}$, the pseudo-differential operator associated with symbol $\theta(\xi) = (1 + |\xi|^2)^{-s/2}$ as $\xi = u + it$ is defined by

$$(V_{\theta}u)(z) = F^{-1}(\hat{\theta}(\xi)\hat{u}(\xi))(z) \quad \text{for } u \in \left[W^0_{M}(\mathbb{C}^n)\right]',$$

(4.1)

Now, we define the Sobolev space $G^{s,p}(\mathbb{C}^n)$ of $L^n$-type to be the set of all distribution $u \in \left[W^0_{M}(\mathbb{C}^n)\right]'$ such that

$$\|u\|_{s,p} = \|V_{-s}u\|_p \quad \text{for } 1 \leq p < \infty.$$

(4.2)

**Theorem 4.1.** Let $u \in \left[W^0_{M}(\mathbb{C}^n)\right]'$. Then

(i) $V_{s}V_{t}u = V_{s+t}u$,

(ii) $V_{0}u = u$.

**Proof.** The proof of the above theorem is obvious from [10, p. 90].

**Theorem 4.2.** $G^{s,p}(\mathbb{C}^n)$ is a Banach space with respect to $\|u\|_{s,p}$.

**Proof.** The proof of the above theorem is usual from [10, p. 81].
Theorem 4.3. $V_t$ is an isometry from $V^{s,p}$ onto $V^{s+t,p}$.

**Proof.** Let $u \in V^{s,p}$. Then from Theorem 4.1 we get $J_{-t}v \in G^{s,p}(\mathbb{C}^n)$ and $v_t v_{-t}v = v$. This implies $G^{s,p}(\mathbb{C}^n)$ is onto.

Theorem 4.4. Let $1 < p < \infty$ and $s \leq t$. Then $G^{t,p}(\mathbb{C}^n) \subseteq G^{s,p}(\mathbb{C}^n)$.

**Proof.** See [10, p. 91]. This is called Sobolev embedding theorem.

Theorem 4.5. Let $s \geq 0$ and $1 \leq p < \infty$. Then

$$\|V_\theta \phi\|_p \leq \|\phi\|_p, \quad \phi \in L^p(\mathbb{R}^n).$$

**Proof.** We have

$$(J_s \phi)^\wedge(\xi) = (1 + |\xi|^2)^{-s/2} \hat{\phi}(\xi), \quad \xi \in \mathbb{C}^n.$$ 

Hence, for $\hat{G}_s(\xi) = (1 + |\xi|^2)^{-s/2}$ we have

$$(G_s * \phi)^\wedge(\xi) = (1 + |\xi|^2)^{-s/2} \hat{\phi}(\xi). \quad (4.3)$$

Hence, for all $\phi \in W^p_{\text{loc}}(\mathbb{R}^n)$,

$$J_s \phi = (G_s * \phi),$$

and using convolution property

$$\|J_s \phi\|_p = \|G_s * \phi\|_p \leq \|G_s\|_1 \|\phi\|_p \leq \|\phi\|_p.$$ 

Theorem 4.6. For symbol $\theta$ in $U^m$, $A_{\theta} : G^{m,p}(\mathbb{C}^n) \to G^{0,p}(\mathbb{C}^n)$ is a bounded linear operator for $1 < p < \infty$.

**Proof.** Consider the bounded linear operators

$$V_{-s} : G^{s,p}(\mathbb{C}^n) \to G^{0,p}(\mathbb{C}^n)$$

$$A_{\theta} V_m : G^{0,p}(\mathbb{C}^n) \to G^{0,p}(\mathbb{C}^n)$$

and

$$V_{s-m} : G^{0,p}(\mathbb{C}^n) \to G^{s-m,p}(\mathbb{C}^n).$$

The first and the third operators are bounded by isometry of pseudo-differential operator of Theorem 4.3 and the second operator is bounded by $L^p(\mathbb{R}^n)$-boundedness property of pseudo-differential operator. Hence the product $V_{s-m} A_{\theta} V_{m-s}$ is a bounded linear operator from $G^{s,p}$ into $G^{s-m,p}$. By Theorem 4.3 operators $V_{m-s}$ and $V_{s-m}$ are isometric and onto. Hence, $A_{\theta} : G^{m,p} \to G^{0,p}$ must be bounded linear operator.

Theorem 4.7. Let $\theta(z, \xi)$ be any symbol in $V^m$, then $A_{\theta} : G^{s,p}(\mathbb{C}^n) \to G^{s-m,p}(\mathbb{C}^n)$ is a bounded linear operator for $1 \leq p < \infty$.

**Proof.** Since $V_{m-s} A_{\theta}$ is a pseudo-differential operator with symbol in $V^s$. Hence, from Theorem 4.6 we can easily prove that

$$\|A_{\theta} u\|_{s-m,p} = \|J_{m-s} A_{\theta} u\|_p \leq C \|u\|_{s,p} \quad \forall u \in G^{s,p}.$$ 

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