

PSEUDO-DIFFERENTIAL OPERATORS ON $W_M^\Omega(\mathbb{C}^n)$ -SPACE

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Abstract. Pseudo-differential operators associated with symbol $\theta(z, \xi)$, $z = x + iy$ and $\xi = u + it$ on $W_M^\Omega(\mathbb{C}^n)$ -space is defined and using the theory of Fourier transformation its various properties are studied. $L^p(\mathbb{R}^n)$ - boundedness is investigated for $1 < p \leq \infty$. Sobolev space associated with distributional space $[W_M^\Omega(\mathbb{C}^n)]'$ is defined and its properties are obtained.

1 Introduction

The spaces $W_M(\mathbb{R}^n)$, $W^\Omega(\mathbb{C}^n)$ and $W_M^\Omega(\mathbb{C}^n)$ were introduced and analyzed by Gurevich [4], Gel'fand and Shilov [3] and Friedman [2]. They applied these W -type spaces for investigating uniqueness and corrected class of Cauchy problem and other problems of partial differential equations by using Fourier transformation tool. Recently, continuous Wavelet transformation on W -spaces are studied by [6, 9] and many interesting properties are obtained.

The theory of pseudo-differential operators is developed by Wong [10], Rodino [1], Pathak [5] and others. They exploited this theory on Schwartz space $S(\mathbb{R}^n)$, Gel'fand and Shilov space of type S and Gevery space by using the technique of Fourier transformation.

They also studied pseudo-differential operators on their respective Sobolev space and got many important results. Sobolev spaces are used to study the minimal-maximal properties, global regularities and spectral properties of pseudo-differential operators on Schwartz space $S(\mathbb{R}^n)$. The Pseudo differential operators on $W_M(\mathbb{R}^n)$ and $W^\Omega(\mathbb{C}^n)$ are studied by author and others and obtained many important results, see [7, 8].

Our main aim in this paper is to introduce more general symbol and precise study of pseudo-differential operators on $W_M^\Omega(\mathbb{C}^n)$ -space and to study many properties because its dual space $[W_M^\Omega(\mathbb{C}^n)]'$ is more general than $[W_M(\mathbb{R}^n)]'$ and Schwartz distributional space $S'(\mathbb{R}^n)$.

The present article is divided into three sections. Section 2 gives the various definitions of pseudo-differential operators, symbol, W -type spaces, Fourier transformation and Sobolev space. Section 3 contains a study of properties of pseudo-differential operators on $W_M^\Omega(\mathbb{C}^n)$ space and $L^p(\mathbb{R}^n)$ -boundedness result of pseudo-differential operators. In the last section, using $L^p(\mathbb{R}^n)$ -boundedness result, the Sobolev space $G^{s,p}(\mathbb{C}^n)$, $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ on $[W_M^\Omega(\mathbb{C}^n)]'$ space is defined and it is proved that the pseudo-differential operator $A_\theta : G^{s,p}(\mathbb{C}^n) \rightarrow G^{s,p}(\mathbb{C}^n)$ and $A_\theta : G^{s,p}(\mathbb{C}^n) \rightarrow G^{s-m,p}(\mathbb{C}^n)$ are bounded linear operator for $s, m \in \mathbb{R}$.

2 Preliminary

Now in this section we recall the definitions of $W_M(\mathbb{R}^n)$, $W^\Omega(\mathbb{C}^n)$ and $W_M^\Omega(\mathbb{C}^n)$ from [2] and [3].

Let M_j and Ω_j be the convex functions such that

$$M_j(x_j) = \int_0^{x_j} \mu_j(\xi_j) d\xi_j \quad (x_j \geq 0) \quad (2.1)$$

and

$$\Omega_j(y_j) = \int_0^{y_j} w_j(\eta_j) d\eta_j \quad (y_j \geq 0) \quad (2.2)$$

for $j = 1, 2, 3, \dots, n$.

We set

$$\begin{aligned} \mu(\xi) &= (\mu_1(\xi_1), \dots, \mu_n(\xi_n)) \\ w(n) &= (w_1(\eta_1), \dots, w_n(\eta_n)) \end{aligned}$$

and

$$M_j(-x_j) = M_j(x_j), \quad M_j(x_j) + M_j(x'_j) \leq M_j(x_j + x'_j) \tag{2.3}$$

$$\Omega_j(-y_j) = \Omega_j(y_j), \quad \Omega_j(y_j) + \Omega_j(y'_j) \leq \Omega_j(y_j + y'_j). \tag{2.4}$$

The space $W_M(\mathbb{R}^n)$ consists of all C^∞ -functions which satisfy the inequalities

$$|D_x^{(k)} \phi(x)| \leq C_k \exp[-M(ax)], \tag{2.5}$$

where $D_x^{(k)} = D_{x_1}^{(k_1)} D_{x_2}^{(k_2)} \dots D_{x_n}^{(k_n)}$,

$$\exp[-M(ax)] = \exp[-M_1(a_1x_1) \dots - M_n(a_nx_n)] \tag{2.6}$$

and constants $C_k, a > 0$ depending on the function ϕ . A function $\phi(z) \in W^\Omega(\mathbb{C}^n)$ if and only if for $b > 0$ there exists a constant $C_k > 0$ such that

$$|z^k \phi(z)| \leq C_k \exp[\Omega(by)], \quad z = x + iy \tag{2.7}$$

where

$$z^k = z_1^{k_1} z_2^{k_2} z_3^{k_3} \dots z_n^{k_n}$$

and

$$\exp[\Omega(by)] = \exp[\Omega_1(b_1y_1) + \dots + \Omega_j(b_jy_j) + \dots + \Omega_n(b_ny_n)], \tag{2.8}$$

the constants C_k and $b > 0$ depend on the function ϕ .

The space $W_M^\Omega(\mathbb{C}^n)$ consists of all entire analytic functions $\phi(z)$ which satisfy

$$|\phi(z)| \leq C \exp[-M[(ax)] + \Omega[(by)]], \tag{2.9}$$

where $z = x + iy$ and $\exp[-M(ax)]$ and $\exp[\Omega(by)]$ have similar meaning like (2.5) and (2.7) and constants C, a and b depend on the function ϕ .

Now, we define the duality of functions $M(x)$ and $\Omega(y)$ in the following way:

Let $M_j(x_j)$ and $\Omega_j(y_j)$ be defined by (2.1) and (2.2) respectively and let $\mu_j(\xi)$ and $w_j(\eta_j)$ be mutually inverse, that is $\mu_j(w_j(\eta_j)) = \eta_j, \eta_j(\mu_j(\xi)) = \xi_j$, then the corresponding functions $M_j(x_j)$ and $\Omega_j(y_j)$ are called dual in sense of Young. In this case the Young inequality is

$$x_j y_j \leq M_j(x_j) + \Omega_j(y_j). \tag{2.10}$$

This inequality holds for any $x_j \geq 0, y_j \geq 0$ and equality holds if and only if $y_j = \mu_j(x_j)$, where x_j varies in the interval $x_j^0 < x_j < \infty$ and y_j varies in the interval $y_j^0 < y_j < \infty$. That equality will be

$$x_j y_j = M_j^0(x_j) + \Omega_j(y_j) \tag{2.11}$$

$$x_j y_j = M_j(x_j) + \Omega_j^0(y_j) \tag{2.12}$$

for $M_j(x_j) < M_j^0(x_j)$ and $\Omega_j^0(y_j) > \Omega_j(y_j)$.

From [2, pp.132-133, Theorem 12] and [2, p.134, Theorem 13 and Theorem 15] the Fourier transformation of a function $\phi \in W_M^\Omega(\mathbb{C}^n)$ is defined by

$$\hat{\phi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \phi(z) dx \tag{2.13}$$

for $z = x + iy$ and $\xi = u + it$.

From [1] and [2] the Fourier duality relation is given by

$$\begin{aligned} F[W^\Omega(\mathbb{C}^n)] &= W_M(\mathbb{R}^n), & F[W_M(\mathbb{R}^n)] &= W^\Omega(\mathbb{C}^n) \text{ and} \\ F[W_M^\Omega(\mathbb{C}^n)] &= W_{M^0}^\Omega(\mathbb{C}^n). \end{aligned}$$

Now, we recall the definitions of translation T and modulation M of function f on \mathbb{R}^n from [10, p. 14]:

Let f be a measurable function defined on \mathbb{R}^n . For any fixed $y \in \mathbb{R}^n$, we define $T_y f$ and $M_y f$ by

$$(T_y f)(x) = f(x + y) \tag{2.14}$$

and

$$(M_y f)(x) = e^{ixy} f(x). \tag{2.15}$$

Next, we define a symbol associated with pseudo-differential operator by the following way:

The function $\theta(z, \xi) \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ which is the set of all entire analytic functions of $z = x + iy$ and $\xi = u + it$ is said to be class V^m iff for any two multiindices α and β , there is a positive constant $C_{\alpha, \beta}$ depending on α and β only such that

$$|(D_z^{(\alpha)} D_\xi^{(\beta)} \theta)(z, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\beta|}, \tag{2.16}$$

where $m \in \mathbb{R}$ and $z, \xi \in \mathbb{C}^n$. If we take $y = 0, t = 0$ then the symbol will to the well known class of S^m .

Theorem 2.1. Let $\phi \in W_M^\Omega(\mathbb{C}^n)$ and let $\sigma(z, \xi)$ be an entire function in (z, ξ) and satisfy

$$|\sigma(z, \xi)| \leq C(1 + |\xi|)^m,$$

then $\phi(\xi)\sigma(z, \xi) \in W_M^\Omega(\mathbb{C}^n)$.

Proof. Let $\phi \in W_M^\Omega(\mathbb{C}^n)$ and $|\sigma(z, \xi)| \leq C(1 + |\xi|)^m$. Then

$$\begin{aligned} |\sigma(z, \xi)\phi(\xi)| &= |\sigma(z, \xi)| |\phi(\xi)| \\ &\leq C(1 + |\xi|)^m \exp[-M(au) + \Omega(bt)]. \end{aligned}$$

Since $(1 + |\xi|)^m \leq \exp[-M(a_0u) + \Omega(b_0t)]$, $m \in \mathbb{R}$ then using the definition of $W_M^\Omega(\mathbb{C}^n)$ space we find that

$$|\sigma(z, \xi)\phi(\xi)| \leq C \exp[-M(a_0u) + \Omega(b_0t)] \exp[M(au) + \Omega(bt)].$$

By the definition of convex function (2.1) and (2.2), we get

$$|\sigma(z, \xi)\phi(\xi)| \leq C \exp[[-M[(a - a_0)u] + \Omega[(b + b_0t)]]].$$

This implies that

$$\phi(\xi)\sigma(z, \xi) \in W_M^\Omega(\mathbb{C}^n).$$

Using this argument and argument of Fourier transform in $W_M^\Omega(\mathbb{C}^n)$, we can define the partial differential operator (2.17) and pseudo differential operator (2.18).

A linear partial differential operator $P(z, D)$ as $z = x + iy$ on \mathbb{C}^n is given by

$$P(z, D) = \sum_{|\alpha| \leq m} a_\alpha(z) D^{(\alpha)}.$$

If we replace $D^{(\alpha)}$ by a monomial $\xi^\alpha \in \mathbb{R}^n$ then we get a symbol

$$P(z, \xi) = \sum_{|\alpha| \leq m} a_\alpha(z) \xi^\alpha.$$

We take $\phi \in W_M^\Omega(\mathbb{C}^n)$ then we get

$$(P(z, D) \phi)(z) = \sum_{|\alpha| \leq m} a_\alpha(z) (D^{(\alpha)} \phi)(z)$$

By the property of Fourier transformation and using the technique of [3] we get

$$\begin{aligned} (P(z, D) \phi)(z) &= \sum_{|\alpha| \leq m} a_\alpha(z) (D^{(\alpha)} \hat{\phi})^\vee(z) \\ &= \sum_{|\alpha| \leq m} a_\alpha(z) (\xi^\alpha \hat{\phi})^\vee(z) \end{aligned}$$

$$\begin{aligned} (P(z, D) \phi)(z) &= \sum_{|\alpha| \leq m} a_\alpha(z) (2\pi)^{-n/2} \int_{\mathbb{R}^n} \xi^\alpha e^{i\langle z, \xi \rangle} \hat{\phi}(\xi) du \\ &= \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \left(\sum_{|\alpha| \leq m} a_\alpha(z) \xi^\alpha \right) \hat{\phi}(\xi) du \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} P(z, \xi) \hat{\phi}(\xi) du. \end{aligned}$$

Hence,

$$(P(z, D) \phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} P(z, \xi) \hat{\phi}(\xi) du. \tag{2.17}$$

In (2.17) if we replace $P(z, \xi)$ by more general symbol $\theta(z, \xi)$ which are no longer polynomial in ξ . The operator is so called pseudo differential operator.

The pseudo-differential operators associated with symbol $\theta(z, \xi) \in V^m$ is defined by

$$(A_\theta \phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \theta(z, \xi) \hat{\phi}(\xi) du \tag{2.18}$$

as $\xi = u + it \in \mathbb{C}^n$ and $z \in \mathbb{C}^n$ and $\phi \in W_M^\Omega(\mathbb{C}^n)$.

For $s \in \mathbb{R}$, the pseudo-differential operators V_s associated with symbol $\theta(\xi) = (1 + |\xi|^2)^{-s/2}$ as $\xi = u + it$ is defined by

$$(V_s f)(z) = F^{-1}((1 + |\xi|^2)^{-s/2} \hat{f})(z), \quad \text{for } f \in W_M^\Omega(\mathbb{C}^n). \tag{2.19}$$

Now, the Sobolev space $G^{s,p}(\mathbb{C}^n)$ of $L^p(\mathbb{R}^n)$ -type is defined to be the set of all $f \in [W_M^\Omega(\mathbb{C}^n)]'$ such that

$$\|f\|_{s,p} = \|(V_s f)(z)\|_p \quad \text{for } 1 \leq p < \infty. \tag{2.20}$$

The notations and terminologies of this paper are taken from Wong [10, pp 1-4] and Friedman [2].

3 Properties of Psuedo-differential Operators

In this section we study the various properties of pseudo-differential operators A_θ associated with symbol $\theta(z, \xi)$ on $W_M^\Omega(\mathbb{C}^n)$ -space.

Theorem 3.1. Let $\theta(z, \xi)$ be the symbol belong to V^m . Then A_θ maps $W_M^\Omega(\mathbb{C}^n)$ into itself.

Proof. Let $\phi \in W_M^\Omega(\mathbb{C}^n)$. Then, for any multi-indices α and β , we have to show that

$$\sup_{z \in \mathbb{C}^n} |\exp[M[(ax)] - \Omega[(by)]] (A_\theta \phi)(z)| < \infty$$

Now from (2.18) the pseudo-differential operator can be written as

$$z^\beta (A_\theta \phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} D_\xi^{(\beta)} e^{i\langle z, \xi \rangle} \theta(z, \xi) \hat{\phi}(\xi) du, \quad z, \xi \in \mathbb{C}^n.$$

Using integration by parts we have

$$\begin{aligned} z^\beta (A_\theta \phi)(z) &= (2\pi)^{-n/2} (-1)^{|\beta|} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} D_\xi^{(\beta)} [\theta(z, \xi) \hat{\phi}(\xi)] du \\ &= (2\pi)^{-n/2} (-1)^{|\beta|} \int_{\mathbb{R}^n} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} (D_\xi^{(\beta-\gamma)} \theta)(z, \xi) D_\xi^{(\gamma)} \hat{\phi}(\xi) du \\ &= (2\pi)^{-n/2} (-1)^{|\beta|} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} (D_\xi^{(\beta-\gamma)} \theta)(z, \xi) D_\xi^{(\gamma)} \hat{\phi}(\xi) du \\ &= (2\pi)^{-n/2} (-1)^{|\beta|} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \int_{\mathbb{R}^n} D_z^{(\alpha)} \left(e^{i\langle z, \xi+1 \rangle} e^{-i\langle z, 1 \rangle} \right) \prod_{j=1}^n [(1 + \xi_j)]^{-\alpha_j} \\ &\quad (D_\xi^{(\beta-\gamma)} \theta)(z, \xi) D_\xi^{(\gamma)} \hat{\phi}(\xi) du. \end{aligned}$$

Again using integration by parts we have

$$\begin{aligned}
z^\beta (A_\theta \phi)(z) &= (2\pi)^{-n/2} (-1)^{|\alpha|+|\beta|} \sum_{|\gamma| \leq \beta} \binom{\beta}{\gamma} \int_{\mathbb{R}^n} e^{i\langle z, \xi+1 \rangle} D_z^{(\alpha)} (e^{-i\langle z, 1 \rangle} D_\xi^{(\beta-\gamma)} \theta(z, \xi)) \\
&\quad \left(\prod_{j=1}^n (1 + \xi_j)^{-\alpha_j} \right) D_\xi^{(\gamma)} \hat{\phi}(\xi) du \\
&= (2\pi)^{-n/2} (-1)^{|\alpha|+|\beta|} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} \int_{\mathbb{R}^n} e^{i\langle z, \xi+1 \rangle} (D_z^{(\alpha-\delta)} D_\xi^{(\beta-\gamma)} \theta)(z, \xi) \\
&\quad \left(\prod_{j=1}^n (1 + \xi_j)^{-\alpha_j} \right) D_z^\delta e^{-i\langle z, 1 \rangle} D_\xi^{(\gamma)} \hat{\phi}(\xi) du \\
&= (2\pi)^{-n/2} (-1)^{|\alpha|+|\beta|} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} \int_{\mathbb{R}^n} e^{i\langle z, \xi+1 \rangle} (D_z^{(\alpha-\delta)} D_\xi^{(\beta-\gamma)} \theta) \\
&\quad (z, \xi) \left(\prod_{j=1}^n (1 + \xi_j)^{-\alpha_j} \right) (-1)^{|\delta|} e^{-i\langle z, 1 \rangle} D_\xi^{(\gamma)} \hat{\phi}(\xi) du
\end{aligned}$$

$$\begin{aligned}
z^\beta (A_\theta \phi)(z) &= (2\pi)^{-n/2} (-1)^{|\alpha|+|\beta|} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} (-1)^{|\delta|} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} (i)^{-|\delta|} \\
&\quad (D_z^{(\alpha-\delta)} D_\xi^{(\beta-\gamma)} \theta)(z, \xi) \left(\prod_{j=1}^n (1 + \xi_j)^{-\alpha_j} \right) D_\xi^{(\gamma)} \hat{\phi}(\xi) du.
\end{aligned}$$

Then

$$\begin{aligned}
|z^\beta (A_\theta \phi)(z)| &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} \int_{\mathbb{R}^n} |e^{i\langle z, \xi \rangle} (D_z^{(\alpha-\delta)} D_\xi^{(\beta-\gamma)} \theta)(z, \xi)| \\
&\quad (1 + |\xi|)^{-|\alpha|} |D_\xi^{(\gamma)} \hat{\phi}(\xi)| du \\
&\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} \int_{\mathbb{R}^n} |e^{i\langle (x+iy), (u+it) \rangle}| \\
&\quad |D_z^{(\alpha-\delta)} D_\xi^{(\beta-\gamma)} \theta(z, \xi)| (1 + |\xi|)^{-|\alpha|} |D_\xi^{(\gamma)} \hat{\phi}(\xi)| du \\
&\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} \int_{\mathbb{R}^n} |\exp(-\langle y, u \rangle - \langle x, t \rangle)| \\
&\quad |(D_z^{(\alpha-\delta)} D_\xi^{(\beta-\gamma)} \theta)(z, \xi)| (1 + |\xi|)^{-|\alpha|} |D_\xi^{(\gamma)} \hat{\phi}(\xi)| du.
\end{aligned}$$

Now,

$$\begin{aligned}
|z^\beta (A_\theta \phi)(z)| &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} \int_{\mathbb{R}^n} |\exp[\langle y, u \rangle - \langle x, t \rangle]| \\
&\quad |(D_z^{(\alpha-\delta)} D_\xi^{(\beta-\gamma)} \theta)(z, \xi)| (1 + |\xi|)^{-|\alpha|} |D_\xi^{(\gamma)} \hat{\phi}(\xi)| du \\
&\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} \int_{\mathbb{R}^n} |\exp[\langle y, u \rangle - \langle x, t \rangle]| \\
&\quad C_{\alpha-\delta, \beta-\gamma} (1 + |\xi|)^{m-|\beta|+|\gamma|-|\alpha|} |D_\xi^{(\gamma)} \hat{\phi}(\xi)| du
\end{aligned}$$

Using inequality $(1 + |\xi|)^{m-|\beta|+|\gamma|-|\alpha|} \leq \exp[M^0(a_1 u) + \Omega^0(b_1 t)]$ for $m - |\beta| + |\gamma| - |\alpha| > 0$

and (2.9) we have

$$|z^\beta(A_\theta\phi)(z)| \leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} C_{\alpha-\delta, \beta-\gamma} \int_{\mathbb{R}^n} |\exp[\langle y, u \rangle - \langle x, t \rangle]| \exp[M^0(a_1u) + \Omega^0(b_1t)] \exp[-M^0(a'' \cdot u) + \Omega^0(b''t)] du.$$

Using (2.3) and (2.4) we have

$$\begin{aligned} |z^\beta(A_\theta\phi)(z)| &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} C_{\alpha-\delta, \beta-\gamma} \int_{\mathbb{R}^n} \exp[|\langle y, u \rangle| - M^0[(a'' - a_1)u]] \exp[-\langle x, t \rangle + \Omega^0(b_1 + b'')t] du \\ &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} C_{\alpha-\delta, \beta-\gamma} \exp[-\langle x, t \rangle + \Omega^0(b_1 + b'')t] \int_{\mathbb{R}^n} \exp[|\langle y, u \rangle| - M^0[(a'' - a_1)u]] du \\ &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} C'_{\alpha-\delta, \gamma-\beta} \exp[-\langle x, t \rangle + \Omega^0[(b'' + b_1)t]] \int_{\mathbb{R}^n} \exp[|\langle y, u \rangle| - M^0[(a'' - a_1)u]] du. \end{aligned}$$

Using (2.11) and (2.12) and the arguments of [2, p.134]

$$\begin{aligned} |z^\beta(A_\theta\phi)(z)| &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \sum_{|\delta| \leq \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} C'_{\alpha-\delta, \beta-\delta} \exp[-M[(b'' + b_1)^{-1}x] + \Omega[(a'' - 2a_1)^{-1}y]] \int_{\mathbb{R}^n} \exp[-M^0(a_1u)] du \\ &\leq C'_{\alpha, \beta} \exp[-M[(b_1 + b'')^{-1}x] + \Omega[(a'' - 2a_1)^{-1}y]]. \end{aligned}$$

Hence

$$|\exp[M[(b_1 + b'')^{-1}x] - \Omega[(a'' - 2a_1)^{-1}y]](A_\theta\phi)(z)| \leq C'_{\alpha, \beta}(1 + |z|^\beta)^{-1}.$$

Thus

$$\begin{aligned} \sup_{z \in \mathbb{C}^n} |\exp[M(b_1 + b'')^{-1}x] - \Omega[(a'' - 2a_1)^{-1}y]](A_\theta\phi)(z)| &\leq C'_{\alpha, \beta} \\ &< \infty. \end{aligned}$$

This implies that

$$(A_\theta\phi)(z) \in W_M^\Omega(\mathbb{C}^n).$$

Theorem 3.2. A_θ is continuous linear mapping $W_M^\Omega(\mathbb{C}^n)$ into itself.

Proof. If the functions $\phi_\nu(z)$ converge uniformly to zero as $\nu \rightarrow \infty$ in any bounded domain of the z -plane and in addition satisfy the inequalities.

$$|\phi_\nu(z)| \leq C \exp[-M[(ax)] + \Omega[(by)]],$$

then the sequence $\phi_\nu(z) \in W_M^\Omega(\mathbb{C}^n)$ is said to converge to zero as $\nu \rightarrow \infty$, where the constants C , a and b do not depend on the index ν .

Since from Theorem 3.1 $A_\theta\phi$ is a mapping from $W_M^\Omega(\mathbb{C}^n)$ into itself. Using above results, $A_\theta\phi_\nu \in W_M^\Omega(\mathbb{C}^n)$ converge to zero uniformly in any bounded domain of the z -plane as $\nu \rightarrow \infty$ and satisfies the above inequality. Therefore, the sequence $A_\theta\phi \in W_M^\Omega(\mathbb{C}^n)$ is converges to zero as $\nu \rightarrow \infty$. This shows that A_θ maps continuously into itself.

Now, we define the pseudo-differential operator A_θ on $[W_M^\Omega(\mathbb{C}^n)]'$ -space by

$$\langle A_\theta f, \phi \rangle = \langle f, \overline{A_\theta^* \phi} \rangle, \quad \phi \in W_M^\Omega(\mathbb{C}^n). \tag{3.1}$$

Theorem 3.3. A_θ is a linear mapping from $[W_M^\Omega(\mathbb{C}^n)]'$ into itself.

Proof. Let $f \in [W_M^\Omega(\mathbb{C}^n)]'$. Then, for any sequence $\{\phi_\nu\}$ of functions in $W_M^\Omega(\mathbb{C}^n)$ converging to zero in $W_M^\Omega(\mathbb{C}^n)$, as $\nu \rightarrow \infty$. From (2.20) we have

$$\langle A_\theta f, \phi_\nu \rangle = \langle f, \overline{A_\theta^* \phi_\nu} \rangle, \quad \nu = 1, 2, 3, \dots \tag{3.2}$$

By the arguments of Theorem 3.2, we conclude that $\langle A_\theta f, \phi_\nu \rangle \rightarrow 0$ as $\nu \rightarrow \infty$. Hence $A_\theta f \in [W_M^\Omega(\mathbb{C}^n)]'$.

Definition 3.4. A sequence of distributions $\{f_\nu\}$ in $[W_M^\Omega(\mathbb{C}^n)]'$ is said to converge to zero in $[W_M^\Omega(\mathbb{C}^n)]'$ if $\langle f_\nu, \phi \rangle \rightarrow 0$ as $\nu \rightarrow \infty$ for all $\phi \in W_M^\Omega(\mathbb{C}^n)$.

Theorem 3.5. A_θ maps continuously $[W_M^\Omega(\mathbb{C}^n)]'$ into itself.

Proof. Let $\phi \in [W_M^\Omega(\mathbb{C}^n)]'$. Then, using (3.2) and the fact that $f_\nu \rightarrow 0$ in $[W_M^\Omega(\mathbb{C}^n)]'$ as $\nu \rightarrow \infty$,

$$\langle A_\theta f_\nu, \phi \rangle = \langle f_\nu, \overline{A_\theta^* \phi} \rangle \rightarrow 0$$

as $\nu \rightarrow \infty$. Hence $A_\theta f_\nu \rightarrow 0$ in $[W_M^\Omega(\mathbb{C}^n)]'$ as $\nu \rightarrow \infty$, and the proof is complete.

Theorem 3.6. Let $\theta \in C^k(\mathbb{C}^n)$, $k \geq n/2$, be such that there exists a positive constant B such that

$$|(D_\xi^{(\alpha)} \theta)(\xi)| \leq C_{\alpha,n} (1 + |\xi|)^{-|\alpha|}, \quad \xi \neq 0 \tag{3.3}$$

for multi-indices α with $|\alpha| \leq k$. Then, for $1 \leq p < \infty$, there exists a positive constant B , depending on α and N , such that

$$\|(A\phi)(z)\|_p \leq M'_{\alpha,n} \|\phi\|_p, \quad \phi \in W_M^\Omega(\mathbb{C}^n), \tag{3.4}$$

where

$$(A\phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \theta(\xi) \hat{\phi}(\xi) du, \tag{3.5}$$

$\xi = u + it$, and $\hat{\phi}$ denotes the Fourier transformation of ϕ .

Proof. (3.5) can be written as

$$(A\phi)(z) = (2\pi)^{-n/2} F^{-1}[\theta(\xi) \hat{\phi}(\xi)](z) \tag{3.6}$$

where F^{-1} denotes the inverse Fourier transformation of a function z as $z = x + iy$.

Now, we assume that

$$F^{-1}[\theta(\xi) \hat{\phi}(\xi)](z) = (f * g)(z). \tag{3.7}$$

Then by convolution property of Fourier transformation, we have

$$\begin{aligned} \theta(\xi) \hat{\phi}(\xi) &= F[(f * g)](\xi) \\ &= \hat{f}(\xi) \cdot \hat{g}(\xi). \end{aligned}$$

This implies that

$$f(z) = F^{-1}[\theta(\xi)](z), \quad g(z) = \phi(z).$$

Thus, the expression (3.7) yields

$$(A\phi)(z) = (2\pi)^{-n/2} (F^{-1}[\theta(\xi)] * \phi)(z).$$

Using convolution property $\|f * \phi\|_p \leq \|f\|_1 \|\phi\|_p$ for $f \in L^1(\mathbb{R}^n)$ and $\phi \in L^p(\mathbb{R}^n)$ we have

$$\begin{aligned} \|(A\phi)(z)\|_p &= (2\pi)^{-n/2} \|(F^{-1}[\theta(\xi)] * \phi)(x)\|_p \\ &\leq (2\pi)^{-n/2} \|F^{-1}[\theta(\xi)]\|_1 \|\phi\|_p. \end{aligned} \tag{3.8}$$

Next, we have to prove that

$$F^{-1}[\theta(\xi)] \in L^1(\mathbb{R}^n).$$

Thus, from [3, p. 24] we have

$$F^{-1}[\theta(\xi)](z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \theta(\xi) du.$$

By property of Fourier transformation the above expression gives

$$(z)^\alpha F^{-1}[\theta(\xi)](z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} D_\xi^{(\alpha)}(e^{-i\langle z, \xi \rangle})\theta(\xi) du.$$

Integration by parts, above expression can be obtained

$$\begin{aligned} (z)^\alpha F^{-1}[a(\xi)](z) &= (2\pi)^{-n/2}(-1)^{|\alpha|} \int_{\mathbb{R}^n} e^{-i\langle z, \xi \rangle} (D_\xi^{(\alpha)}\theta)(\xi) du \\ &= (2\pi)^{-n/2}(-1)^{|\alpha|} \int_{\mathbb{R}^n} \exp[-\langle x, u \rangle - \langle y, t \rangle] (D_\xi^{(\alpha)}\theta)(\xi) du. \end{aligned}$$

Therefore,

$$\begin{aligned} |(z)^\alpha F^{-1}[a(\xi)](z)| &\leq B C_n |\exp[-\langle y, t \rangle]| \int_{\mathbb{R}^n} |\exp[-\langle x, u \rangle]| (1 + |\xi|)^{-|\alpha|} du \\ &\leq B_n |\exp[-\langle y, t \rangle]| \sup_u |\exp[-\langle x, u \rangle]| \int_{\mathbb{R}^n} (1 + |\xi|)^{-|\alpha|} du \\ &\leq B_{\alpha, n} |\exp[-\langle y, t \rangle]| \\ &\leq B_{\alpha, n}. \end{aligned}$$

This implies that

$$|F^{-1}[\theta(\xi)]| \leq B_{\alpha, n} \|(1 + |z|^n)^{-1}\|_1. \tag{3.9}$$

From (3.8) - (3.9), we find the required result (3.4)

Theorem 3.7. Let $\phi \in W_M^\Omega(\mathbb{C}^n)$ and symbol $\theta_m(z, \xi)$ has compact support in z . Then, pseudo-differential operators $A_{\theta_m}\phi$ can be expressed as

$$(A_{\theta_m}\phi)(z) = (2\pi)^{-n} \left(\int_{\mathbb{R}^n} e^{-i\langle \lambda, z \rangle} \left(\int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \hat{\theta}_m(\lambda, \xi) \hat{\phi}(\xi) du \right) dv \right)$$

where

$$(A_\lambda\phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle z, \lambda \rangle} \hat{\theta}_m(\lambda, \xi) \hat{\phi}(\xi) du \tag{3.10}$$

as $z = x + iy, \lambda = v + iv'$ and

$$\hat{\theta}_m(\lambda, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle \lambda, z \rangle} \theta_m(z, \xi) dx, \quad \lambda, \xi \in \mathbb{C}^n.$$

Proof. Since

$$(A_\theta\phi)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \theta_m(z, \xi) \hat{\phi}(\xi) du,$$

then, by using the property of Fourier transformation we have

$$(A_{\theta_m}\phi)(z) = (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} e^{-i\langle \lambda, z \rangle} \left((2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \hat{\theta}_m(\lambda, \xi) dv \right) \hat{\phi}(\xi) du \right)$$

as $\lambda = v + iv'$. By Fubini's theorem and (3.10) we get

$$(A_{\theta_m}\phi)(z) = (2\pi)^{-n} \left(\int_{\mathbb{R}^n} e^{-i\langle \lambda, z \rangle} \left(\int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \hat{\theta}_m(\lambda, \xi) \hat{\phi}(\xi) du \right) dv \right). \tag{3.11}$$

Lemma 3.8. For all multi-indices α and β and positive integers N , there is a positive constant $C_{\alpha, N}$, depending on α and N such that

$$|(D_\xi^{(\alpha)}\hat{\theta}_m)(\lambda, \xi)| \leq C_{\alpha, N} (1 + |\lambda|^{|\beta|})^{-1} (1 + |\xi|)^{-|\alpha|}$$

for $\xi = u + it$ and $\lambda = v + iv'$.

Proof. The Fourier transformation of θ_m with respect to $\lambda = v + iv'$ is given by

$$\hat{\theta}_m(\lambda, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle z, \lambda \rangle} \theta_m(z, \xi) dx.$$

Then

$$(i\lambda)^\beta D_\xi^{(\alpha)} \hat{\theta}_m(\lambda, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \partial_z^{(\beta)} [e^{-i\langle z, \lambda \rangle}] D_\xi^{(\alpha)} \theta_m(z, \xi) dx.$$

Integration by parts we have

$$(i\lambda)^\beta D_\xi^{(\alpha)} \hat{\theta}_m(\lambda, \xi) = (2\pi)^{-n/2} (-1)^{|\beta|} \int_{\mathbb{R}^n} e^{i\langle z, \lambda \rangle} \partial_z^{(\beta)} D_\xi^{(\alpha)} \theta_m(z, \xi) dx$$

as $z = x + iy$.

Hence,

$$\begin{aligned} (i\lambda)^\beta D_\xi^{(\alpha)} \hat{\theta}_m(\lambda, \xi) &= (2\pi)^{-n/2} (-1)^\beta \int_{\mathbb{R}^n} e^{i\langle z, \lambda \rangle} \partial_z^{(\beta)} D_\xi^{(\alpha)} [\eta(z - m)\theta(z, \xi)] dx \\ &= (2\pi)^{-n/2} (-1)^\beta \int_{\mathbb{R}^n} e^{i\langle z, \lambda \rangle} \sum_{|\gamma| \leq \beta} \binom{\beta}{\gamma} D_z^{(\gamma)} \eta(z - m) \\ &\quad \partial_z^{(\beta-\gamma)} D_\xi^{(\alpha)} \theta(z, \xi) dx. \end{aligned}$$

Now

$$\begin{aligned} &|\lambda^\beta D_\xi^{(\alpha)} \hat{\theta}_m(\lambda, \xi)| \\ &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \binom{\beta}{\gamma} \int_{\mathbb{R}^n} |\exp[-\langle x, v \rangle - \langle y, v' \rangle]| |D_\xi^{(\gamma)} \eta(z - m)| |D_z^{(\beta-\gamma)} D_\xi^{(\alpha)} \theta(z, \xi)| dx \\ &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \binom{\beta}{\gamma} \int_{\mathbb{R}^n} |\exp[-\langle x, v \rangle - \langle y, v' \rangle]| |\partial_z^{(\gamma)} \eta(z - m)| C_{\beta-\gamma, \alpha} (1 + |\xi|)^{-|\alpha|} dx \\ &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \binom{\beta}{\gamma} C_{\beta-\gamma, \alpha} \int_{\mathbb{R}^n} |\exp[-\langle x, v \rangle - \langle y, v' \rangle]| |\partial_z^{(\gamma)} \eta(z - m)| (1 + |\xi|)^{-|\alpha|} dx \\ &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \binom{\beta}{\gamma} C_{\beta-\gamma, \alpha} (1 + |\xi|)^{-|\alpha|} \int_{\mathbb{R}^n} |\exp[-\langle x, v \rangle] \partial_z^{(\gamma)} \eta(z - m)| dx. \end{aligned}$$

Then

$$\begin{aligned} &|\lambda^\beta D_\xi^{(\alpha)} \hat{\theta}_m(\lambda, \xi)| \\ &\leq (2\pi)^{-n/2} \sum_{|\gamma| \leq \beta} \binom{\beta}{\gamma} C_{\beta-\gamma, \alpha} (1 + |\xi|)^{-|\alpha|} \int_{\mathbb{R}^n} |\partial_z^{(\gamma)} \eta(z - m)| dx \\ &\leq (2\pi)^{-n/2} (1 + |\xi|)^{-|\alpha|} \sum_{|\gamma| \leq \beta} \binom{\beta}{\gamma} C_\gamma C_{\beta-\gamma, \alpha} \\ &\leq (2\pi)^{-n/2} (1 + |\xi|)^{-|\alpha|} C_\beta \\ &\leq C_{\beta, n} (1 + |\xi|)^{-|\alpha|}. \end{aligned}$$

Hence, for large arbitrary positive integers N , we have

$$\left| (D_\xi^{(\alpha)} \hat{\theta}_m)(\lambda, \xi) \right| \leq C_{n, \beta} (1 + |\lambda|^N)^{-1} (1 + |\xi|)^{-|\alpha|}.$$

as $\xi = u + it$.

Theorem 3.9. Let $\theta \in V^0$. Then we get the following relation

$$\int_{Q_m} |(A_\theta \phi)(z)|^p dx \leq C_N^p \|\phi\|_p^p \quad \forall \phi \in W_M^\Omega(\mathbb{C}^n).$$

Proof. From Wong [10, p. 80], we can write

$$\left(\int_{Q_m} |(A_\theta \phi)(z)|^p dx \right) \leq \left(\int_{\mathbb{R}^n} |(A_{\theta_m} \phi)(z)|^p dx \right). \tag{3.12}$$

Using Lemma 3.8 and Theorem 3.6, we find that

$$\|A_\lambda \phi\|_p \leq C_N(1 + |\lambda|)^{-N} \|\phi\|_p \quad \forall \phi \in W_M^\Omega(\mathbb{C}^n). \tag{3.13}$$

Using (3.11), (3.13) and Minkowski's inequality in the integral form we obtain

$$\begin{aligned} \|A_{\theta_m} \phi\|_p &= (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{i\langle z, \lambda \rangle} (A_\lambda \phi)(z) dv \right|^p dx \right)^{1/p} \\ &= (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \exp[-\langle x, v' \rangle - \langle y, v \rangle] (A_\lambda \phi)(z) dv \right|^p dx \right)^{1/p} \\ &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\exp[-\langle x, v' \rangle]| (A_\lambda \phi)(z)^p dx \right)^{1/p} dv \\ &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |(A_\lambda \phi)(z)|^p dx \right)^{1/p} dv \\ &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|(A_\lambda \phi)(z)\|_p dv. \end{aligned}$$

Using (3.13) we get

$$\begin{aligned} \|A_{\theta_m} \phi\|_p &\leq (2\pi)^{-n/2} C_N \left(\int_{\mathbb{R}^n} (1 + |\lambda|)^{-N} dv \right) \|\phi\|_p \\ &\leq (2\pi)^{-n/2} C_N \|\phi\|_p \quad \phi \in W_M^\Omega(\mathbb{C}^n). \end{aligned}$$

Hence from (3.12) and (3.13) we have

$$\int_{Q_m} |(A_\theta \phi)(z)|^p dx \leq C_{N,n}^p \|\phi\|_p^p, \quad \phi \in W_M^\Omega(\mathbb{C}^n). \tag{3.14}$$

Now, we represent A_θ as a singular integral operator.

Lemma 3.10. Let $K(z, w) = \int_{\mathbb{R}^n} e^{i\langle z, w \rangle} \theta(z, w) ds, z = x + iy \in \mathbb{C}^n, w = s + iv \in \mathbb{C}^n$ in the distributional sense. Then

- (i) for each $z \in \mathbb{C}^n, K(z, w)$ is a function defined on $\mathbb{R}^n,$
- (ii) for each sufficiently large positive integer $N,$ there is a positive constant C_N such that

$$|K(z - w, w)| \leq C_N(1 + |z - w|^N)^{-1}, \tag{3.15}$$

- (iii) for each fixed $z = x + iy$ and $\phi \in W_M^\Omega(\mathbb{C}^n)$ vanishing in the neighbourhood of $\mathbb{C}^n,$ we find that

$$(A_\theta \phi)(z) = \int_{\mathbb{R}^n} K(z - w, w) \phi(w) ds. \tag{3.16}$$

Proof. (i) can be defined by using the arguments of [10, p. 26] and [1, pp. 23-24].

To prove (ii), let α be a multi-index with length greater than $w.$ Then by the property of Fourier transformation $(D^{(\alpha)} u)^\wedge = \xi^{|\alpha|} \tilde{u}$ we have

$$(iw)^\alpha K(z, w) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} e^{i\langle \xi, w \rangle} D_\xi^{(\alpha)} \theta(z, \xi) du.$$

Therefore, using (2.16) and tools of theorem (3.6) we have

$$|K(z, w)| \leq C'_\alpha (1 + |w|^\alpha)^{-1}.$$

For large positive integer N we can obtain

$$|K(z, z - w)| \leq C'_\alpha (1 + |z - w|^N)^{-1}.$$

To prove (iii), we define the distribution L_z on $W_M^\Omega(\mathbb{C}^n)$ by

$$\langle L_z, \psi \rangle = \int_{\mathbb{R}^n} \theta(z, \xi) \psi(\xi) \, du,$$

where $z = x + iy$, $\xi = u + i\tau$ and $w = s + iv$. By the definition of pseudo-differential operator (2.18)

$$\begin{aligned} (A_\theta \phi)(z) &= \int_{\mathbb{R}^n} e^{i\langle z, \xi \rangle} \theta(z, \xi) \hat{\phi}(\xi) \, du \\ &= L_z(M_z \hat{\phi}) \end{aligned} \tag{3.17}$$

Using Gelfand and Shilov [3] technique of integration we get

$$\begin{aligned} (A_\theta \phi)(z) &= L_z(T_z \phi) \\ &= \hat{L}_z(T_z \phi) \end{aligned} \tag{3.18}$$

From (i) we have

$$\hat{L}_z(\psi) = \int_{\mathbb{R}^n} \theta(z, -w) \psi(w) \, ds.$$

Hence

$$\begin{aligned} (A_\theta \phi)(z) &= \int_{\mathbb{R}^n} \theta(z, -w) (T_z \phi)(w) \, ds \\ &= \int_{\mathbb{R}^n} \theta(z, -w) \phi(z + w) \, ds \\ &= \int_{\mathbb{R}^n} \theta(z, z - w) \phi(w) \, ds. \end{aligned}$$

This completes the proof of the theorem.

Theorem 3.11. Let $\theta(z, \xi)$ be a symbol in V^0 . Then $A_\theta : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is a bounded linear operator for $1 < p < \infty$.

Proof. From Theorem 3.6, Theorem 3.7, Theorem 3.9 and Lemma 3.10 we can show that the pseudo-differential operator A_θ is a bounded linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.
□

4 The Sobolev Space

In this section, we study the pseudo-differential operators on Sobolev type space $G^{s,p}(\mathbb{C}^n)$ which is defined in Section 2.

For $s \in \mathbb{R}$, the pseudo-differential operator associated with symbol $\theta(\xi) = (1 + |\xi|^2)^{-s/2}$ as $\xi = u + it$ is defined by

$$(V_\theta u)(z) = F^{-1}(\theta(\xi) \hat{u}(\xi))(z) \quad \text{for } u \in [W_M^\Omega(\mathbb{C}^n)]'. \tag{4.1}$$

Now, we define the Sobolev space $G^{s,p}(\mathbb{C}^n)$ of L^p -type to be the set of all distribution $u \in [W_M^\Omega(\mathbb{C}^n)]'$ such that

$$\|u\|_{s,p} = \|V_{-s} u\|_p \quad \text{for } 1 \leq p < \infty. \tag{4.2}$$

Theorem 4.1. Let $u \in [W_M^\Omega(\mathbb{C}^n)]'$. Then

- (i) $V_s V_t u = V_{s+t} u$,
- (ii) $V_0 u = u$.

Proof. The proof of the above theorem is obvious from [10, p. 90].

Theorem 4.2. $G^{s,p}(\mathbb{C}^n)$ is a Banach space with respect to $\|u\|_{s,p}$.

Proof. The proof of the above theorem is usual from [10, p. 81].

Theorem 4.3. V_t is an isometry from $V^{s,p}$ onto $V^{s+t,p}$.

Proof. Let $u \in V^{s,p}$. Then from Theorem 4.1 we get $J_{-t}v \in G^{s,p}(\mathbb{C}^n)$ and $v_t v_{-t}v = v$. This implies $G^{s,p}(\mathbb{C}^n)$ is onto.

Theorem 4.4. Let $1 < p < \infty$ and $s \leq t$. Then $G^{t,p}(\mathbb{C}^n) \subseteq G^{s,p}(\mathbb{C}^n)$.

Proof. See [10, p. 91]. This is called Sobolev embedding theorem.

Theorem 4.5. Let $s \geq 0$ and $1 \leq p < \infty$. Then

$$\|V_s \phi\|_p \leq \|\phi\|_p, \quad \phi \in L^p(\mathbb{R}^n).$$

Proof. We have

$$(J_s \phi)\hat{(\xi)} = (1 + |\xi|^2)^{-s/2} \hat{\phi}(\xi), \quad \xi \in \mathbb{C}^n.$$

Hence, for $\hat{G}_s(\xi) = (1 + |\xi|^2)^{-s/2}$ we have

$$(G_s * \phi)\hat{(\xi)} = (1 + |\xi|^2)^{-s/2} \hat{\phi}(\xi). \tag{4.3}$$

Hence, for all $\phi \in W_M^\Omega(\mathbb{R}^n)$,

$$J_s \phi = (G_s * \phi),$$

and using convolution property

$$\begin{aligned} \|J_s \phi\|_p &= \|G_s * \phi\|_p \\ &\leq \|G_s\|_1 \|\phi\|_p \\ &\leq \|\phi\|_p. \end{aligned}$$

Theorem 4.6. For symbol θ in U^m , $A_\theta : G^{m,p}(\mathbb{C}^n) \rightarrow G^{0,p}(\mathbb{C}^n)$ is a bounded linear operator for $1 < p < \infty$.

Proof. Consider the bounded linear operators

$$\begin{aligned} V_{-s} &: G^{s,p}(\mathbb{C}^n) \rightarrow G^{0,p}(\mathbb{C}^n) \\ A_\theta V_m &: G^{0,p}(\mathbb{C}^n) \rightarrow G^{0,p}(\mathbb{C}^n) \end{aligned}$$

and

$$V_{s-m} : G^{0,p}(\mathbb{C}^n) \rightarrow G^{s-m,p}(\mathbb{C}^n).$$

The first and the third operators are bounded by isometry of pseudo-differential operator of Theorem 4.3 and the second operator is bounded by $L^p(\mathbb{R}^n)$ -boundedness property of pseudo-differential operator. Hence the product $V_{s-m} A_\theta V_{m-s}$ is a bounded linear operator from $G^{s,p}$ into $G^{s-m,p}$. By Theorem 4.3 operators V_{m-s} and V_{s-m} are isometric and onto. Hence, $A_0 : G^{m,p} \rightarrow G^{0,p}$ must be bounded linear operator.

Theorem 4.7. Let $\theta(z, \xi)$ be any symbol in V^m , then $A_\theta : G^{s,p}(\mathbb{C}^n) \rightarrow G^{s-m,p}(\mathbb{C}^n)$ is a bounded linear operator for $1 \leq p < \infty$.

Proof. Since $V_{m-s} A_\theta$ is a pseudo-differential operator with symbol in V^s . Hence, from Theorem 4.6 we can easily prove that

$$\|A_\theta u\|_{s-m,p} = \|J_{m-s} A_\theta u\|_p \leq C \|u\|_{s,p} \quad \forall u \in G^{s,p}.$$

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