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PSEUDO-DIFFERENTIAL OPERATORS ON $W_M^\Omega(\mathbb{C}^n)$-SPACE

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Abstract. Pseudo-differential operators associated with symbol $\theta(z,\xi), z = x + iy$ and $\xi = u + it$ on $W_M^\Omega(\mathbb{C}^n)$-space is defined and using the theory of Fourier transformation its various properties are studied. $L^p(\mathbb{R}^n)$- boundedness is investigated for $1 < p \leq \infty$. Sobolev space associated with distributional space $[W_M^\Omega(\mathbb{C}^n)]'$ is defined and its properties are obtained.

1 Introduction

The spaces $W_M(\mathbb{R}^n)$, $W^\Omega(\mathbb{C}^n)$ and $W_M^\Omega(\mathbb{C}^n)$ were introduced and analyzed by Gurevich [4], Gel'fand and Shilov [3] and Friedman [2]. They applied these W-type spaces for investigating uniqueness and corrected class of Cauchy problem and other problems of partial differential equations by using Fourier transformation tool. Recently, continuous Wavelet transformation on W-spaces are studied by [6, 9] and many interesting properties are obtained.

The theory of pseudo-differential operators is developed by Wong [10], Rodino [1], Pathak [5] and others. They exploited this theory on Schwartz space $S(\mathbb{R}^n)$, Gel'fand and Shilov space of type S and Gevery space by using the technique of Fourier transformation.

They also studied psuedo-differential operators on their respective Sobolev space and got many important results. Sobolev spaces are used to study the minimal-maximal properties, global regularities and spectral properties of pseudo-differential operators on Schwartz space $S(\mathbb{R}^n)$. The Pseudo differential operators on $W_M(R^n)$ and $W^\Omega(C^n)$ are studied by author and others and obtained many important results, see [7, 8].

Our main aim in this paper is to introduce more general symbol and precise study of pseudo-differential operators on $W_M^\Omega(\mathbb{C}^n)$-space and to study many properties because its dual space $[W_M^\Omega(\mathbb{C}^n)]'$ is more general than $[W_M(\mathbb{R}^n)]'$ and Schwartz distributional space $S'(\mathbb{R}^n)$.

The present article is divided into three sections. Section 2 gives the various definitions of pseudo-differential operators, symbol, W-type spaces, Fourier transformation and Sobolev space. Section 3 contains a study of properties of pseudo-differential operators on $W_M^\Omega(\mathbb{C}^n)$ space and $L^p(\mathbb{R}^n)$-boundedness result of pseudo differential operators. In the last section, using $L^p(\mathbb{R}^n)$-boundedness result, the Sobolev space $G^{s,p}(\mathbb{C}^n)$, $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ on $[W_M^\Omega(\mathbb{C}^n)]'$ space is defined and it is proved that the pseudo-differential operator $A_\theta : G^{s,p}(\mathbb{C}^n) \to G^{o,p}(\mathbb{C}^n)$ and $A_\theta : G^{s,p}(\mathbb{C}^n) \to G^{s-m,p}(\mathbb{C}^n)$ are bounded linear operator for $s, m \in \mathbb{R}$.

2 Preliminary

Now in this section we recall the definitions of $W_M(\mathbb{R}^n)$, $W^\Omega(\mathbb{C}^n)$ and $W_M^\Omega(\mathbb{C}^n)$ from [2] and [3].

Let M_j and Ω_j be the convex functions such that

$$M_j(x_j) = \int_0^{x_j} \mu_j(\xi_j)\, d\xi_j \qquad (x_j \geq 0) \tag{2.1}$$

and

$$\Omega_j(y_j) = \int_0^{y_j} w_j(\eta_j)\, d\eta_j \qquad (y_j \geq 0) \tag{2.2}$$

for $j = 1, 2, 3, \cdots n$.
We set

$$\begin{aligned} \mu(\xi) &= (\mu_1(\xi_1), \cdots, \mu_n(\xi_n)) \\ w(n) &= (w_1(\eta_1), \cdots, w_n(\eta_n)) \end{aligned}$$

and

$$M_j(-x_j) = M_j(x_j), \qquad M_j(x_j) + M_j(x_j') \leq M_j(x_j + x_j') \tag{2.3}$$

$$\Omega_j(-y_j) = \Omega_j(y_j), \qquad \Omega_j(y_j) + \Omega_j(y_j') \leq \Omega_j(y_j + y_j'). \tag{2.4}$$

The space $W_M(\mathbb{R}^n)$ consists of all $\mathbb{C}^\infty$-functions which satisfy the inequalities

$$|D_x^{(k)}\phi(x)| \leq C_k \exp[-M(ax)], \tag{2.5}$$

where $D_x^{(k)} = D_{x_1}^{(k_1)} D_{x_2}^{(k_2)} \cdots D_{x_n}^{(k_n)}$,

$$\exp[-M(ax)] = \exp[-M_1(a_1x_1) \cdots - M_n(a_nx_n)] \tag{2.6}$$

and constants $C_k, a > 0$ depending on the function ϕ. A function $\phi(z) \in W^\Omega(\mathbb{C}^n)$ if and only if for $b > 0$ there exists a constant $C_k > 0$ such that

$$|z^k\phi(z)| \leq C_k \exp[\Omega(by)], \qquad z = x + iy \tag{2.7}$$

where

$$z^k = z_1^{k_1} z_2^{k_2} z_3^{k_3} \cdots z_n^{k_n}$$

and

$$\exp[\Omega(by)] = \exp[\Omega_1(b_1y_1) + \cdots + \Omega_j(b_jy_j) + \cdots + \Omega_n(b_ny_n)], \tag{2.8}$$

the constants C_k and $b > 0$ depend on the function ϕ.
The space $W_M^\Omega(C^n)$ consists of all entire analytic functions $\phi(z)$ which satisfy

$$|\phi(z)| \leq C \exp[-M[(ax)] + \Omega[(by)]], \tag{2.9}$$

where $z = x + iy$ and $\exp[-M(ax)]$ and $\exp[\Omega(by)]$ have similar meaning like (2.5) and (2.7) and constants C, a and b depend on the function ϕ.

Now, we define the duality of functions $M(x)$ and $\Omega(y)$ in the following way:
Let $M_j(x_j)$ and $\Omega_j(y_j)$ be defined by (2.1) and (2.2) respectively and let $\mu_j(\xi)$ and $w_j(\eta_j)$ be mutually inverse, that is $\mu_j(w_j(\eta_j)) = \eta_j,\ \eta_j(\mu_j(\xi)) = \xi_j$, then the corresponding functions $M_j(x_j)$ and $\Omega_j(y_j)$ are called dual in sense of Young. In this case the Young inequality is

$$x_jy_j \leq M_j(x_j) + \Omega_j(y_j). \tag{2.10}$$

This inequality holds for any $x_j \geq 0$, $y_j \geq 0$ and equality holds if and only if $y_j = \mu_j(x_j)$, where x_j varies in the interval $x_j^0 < x_j < \infty$ and y_j varies in the interval $y_j^0 < y_j < \infty$. That equality will be

$$x_jy_j = M_j^0(x_j) + \Omega_j(y_j) \tag{2.11}$$

$$x_jy_j = M_j(x_j) + \Omega_j^0(y_j) \tag{2.12}$$

for $M_j(x_j) < M_j^0(x_j)$ and $\Omega^0(y_j) > \Omega_j(y_j)$.

From [2, pp.132-133,Theorem 12] and [2, p.134,Theorem 13 and Theorem 15] the Fourier transformation of a function $\phi \in W_M^\Omega(\mathbb{C}^n)$ is defined by

$$\hat{\phi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\phi(z)\, dx \tag{2.13}$$

for $z = x + iy$ and $\xi = u + it$.

From [1] and [2] the Fourier duality relation is given by

$$F[W^\Omega(\mathbb{C}^n)] = W_M(\mathbb{R}^n), \qquad F[W_M(\mathbb{R}^n)] = W^\Omega(\mathbb{C}^n) \text{ and}$$

$$F[W_M^\Omega(\mathbb{C}^n)] = W_{M^0}^{\Omega^0}(\mathbb{C}^n).$$

Now, we recall the definitions of translation T and modulation M of function f on $\mathbb{R}^n$ from [10, p. 14]:

Let f be a measurable function defined on $\mathbb{R}^n$. For any fixed $y \in \mathbb{R}^n$, we define $T_y f$ and $M_y f$ by

$$(T_y f)(x) = f(x+y) \tag{2.14}$$

and

$$(M_y f)(x) = e^{ixy} f(x). \tag{2.15}$$

Next, we define a symbol associated with pseudo-differential operator by the following way:

The function $\theta(z,\xi) \in \mathbb{C}^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ which is the set of all entire analytic functions of $z = x+iy$ and $\xi = u+it$ is said to be class V^m iff for any two multiindices α and β, there is a positive constant $C_{\alpha,\beta}$ depending on α and β only such that

$$|(D_z^{(\alpha)} D_\xi^{(\beta)} \theta)(z,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}, \tag{2.16}$$

where $m \in \mathbb{R}$ and $z, \xi \in \mathbb{C}^n$. If we take $y=0$, $t=0$ then the symbol will to the well known class of S^m.

Theorem 2.1. Let $\phi \in W_M^\Omega(\mathbb{C}^n)$ and let $\sigma(z,\xi)$ be an entire function in (z,ξ) and satisfy

$$|\sigma(z,\xi)| \leq C(1+|\xi|)^m,$$

then $\quad \phi(\xi)\sigma(z,\xi) \in W_M^\Omega(\mathbb{C}^n)$.

Proof. Let $\phi \in W_M^\Omega(\mathbb{C}^n)$ and $|\sigma(z,\xi)| \leq C(1+|\xi|)^m$. Then

$$\begin{aligned} |\sigma(z,\xi)\phi(\xi)| &= |\sigma(z,\xi)||\phi(\xi)| \\ &\leq C(1+|\xi|)^m exp[-M(au) + \Omega(bt)]. \end{aligned}$$

Since $(1+|\xi|)^m \leq exp[-M(a_0 u) + \Omega(b_0 t)]$, $m \in \mathbb{R}$ then using the definition of $W_M^\Omega(\mathbb{C}^n)$ space we find that

$$|\sigma(z,\xi)\phi(\xi)| \leq C\ exp[-M(a_0 u) + \Omega(b_0 t)]\ exp[M(au) + \Omega(bt)].$$

By the definition of convex function (2.1) and (2.2), we get

$$|\sigma(z,\xi)\phi(\xi)| \leq C\ exp\left[[-M[(a-a_0)u] + \Omega[(b+b_0 t)]\right].$$

This implies that

$$\phi(\xi)\sigma(z,\xi) \in W_M^\Omega(\mathbb{C}^n).$$

Using this argument and argument of Fourier transform in $W_M^\Omega(\mathbb{C}^n)$, we can define the partial differential operator (2.17) and pseudo differential operator (2.18).

A linear partial differential operator $P(z,D)$ as $z = x+iy$ on $\mathbb{C}^n$ is given by

$$P(z,\ D) = \sum_{|\alpha| \leq m} a_\alpha(z)\ D^{(\alpha)}.$$

If we replace $D^{(\alpha)}$ by a monomial $\xi^\alpha \ \in \mathbb{R}^n$ then we get a symbol

$$P(z,\ \xi) = \sum_{|\alpha| \leq m} a_\alpha(z)\ \xi^\alpha.$$

We take $\phi \in W_M^\Omega(\mathbb{C}^n)$ then we get

$$\left(P(z,\ D)\ \phi\right)(z) = \sum_{|\alpha| \leq m} a_\alpha(z)(D^{(\alpha)}\phi)\ (z)$$

By the property of Fourier transformation and using the technique of [3] we get

$$\begin{aligned} \left(P(z,\ D)\ \phi\right)(z) &= \sum_{|\alpha| \leq m} a_\alpha(z)(D^{(\alpha)}\hat{\phi})^\vee(z) \\ &= \sum_{|\alpha| \leq m} a_\alpha(z)(\xi^\alpha \hat{\phi})^\vee(z) \end{aligned}$$

$$\begin{aligned}(P(z,\ D)\ \phi)\,(z) &= \sum_{|\alpha|\leq m} a_\alpha(z)\ (2\ \pi)^{-n/2}\int_{\mathbb{R}^n}\xi^\alpha e^{i\langle z,\xi\rangle}\hat{\phi}(\xi)\ du\\ &= \int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\left(\sum_{|\alpha|\leq m} a_\alpha(z)\xi^\alpha\right)\hat{\phi}(\xi)\ du\\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\ P(z,\ \xi)\hat{\phi}(\xi)\ du.\end{aligned}$$

Hence,

$$(P(z,\ D)\ \phi)\,(z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\ P(z,\ \xi)\hat{\phi}(\xi)\ du. \tag{2.17}$$

In (2.17) if we replace $P(z,\xi)$ by more general symbol $\theta(z,\xi)$ which are no longer polynomial in ξ. The operator is so called pseudo differential operator.
The pseudo-differential operators associated with symbol $\theta(z,\xi)\in V^m$ is defined by

$$(A_\theta\phi)(z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\theta(z,\xi)\hat{\phi}(\xi)\ du \tag{2.18}$$

as $\xi = u+it\in\mathbb{C}^n$ and $z\in\mathbb{C}^n$ and $\phi\in W_M^{\Omega}(\mathbb{C}^n)$.

For $s\in\mathbb{R}$, the pseudo-differential operators V_s associated with symbol $\theta(\xi) = (1+|\xi|^2)^{-s/2}$ as $\xi = u+it$ is defined by

$$(V_s f)(z) = F^{-1}((1+|\xi|^2)^{-s/2}\hat{f})(z), \qquad \text{for } f\in W_M^{\Omega}(\mathbb{C}^n). \tag{2.19}$$

Now, the Sobolev space $G^{s,p}(\mathbb{C}^n)$ of $L^p(\mathbb{R}^n)$-type is defined to be the set of all $f\in[W_M^{\Omega}(\mathbb{C}^n)]'$ such that

$$\|f\|_{s,p} = \|(V_s f)(z)\|_p \qquad \text{for } 1\leq p<\infty. \tag{2.20}$$

The notations and terminologies of this paper are taken from Wong [10, pp 1-4] and Friedman [2].

3 Properties of Psueudo-differential Operators

In this section we study the various properties of pseudo-differential operators A_θ associated with symbol $\theta(z,\xi)$ on $W_M^{\Omega}(\mathbb{C}^n)$-space.

Theorem 3.1. Let $\theta(z,\xi)$ be the symbol belong to V^m. Then A_θ maps $W_M^{\Omega}(\mathbb{C}^n)$ into itself.

Proof. Let $\phi\in W_M^{\Omega}(\mathbb{C}^n)$. Then, for any multi-indices α and β, we have to show that

$$\sup_{z\in\mathbb{C}^n}|\exp[M[(ax)]-\Omega[(by)]](A_\theta\phi)(z)|<\infty$$

Now from (2.18) the pseudo-differential operator can be written as

$$z^\beta(A_\theta\phi)(z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} D_\xi^{(\beta)}e^{i\langle z,\xi\rangle}\theta(z,\xi)\hat{\phi}(\xi)\ du, \qquad z,\xi\in\mathbb{C}^n.$$

Using integration by parts we have

$$\begin{aligned}z^\beta(A_\theta\phi)(z) &= (2\pi)^{-n/2}(-1)^{|\beta|}\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}D_\xi^{(\beta)}[\theta(z,\xi)\hat{\phi}(\xi)]du\\ &= (2\pi)^{-n/2}(-1)^{|\beta|}\int_{\mathbb{R}^n}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}(D_\xi^{(\beta-\gamma)}\theta)(z,\xi)D_\xi^{(\gamma)}\hat{\phi}(\xi)\ du\\ &= (2\pi)^{-n/2}(-1)^{|\beta|}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}(D_\xi^{(\beta-\gamma)}\theta)(z,\xi)D_\xi^{(\gamma)}\hat{\phi}(\xi)\ du\\ &= (2\pi)^{-n/2}(-1)^{|\beta|}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\int_{\mathbb{R}^n} D_z^{(\alpha)}\left(e^{i\langle z,\xi+1\rangle}e^{-i\langle z,1\rangle}\right)\prod_{j=1}^n[(1+\xi_j)]^{-\alpha_j}\\ &\qquad (D_\xi^{(\beta-\gamma)}\theta)(z,\xi)D_\gamma^{(\gamma)}\hat{\phi}(\xi)\ du.\end{aligned}$$

Again using integration by parts we have

$$\begin{aligned}
z^{\beta}(A_{\theta}\phi)(z) &= (2\pi)^{-n/2}(-1)^{|\alpha|+|\beta|}\sum_{|\gamma|\le\beta}\binom{\beta}{\gamma}\int_{\mathbb{R}^n} e^{i\langle z,\xi+1\rangle}D_z^{(\alpha)}(e^{-i\langle z,1\rangle}D_\xi^{(\beta-\gamma)}\theta(z,\xi)) \\
&\qquad (\prod_{j=1}^{n}(1+\xi_j)^{-\alpha_j})D_\xi^{(\gamma)}\hat{\phi}(\xi)\,du \\
&= (2\pi)^{-n/2}(-1)^{|\alpha|+|\beta|}\sum_{|\gamma|\le\beta}\sum_{|\delta|\le\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}\int_{\mathbb{R}^n} e^{i\langle z,\xi+1\rangle}(D_z^{(\alpha-\delta)}D_\xi^{(\beta-\gamma)}\theta)(z,\xi) \\
&\qquad (\prod_{j=1}^{n}(1+\xi_j)^{-\alpha_j})D_z^{\delta}e^{-i\langle z,1\rangle}D_\xi^{(\gamma)}\hat{\phi}(\xi)\,du \\
&= (2\pi)^{-n/2}(-1)^{|\alpha|+|\beta|}\sum_{|\gamma|\le\beta}\sum_{|\delta|\le\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}\int_{\mathbb{R}^n} e^{i\langle z,\xi+1\rangle}(D_z^{(\alpha-\delta)}D_\xi^{(\beta-\gamma)}\theta) \\
&\qquad (z,\xi)(\prod_{j=1}^{n}(1+\xi_j)^{-\alpha_j})(-1)^{|\delta|}e^{-i\langle z,1\rangle}D_\xi^{(\gamma)}\hat{\phi}(\xi)\,du
\end{aligned}$$

$$\begin{aligned}
z^{\beta}(A_{\theta}\phi)(z) &= (2\pi)^{-n/2}(-1)^{|\alpha|+|\beta|}\sum_{|\gamma|\le\beta}\sum_{|\delta|\le\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}(-1)^{|\delta|}\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}(i)^{-|\delta|} \\
&\qquad (D_z^{(\alpha-\delta)}D_\xi^{(\beta-\gamma)}\theta)(z,\xi)\left(\prod_{j=1}^{n}(1+\xi_j)^{-\alpha_j}\right)D_\xi^{(\gamma)}\hat{\phi}(\xi)\,du.
\end{aligned}$$

Then

$$\begin{aligned}
|z^{\beta}(A_{\theta}\phi)(z)| &\le (2\pi)^{-n/2}\sum_{|\gamma|\le\beta}\sum_{|\delta|\le\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}\int_{\mathbb{R}^n}|e^{i\langle z,\xi\rangle}(D_z^{(\alpha-\delta)}D_\xi^{(\beta-\gamma)}\theta)(z,\xi)| \\
&\qquad (1+|\xi|)^{-|\alpha|}\,|D_\xi^{(\gamma)}\hat{\phi}(\xi)|\,du \\
&\le (2\pi)^{-n/2}\sum_{|\gamma|\le\beta}\sum_{|\delta|\le\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}\int_{\mathbb{R}^n}|e^{i\langle(x+iy),(u+it)\rangle}| \\
&\qquad |D_z^{(\alpha-\delta)}D_z^{(\beta-\gamma)}\theta(z,\xi)|\,(1+|\xi|)^{-|\alpha|}\,|D_\xi^{(\gamma)}\hat{\phi}(\xi)|\,du \\
&\le (2\pi)^{-n/2}\sum_{|\gamma|\le\beta}\sum_{|\delta|\le\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}\int_{\mathbb{R}^n}|\exp(-\langle y,u\rangle-\langle x,t\rangle)| \\
&\qquad |(D_z^{(\alpha-\delta)}D_\xi^{(\beta-\gamma)}\theta)(z,\xi)|\,(1+|\xi|)^{-|\alpha|}\,|D_\xi^{(\gamma)}\hat{\phi}(\xi)|\,du.
\end{aligned}$$

Now,

$$\begin{aligned}
|z^{\beta}(A_{\theta}\phi)(z)| &\le (2\pi)^{-n/2}\sum_{|\gamma|\le\beta}\sum_{|\delta|\le\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}\int_{\mathbb{R}^n}|\exp[\langle y,u\rangle-\langle x,t\rangle]| \\
&\qquad |(D_z^{(\alpha-\delta)}D_\xi^{(\beta-\gamma)}\theta)(z,\xi)|\,(1+|\xi|)^{-|\alpha|}\,|D_\xi^{(\gamma)}\hat{\phi}(\xi)|\,du \\
&\le (2\pi)^{-n/2}\sum_{|\gamma|\le\beta}\sum_{|\delta|\le\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}\int_{\mathbb{R}^n}|\exp[\langle y,u\rangle-\langle x,t\rangle]| \\
&\qquad C_{\alpha-\delta,\beta-\gamma}(1+|\xi|)^{m-|\beta|+|\gamma|-|\alpha|}\,|D_\xi^{(\gamma)}\hat{\phi}(\xi)|\,du
\end{aligned}$$

Using inequality $(1+|\xi|)^{m-|\beta|+|\gamma|-|\alpha|}\le\exp[M^0(a_1u)+\Omega^0(b_1t)]$ for $m-|\beta|+|\gamma|-|\alpha|>0$

and (2.9) we have

$$\begin{aligned}|z^\beta(A_\theta\phi)(z)| \quad \leq \quad & (2\pi)^{-n/2}\sum_{|\gamma|\leq\beta}\sum_{|\delta|\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}C_{\alpha-\delta,\beta-\gamma}\\ & \int_{\mathbb{R}^n}|\exp[\langle y,u\rangle-\langle x,t\rangle]|\ \exp[M^0(a_1u)+\Omega^0(b_1t)]\\ & \exp[-M^0(a''.u)+\Omega^0(b''t)]\,du.\end{aligned}$$

Using (2.3) and (2.4) we have

$$\begin{aligned}|z^\beta(A_\theta\phi)(z)| \quad \leq \quad & (2\pi)^{-n/2}\sum_{|\gamma|\leq\beta}\sum_{|\delta|\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}C\,C_{\alpha-\delta,\beta-\gamma}\\ & \int_{\mathbb{R}^n}\exp[|\langle y,u\rangle|-M^0[(a''-a_1)u]]\exp[-\langle x,t\rangle+\Omega^0(b_1+b'')t]\,du\\ \leq \quad & (2\pi)^{-n/2}\sum_{|\gamma|\leq\beta}\sum_{|\delta|\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}C\cdot C_{\alpha-\delta,\beta-\gamma}\\ & \exp[-\langle x,t\rangle+\Omega^0(b_1+b'')t]\int_{\mathbb{R}^n}\exp[|\langle y,u\rangle|-M^0[(a''-a_1)u]]\,du\\ \leq \quad & (2\pi)^{-n/2}\sum_{|\gamma|\leq\beta}\sum_{|\delta|\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}C'_{\alpha-\delta,\gamma-\beta}\\ & \exp[-\langle x,t\rangle+\Omega^0[(b''+b_1)t]]\int_{\mathbb{R}^n}\exp[|\langle y,u\rangle|-M^0[(a''-a_1)u]]\,du.\end{aligned}$$

Using (2.11) and (2.12) and the arguements of [2, p.134]

$$\begin{aligned}|z^\beta(A_\theta\phi)(z)| \quad \leq \quad & (2\pi)^{-n/2}\sum_{|\gamma|\leq\beta}\sum_{|\delta|\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}C'_{\alpha-\delta,\beta-\delta}\\ & \exp[-M[(b''+b_1)^{-1}x]+\Omega[(a''-2a_1)^{-1}y]]\\ & \int_{\mathbb{R}^n}\exp[-M^0(a_1u)]\,du\\ \leq \quad & C'_{\alpha,\beta}\exp[-M[(b_1+b'')^{-1}x]+\Omega[(a''-2a_1)^{-1}y].\end{aligned}$$

Hence

$$|\exp[M[(b_1+b'')^{-1}x]-\Omega[(a''-2a_1)^{-1}y]](A_\theta\phi)(z)|\leq C'_{\alpha,\beta}(1+|z|^\beta)^{-1}.$$

Thus

$$\begin{aligned}\sup_{z\in\mathbb{C}^n}|\exp[M(b_1+b'')^{-1}x]-\Omega[(a''-2a_1)^{-1}y](A_\theta\phi)(z)| \quad & \leq \quad C'_{\alpha,\beta}\\ & < \quad \infty.\end{aligned}$$

This implies that

$$(A_\theta\phi)(z)\in W_M^\Omega(\mathbb{C}^n).$$

Theorem 3.2. A_θ is continuous linear mapping $W_M^\Omega(\mathbb{C}^n)$ into itself.

Proof. If the functions $\phi(z)$ converge uniformly to zero as $\nu\to\infty$ in any bounded domain of the z-plane and in addition satisfy the inequalities.

$$|\phi_\nu(z)|\leq C\exp[-M[(ax)]+\Omega[(by)]],$$

then the sequence $\phi_\nu(z)\in W_M^\Omega(\mathbb{C}^n)$ is said to converge to zero as $\nu\to\infty$, where the constants C, a and b do not depend on the index ν.

Since from Theorem 3.1 $A_\theta\phi$ is a mapping from $W_M^\Omega(\mathbb{C}^n)$ into itself. Using above results, $A_\theta\phi_\nu\in W_M^\Omega(\mathbb{C}^n)$ converge to zero uniformly in any bounded domain of the z-plane as $\nu\to\infty$ and satisfies the above inequality. Therefore, the sequence $A_\theta\phi\in W_M^\Omega(\mathbb{C}^n)$ is converges to zero as $\nu\to\infty$. This shows that A_θ maps continuously into itself.

Now, we define the pseudo-differential operator A_θ on $[W_M^\Omega(\mathbb{C}^n)]'$-space by

$$\langle A_\theta f,\phi\rangle=\langle f,\overline{A^*_\phi\phi}\rangle,\qquad \phi\in W_M^\Omega(\mathbb{C}^n). \tag{3.1}$$

Theorem 3.3. A_θ is a linear mapping from $[W_M^\Omega(\mathbb{C}^n)]'$ into itself.

Proof. Let $f \in [W_M^\Omega(\mathbb{C}^n)]'$. Then, for any sequence $\{\phi_\nu\}$ of functions in $W_M^\Omega(\mathbb{C}^n)$ converging to zero in $W_M^\Omega(\mathbb{C}^n)$, as $\nu \to \infty$. From (2.20) we have

$$\langle A_\theta f, \phi_\nu\rangle = \langle f, \overline{A_\phi^* \phi_\nu}\rangle, \qquad \nu = 1, 2, 3, \cdots \tag{3.2}$$

By the arguements of Theorem 3.2, we conclude that $\langle A_\theta f, \phi_\nu\rangle \to 0$ as $\nu \to \infty$. Hence $A_\theta f \in [W_M^\Omega(\mathbb{C}^n)]'$.

Definition 3.4. A sequence of distributions $\{f_\nu\}$ in $[W_M^\Omega(\mathbb{C}^n)]'$ is said to converge to zero in $[W_M^\Omega(\mathbb{C}^n)]'$ if $\langle f_\nu, \phi\rangle \to 0$ as $\nu \to \infty$ for all $\phi \in W_M^\Omega(\mathbb{C}^n)$.

Theorem 3.5. A_θ maps continuously $[W_M^\Omega(\mathbb{C}^n)]'$ into itself.

Proof. Let $\phi \in [W_M^\Omega(\mathbb{C}^n)]$. Then, using (3.2) and the fact that $f_\nu \to 0$ in $[W_M^\Omega(\mathbb{C}^n)]'$ as $\nu \to \infty$,

$$\langle A_\theta f_\nu, \phi\rangle = \langle f_\nu, A_\theta^* \overline{\phi}\rangle \to 0$$

as $\nu \to \infty$. Hence $A_\theta f_\nu \to 0$ in $[W_M^\Omega(\mathbb{C}^n)]'$ as $\nu \to \infty$, and the proof is complete.

Theorem 3.6. Let $\theta \in C^k(\mathbb{C}^n)$, $k \geq n/2$, be such that there exists a positive constant B such that

$$|(D_\xi^{(\alpha)}\theta)(\xi)| \leq C_{\alpha,n}(1 + |\xi|)^{-|\alpha|}, \qquad \xi \neq 0 \tag{3.3}$$

for multi-indices α with $|\alpha| \leq k$. Then, for $1 \leq p < \infty$, there exists a positive constant B, depending on α and N, such that

$$\|(A\phi)(z)\|_p \leq M'_{\alpha,n}\|\phi\|_p, \qquad \phi \in W_M^\Omega(\mathbb{C}^n), \tag{3.4}$$

where

$$(A\phi)(z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\theta(\xi)\hat{\phi}(\xi)\, du, \tag{3.5}$$

$\xi = u + it$, and $\hat{\phi}$ denotes the Fourier transformation of ϕ.

Proof. (3.5) can be written as

$$(A\phi)(z) = (2\pi)^{-n/2}F^{-1}[\theta(\xi)\hat{\phi}(\xi)](z) \tag{3.6}$$

where F^{-1} denotes the inverse Fourier transformation of a function z as $z = x + iy$.
Now, we assume that

$$F^{-1}[\theta(\xi)\hat{\phi}(\xi)](z) = (f * g)(z). \tag{3.7}$$

Then by convolution property of Fourier transformation, we have

$$\begin{aligned}\theta(\xi)\hat{\phi}(\xi) &= F[(f * g)](\xi)\\ &= \hat{f}(\xi)\cdot\hat{g}(\xi).\end{aligned}$$

This implies that

$$f(z) = F^{-1}[\theta(\xi)](z), \qquad g(z) = \phi(z).$$

Thus, the expression (3.7) yields

$$(A\phi)(z) = (2\pi)^{-n/2}(F^{-1}[\theta(\xi)] * \phi)(z).$$

Using convolution property $\|f * \phi\|_p \leq \|f\|_1\, \|\phi\|_p$ for $f \in L^1(\mathbb{R}^n)$ and $\phi \in L^p(\mathbb{R}^n)$ we have

$$\begin{aligned}\|(A\phi)(z)\|_p &= (2\pi)^{-n/2}\|(F^{-1}[\theta(\xi)] * \phi)(x)\|_p\\ &\leq (2\pi)^{-n/2}\|F^{-1}[\theta(\xi)]\|_1\, \|\phi\|_p.\end{aligned} \tag{3.8}$$

Next, we have to prove that

$$F^{-1}[\theta(\xi)] \in L^1(\mathbb{R}^n).$$

Thus, from [3, p. 24] we have

$$F^{-1}[\theta(\xi)](z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\theta(\xi)\, du.$$

By property of Fourier transformation the above expression gives

$$(z)^\alpha F^{-1}[\theta(\xi)](z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} D_\xi^{(\alpha)}(e^{-i\langle z,\xi\rangle})\theta(\xi)\, du.$$

Integration by parts, above expression can be obtained

$$\begin{aligned}(z)^\alpha F^{-1}[a(\xi)](z) &= (2\pi)^{-n/2}(-1)^{|\alpha|} \int_{\mathbb{R}^n} e^{-i\langle z,\xi\rangle}(D_\xi^{(\alpha)}\theta)(\xi)\, du \\ &= (2\pi)^{-n/2}(-1)^{|\alpha|} \int_{\mathbb{R}^n} \exp[-\langle x,u\rangle - \langle y,t\rangle](D_\xi^{(\alpha)}\theta)(\xi)\, du.\end{aligned}$$

Therefore,

$$\begin{aligned}|(z)^\alpha F^{-1}[a(\xi)](z)| &\leq B\, C_n|\exp[-\langle y,t\rangle]| \int_{\mathbb{R}^n} |\exp[-\langle x,u\rangle]|\, (1+|\xi|)^{-|\alpha|}\, du \\ &\leq B_n\, |\exp[-\langle y,t\rangle]|\, \sup_u |\exp[-\langle x,u\rangle]| \int_{\mathbb{R}^n} (1+|\xi|)^{-|\alpha|}\, du \\ &\leq B_{\alpha,n}|\exp[-\langle y,t\rangle]| \\ &\leq B_{\alpha,n}.\end{aligned}$$

This implies that

$$|F^{-1}[\theta(\xi)]| \leq B_{\alpha,n}\|(1+|z|^n)^{-1}\|_1. \tag{3.9}$$

From (3.8) - (3.9), we find the required result (3.4)

Theorem 3.7. Let $\phi \in W_M^{\Omega}(\mathbb{C}^n)$ and symbol $\theta_m(z,\xi)$ has compact support in z. Then, pseudo-differential operators $A_{\theta_m}\phi$ can be expressed as

$$(A_{\theta_m}\phi)(z) = (2\pi)^{-n} \left(\int_{\mathbb{R}^n} e^{-i\langle\lambda,z\rangle} \left(\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle} \hat{\theta}_m(\lambda,\xi)\hat{\phi}(\xi) du \right) dv \right)$$

where

$$(A_\lambda\phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle z,\lambda\rangle}\hat{\theta}_m(\lambda,\xi)\hat{\phi}(\xi)\, du \tag{3.10}$$

as $z = x+iy$, $\lambda = v+iv'$ and

$$\hat{\theta}_m(\lambda,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle\lambda,z\rangle}\theta_m(z,\xi)\, dx, \qquad \lambda,\xi \in \mathbb{C}^n.$$

Proof. Since

$$(A_\theta\phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\theta_m(z,\xi)\hat{\phi}(\xi)\, du,$$

then, by using the property of Fourier transformation we have

$$(A_{\theta_m}\phi)(z) = (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} e^{-i\langle\lambda,z\rangle} \left((2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle,z,\xi\rangle}\hat{\theta}_m(\lambda,\xi)\, dv \right) \hat{\phi}(\xi)\, du \right)$$

as $\lambda = v+iv'$. By Fubini's theorem and (3.10) we get

$$(A_{\theta_m}\phi)(z) = (2\pi)^{-n} \left(\int_{\mathbb{R}^n} e^{-i\langle\lambda,z\rangle} \left(\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\hat{\theta}_m(\lambda,\xi)\hat{\phi}(\xi)\, du \right)\, dv \right). \tag{3.11}$$

Lemma 3.8. For all multi-indices α and β and positive integers N, there is a positive constant $C_{\alpha,N}$, depending on α and N such that

$$|(D_\xi^{(\alpha)}\hat{\theta}_m)(\lambda,\xi)| \leq C_{\alpha,N}(1+|\lambda|^{|\beta|})^{-1}(1+|\xi|)^{-|\alpha|}$$

for $\xi = u+it$ and $\lambda = v+iv'$.

Proof. The Fourier transformation of θ_m with respect to $\lambda = v+iv'$ is given by

$$\hat{\theta}_m(\lambda,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle z,\lambda\rangle}\theta_m(z,\xi)\, dx.$$

Then

$$(i\lambda)^\beta D_\xi^{(\alpha)}\hat{\theta}_m(\lambda,\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \partial_z^{(\beta)}[e^{-i\langle z,\lambda\rangle}]D_\xi^{(\alpha)}\theta_m(z,\xi)\,dx.$$

Integration by parts we have

$$(i\lambda)^\beta D_\xi^{(\alpha)}\hat{\theta}_m(\lambda,\xi) = (2\pi)^{-n/2}(-1)^{|\beta|}\int_{\mathbb{R}^n} e^{i\langle z,\lambda\rangle}\partial_z^{(\beta)}D_\xi^{(\alpha)}\theta_m(z,\xi)\,dx$$

as $z = x+iy$.
Hence,

$$\begin{aligned}(i\lambda)^\beta D_\xi^{(\alpha)}\hat{\theta}_m(\lambda,\xi) &= (2\pi)^{-n/2}(-1)^\beta\int_{\mathbb{R}^n} e^{i\langle z,\lambda\rangle}\partial_z^{(\beta)}D_\xi^{(\alpha)}[\eta(z-m)\theta(z,\xi)]\,dx\\ &= (2\pi)^{-n/2}(-1)^\beta\int_{\mathbb{R}^n} e^{i\langle z,\lambda\rangle}\sum_{|\gamma|\le\beta}\binom{\beta}{\gamma}D_z^{(\gamma)}\eta(z-m)\\ &\qquad\qquad \partial_z^{(\beta-\gamma)}D_\xi^{(\alpha)}\theta(z,\xi)\,dx.\end{aligned}$$

Now

$$\begin{aligned}&|\lambda^\beta D_\xi^{(\alpha)}\hat{\theta}_m(\lambda,\xi)|\\ &\le (2\pi)^{-n/2}\sum_{|\gamma|\le\beta}\binom{\beta}{\gamma}\int_{\mathbb{R}^n}|\exp-\langle x,v\rangle-\langle y,v'\rangle)|\,|D_\xi^{(\gamma)}\eta(z-m)|\,|D_z^{(\beta-\gamma)}D_\xi^\alpha\theta(z,\xi)|\,dx\\ &\le (2\pi)^{-n/2}\sum_{|\gamma|\le\beta}\binom{\beta}{\gamma}\int_{\mathbb{R}^n}|\exp[-\langle x,v\rangle-\langle y,v'\rangle]|\,|\partial_z^{(\gamma)}\eta(z-m)|C_{\beta-\gamma,\alpha}(1+|\xi|)^{-|\alpha|}\,dx\\ &\le (2\pi)^{-n/2}\sum_{|\gamma|\le\beta}\binom{\beta}{\gamma}C_{\beta-\gamma,\alpha}\int_{\mathbb{R}^n}|\exp[-\langle x,v\rangle-\langle y,v'\rangle)]|\,|\partial_z^{(\gamma)}\eta(z-m)|\,(1+|\xi|)^{-|\alpha|}\,dx\\ &\le (2\pi)^{-n/2}\sum_{|\gamma|\le\beta}\binom{\beta}{\gamma}C_{\beta-\gamma,\alpha}(1+|\xi|)^{-|\alpha|}\int_{\mathbb{R}^n}|\exp[-\langle x,v\rangle]\partial_z^{(\gamma)}\eta(z-m)|\,dx.\end{aligned}$$

Then

$$\begin{aligned}&|\lambda^\beta D_\xi^\alpha\hat{\theta}_m(\lambda,\xi)|\\ &\le (2\pi)^{-n/2}\sum_{|\gamma|\le\beta}\binom{\beta}{\gamma}C_{\beta-\gamma,\alpha}(1+|\xi|)^{-|\alpha|}\int_{\mathbb{R}^n}|\partial_z^{(\gamma)}\eta(z-m)|\,dx\\ &\le (2\pi)^{-n/2}(1+|\xi|)^{-|\alpha|}\sum_{|\gamma|\le\beta}\binom{\beta}{\gamma}C_\gamma C_{\beta-\gamma,\alpha}\\ &\le (2\pi)^{-n/2}(1+|\xi|)^{-|\alpha|}C_\beta\\ &\le C_{\beta,n}(1+|\xi|)^{-|\alpha|}.\end{aligned}$$

Hence, for large arbitrary positive integers N, we have

$$\left|\left(D_\xi^{(\alpha)}\hat{\theta}_m\right)(\lambda,\xi)\right| \le C_{n,\beta}(1+|\lambda|^N)^{-1}(1+|\xi|)^{-|\alpha|}.$$

as $\xi = u+it$.

Theorem 3.9. Let $\theta\in V^0$. Then we get the following relation

$$\int_{Q_m}|(A_\theta\phi)(z)|^p\,dx \le C_N^p\|\phi\|_p^p \qquad \forall\phi\in W_M^\Omega(\mathbb{C}^n).$$

Proof. From Wong [10, p. 80], we can write

$$\left(\int_{Q_m}|(A_\theta\phi)(z)|^p\,dx\right) \le \left(\int_{\mathbb{R}^n}|(A_{\theta_m}\phi)(z)|^p\,dx\right). \tag{3.12}$$

Using Lemma 3.8 and Theorem 3.6, we find that

$$\|A_\lambda \phi\|_p \leq C_N(1+|\lambda|)^{-N}\|\phi\|_p \qquad \forall \phi \in W_M^{\Omega}(\mathbb{C}^n). \tag{3.13}$$

Using (3.11), (3.13) and Minkowski's inequality in the integral form we obtain

$$\begin{aligned}
\|A_{\theta_m}\phi\|_p &= (2\pi)^{-n/2}\left(\int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} e^{i\langle z,\lambda\rangle}(A_\lambda \phi)(z)\ dv|^p\ dx\right)^{1/p} \\
&= (2\pi)^{-n/2}\left(\int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} exp\left[-\langle x,\ v'\rangle - \langle y,\ v\rangle\right](A_\lambda \phi)(z)\ dv|^p\ dx\right)^{1/p} \\
&\leq (2\pi)^{-n/2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |exp[-\langle x,\ v'\rangle]\ (A_\lambda \phi)(z)|^p\ dx\right)^{1/p}\ dv \\
&\leq (2\pi)^{-n/2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |(A_\lambda \phi)(z)|^p\ dx\right)^{1/p} dv \\
&\leq (2\pi)^{-n/2}\int_{\mathbb{R}^n} \left\|(A_\lambda \phi)(z)\right\|_p\ dv.
\end{aligned}$$

Using (3.13) we get

$$\begin{aligned}
\|A_{\theta_m}\phi\|_p &\leq (2\pi)^{-n/2}\ C_N\left(\int_{\mathbb{R}^n}(1+|\lambda|)^{-N}dv\right)\|\phi\|_p \\
&\leq (2\pi)^{-n/2}C_N\left\|\phi\right\|_p \qquad \phi \in W_M^{\Omega}(\mathbb{C}^n).
\end{aligned}$$

Hence from (3.12) and (3.13) we have

$$\int_{Q_m} |(A_\theta \phi)(z)|^p\ dx \leq C_{N,n}^p \|\phi\|_p^p, \qquad \phi \in W_M^{\Omega}(\mathbb{C}^n). \tag{3.14}$$

Now, we represent A_θ as a singular integral operator.

Lemma 3.10. Let $K(z,w) = \int_{\mathbb{R}^n} e^{i\langle z,w\rangle}\theta(z,w)\ ds$, $z = x+iy \in \mathbb{C}^n$ $w = s+iv \ \in \mathbb{C}^n$ in the distributional sense. Then

(i) for each $z \in \mathbb{C}^n$, $K(z,w)$ is a function defined on $\mathbb{R}^n$,

(ii) for each sufficiently large positive integer N, there is a positive constant C_N such that

$$|K(z-w,w)| \leq C_N(1+|z-w|^N)^{-1}, \tag{3.15}$$

(iii) for each fixed $z = x+iy$ and $\phi \in W_M^{\Omega}(\mathbb{C}^n)$ vanishing in the neighbourhood of $\mathbb{C}^n$, we find that

$$(A_\theta \phi)(z) = \int_{\mathbb{R}^n} K(z-w,w)\phi(w)ds. \tag{3.16}$$

Proof. (i) can be defined by using the arguments of [10, p. 26] and [1, pp. 23-24].

To prove (ii), let α be a multi-index with length greater than w. Then by the property of Fourier transformation $(D^{(\alpha)}u)\tilde{} = \xi^{|\alpha|}\tilde{u}$ we have

$$(iw)^\alpha K(z,w) = (-1)^{|\alpha|}\int_{\mathbb{R}^n} e^{i\langle \xi,w\rangle} D_\xi^{(\alpha)}\theta(z,\xi)\ du.$$

Therefore, using (2.16)and tools of theorem (3.6) we have

$$|K(z,w)| \leq C'_\alpha(1+|w|^\alpha)^{-1}.$$

For large positive integer N we can obtain

$$|K(z,z-w)| \leq C'_\alpha(1+|z-w|^N)^{-1}.$$

To prove (iii), we define the distribution L_z on $W_M^{\Omega}(\mathbb{C}^n)$ by

$$\langle L_z, \psi\rangle = \int_{\mathbb{R}^n} \theta(z,\xi)\psi(\xi)\, du,$$

where $z = x + iy$, $\xi = u + i\tau$ and $w = s + iv$. By the definition of pseudo-differential operator (2.18)

$$\begin{aligned}(A_\theta\phi)(z) &= \int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\theta(z,\xi)\hat{\phi}(\xi)\, du \\ &= L_z(M_z\hat{\phi}) \end{aligned} \tag{3.17}$$

Using Gelfand and Shilov [3] technique of integration we get

$$\begin{aligned}(A_\theta\phi)(z) &= L_z(T_z\phi)\hat{} \\ &= \hat{L}_z(T_z\phi)\end{aligned} \tag{3.18}$$

From (i) we have

$$\hat{L}_z(\psi) = \int_{\mathbb{R}^n} \theta(z,-w)\psi(w)\, ds.$$

Hence

$$\begin{aligned}(A_\theta\phi)(z) &= \int_{\mathbb{R}^n} \theta(z,-w)(T_z\phi)(w)ds \\ &= \int_{\mathbb{R}^n} \theta(z,-w)\phi(z+w)\, ds \\ &= \int_{\mathbb{R}^n} \theta(z,z-w)\phi(w)\, ds.\end{aligned}$$

This completes the proof of the theorem.

Theorem 3.11. Let $\theta(z,\xi)$ be a symbol in V^0. Then $A_\theta : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator for $1 < p < \infty$.

Proof. From Theorem 3.6, Theorem 3.7, Theorem 3.9 and Lemma 3.10 we can show that the pseudo-differential operator A_θ is a bounded linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.
□

4 The Sobolev Space

In this section, we study the pseudo-differential operators on Sobolev type space $G^{s,p}(\mathbb{C}^n)$ which is defined in Section 2.

For $s \in \mathbb{R}$, the pseudo-differential operator associated with symbol $\theta(\xi) = (1+|\xi|^2)^{-s/2}$ as $\xi = u + it$ is defined by

$$(V_\theta u)(z) = F^{-1}(\theta(\xi)\hat{u}(\xi))(z) \quad \text{for } u \in \left[W_M^{\Omega}(\mathbb{C}^n)\right]'. \tag{4.1}$$

Now, we define the Sobolev space $G^{s,p}(\mathbb{C}^n)$ of L^p-type to be the set of all distribution $u \in \left[W_M^{\Omega}(\mathbb{C}^n)\right]'$ such that

$$\|u\|_{s,p} = \|V_{-s}u\|_p \qquad \text{for } 1 \le p < \infty. \tag{4.2}$$

Theorem 4.1. Let $u \in [W_M^{\Omega}(\mathbb{C}^n)]'$. Then

(i) $V_sV_t\, u = V_{s+t}u$,

(ii) $V_0 u = u$.

Proof. The proof of the above theorem is obvious from [10, p. 90].

Theorem 4.2. $G^{s,p}(\mathbb{C}^n)$ is a Banach space with respect to $\|u\|_{s,p}$.

Proof. The proof of the above theorem is usual from [10, p. 81].

Theorem 4.3. V_t is an isometry from $V^{s,p}$ onto $V^{s+t,p}$.

Proof. Let $u \in V^{s,p}$. Then from Theorem 4.1 we get $J_{-t}v \in G^{s,p}(\mathbb{C}^n)$ and $v_t v_{-t} v = v$. This implies $G^{s,p}(\mathbb{C}^n)$ is onto.

Theorem 4.4. Let $1 < p < \infty$ and $s \leq t$. Then $G^{t,p}(\mathbb{C}^n) \subseteq G^{s,p}(\mathbb{C}^n)$.

Proof. See [10, p. 91]. This is called Sobolev embedding theorem.

Theorem 4.5. Let $s \geq 0$ and $1 \leq p < \infty$. Then

$$\|V_s\phi\|_p \leq \|\phi\|_p, \qquad \phi \in L^p(\mathbb{R}^n).$$

Proof. We have

$$(J_s\phi)\hat{}(\xi) = (1+|\xi|^2)^{-s/2}\hat{\phi}(\xi), \qquad \xi \in \mathbb{C}^n.$$

Hence, for $\hat{G}_s(\xi) = (1+|\xi|^2)^{-s/2}$ we have

$$(G_s * \phi)\hat{}(\xi) = (1+|\xi|^2)^{-s/2}\hat{\phi}(\xi). \tag{4.3}$$

Hence, for all $\phi \in W_M^\Omega(\mathbb{R}^n)$,

$$J_s\phi = (G_s * \phi),$$

and using convolution property

$$\begin{aligned} \|J_s\phi\|_p &= \|G_s * \phi\|_p \\ &\leq \|G_s\|_1 \, \|\phi\|_p \\ &\leq \|\phi\|_p. \end{aligned}$$

Theorem 4.6. For symbol θ in U^m, $A_\theta : G^{m,p}(\mathbb{C}^n) \to G^{0,p}(\mathbb{C}^n)$ is a bounded linear operator for $1 < p < \infty$.

Proof. Counsider the bounded linear operators

$$V_{-s} : G^{s,p}(\mathbb{C}^n) \to G^{0,p}(\mathbb{C}^n)$$
$$A_\theta V_m : G^{0,p}(\mathbb{C}^n) \to G^{0,p}(\mathbb{C}^n)$$

and

$$V_{s-m} : G^{0,p}(\mathbb{C}^n) \to G^{s-m,p}(\mathbb{C}^n).$$

The first and the third operators are bounded by isometry of pseudo-differential operator of Theorem 4.3 and the second operator is bounded by $L^p(\mathbb{R}^n)$-boundedness property of pseudo-differential operator. Hence the product $V_{s-m}A_\theta V_{m-s}$ is a bounded linear operator from $G^{s,p}$ into $G^{s-m,p}$. By Theorem 4.3 operators V_{m-s} and V_{s-m} are isometric and onto. Hence, A_0 : $G^{m,p} \to G^{0,p}$ must be bounded linear operator.

Theorem 4.7. Let $\theta(z,\xi)$ be any symbol in V^m, then $A_\theta : G^{s,p}(\mathbb{C}^n) \to G^{s-m,p}(\mathbb{C}^n)$ is a bounded linear operator for $1 \leq p < \infty$.

Proof. Since $V_{m-s}A_\theta$ is a pseudo-differential operator with symbol in V^s. Hence, from Theorem 4.6 we can easily prove that

$$\|A_\theta u\|_{s-m,p} = \|J_{m-s}A_\theta u\|_p \leq C\|u\|_{s,p} \qquad \forall\, u \in G^{s,p}.$$

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