

On α -*Centralizers of Semiprime Rings With Involution

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Abstract. Let R be a semiprime ring equipped with an involution $*$ and α be an epimorphism of R . In this paper, we prove that an additive mapping $T : R \rightarrow R$ is a Jordan α -*centralizer if the following holds:

$$2T(xy) = T(x)\alpha(y^*)\alpha(x^*) + \alpha(x^*)\alpha(y^*)T(x), \text{ for all } x, y \in R.$$

1 Introduction

Throughout, R will represent an associative ring with center Z . Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$, and semiprime if $xRx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$ is called an involution and R is called a *-ring.

According B. Zalar [8], an additive mapping $T : R \rightarrow R$ is called a left (resp. right) centralizer of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. If T is both left as well right centralizer, then it is called a centralizer. This concept appears naturally C^* -algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would write that $T : R_R \rightarrow R_R$ is a homomorphism of a ring module R into itself instead of a left centralizer. In case $T : R \rightarrow R$ is a centralizer, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ and $\lambda \in C$, where C is the extended centroid of R .

An additive mapping $T : R \rightarrow R$ is said to be a left (resp. right) Jordan centralizer if $T(x^2) = T(x)x$ (resp. $T(x^2) = xT(x)$) holds for all $x \in R$. Zalar proved in [8] that any left (right) Jordan centralizer on 2-torsion free semiprime ring is a left (right) centralizer. Recently, in [1], E. Albaş introduced the definition of α -centralizer of R , i. e. an additive mapping $T : R \rightarrow R$ is called a left (resp. right) α -centralizer of R if $T(xy) = T(x)\alpha(y)$ (resp. $T(xy) = \alpha(x)T(y)$) holds for all $x, y \in R$, where α is an endomorphism of R . If T is left and right α -centralizer then it is natural to call α -centralizer. Clearly every centralizer is a special case of a α -centralizer with $\alpha = id_R$. Also, an additive mapping $T : R \rightarrow R$ associated with a homomorphism $\alpha : R \rightarrow R$, if $L_a(x) = \alpha(x)a$ and $R_a(x) = \alpha(x)a$ for a fixed element $a \in R$ and for all $x \in R$, then L_a is a left α -centralizer and R_a is a right α -centralizer. Albaş showed Zalar's result holds for α -centralizer. Considerable work has been done on this topic during the last couple of decades (see [1-8], where further references can be found).

On the other hand, it was proved that T is a centralizer if one of the following holds

$$\begin{aligned} 2T(x^2) &= T(x)x + xT(x), \\ 2T(xy) &= T(x)y + xyT(x), \text{ for all } x, y \in R, \end{aligned}$$

where $T : R \rightarrow R$ is an additive mapping respectively in [5] and [7]. These results proved for α -centralizer in [4] and [3].

Inspired by the definition centralizer, the notion of *-centralizer was extended as follow:

Let R be a ring with involution $*$. An additive mapping $T : R \rightarrow R$ is called a left (resp. right) *-centralizer of R if $T(xy) = T(x)y^*$ (resp. $T(xy) = x^*T(y)$) holds for all $x, y \in R$. An additive mapping $T : R \rightarrow R$ is said to be a left (resp. right) Jordan *-centralizer if $T(x^2) = T(x)x^*$ (resp. $T(x^2) = x^*T(x)$) holds for all $x \in R$. In [2], the authors proved that if R is a 2-torsion free semiprime ring and $T : R \rightarrow R$ is an additive mapping such that $2T(x^2) = T(x)\alpha(x^*) + \alpha(x^*)T(x)$, for all $x \in R$, then T is a Jordan α -*centralizer. Motivated this result, we will prove that an additive mapping $T : R \rightarrow R$ is a Jordan α -*centralizer if the following holds:

$$2T(xy) = T(x)\alpha(y^*)\alpha(x^*) + \alpha(x^*)\alpha(y^*)T(x), \text{ for all } x, y \in R$$

This enables us a unified treatment (and extensions) of several results that can be found in the literature.

2 Results

Lemma 2.1. [6, Lemma 1] *Let R be a 2-torsion free semiprime ring. Suppose that the identity $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. Then in this case $(a + c)xb = 0$ satisfied for all $x \in R$.*

Lemma 2.2. [2, Theorem 2.1] *Let R be a 2-torsion free semiprime * -ring. Suppose that α is an automorphism of R . If $T : R \rightarrow R$ is an additive mapping satisfying $2T(x^2) = T(x)\alpha(x^*) + \alpha(x^*)T(x)$ for all $x \in R$, then T is a Jordan α - * centralizer.*

Theorem 2.3. *Let R be a 2-torsion free semiprime * -ring. Suppose that α is an automorphism of R . If $T : R \rightarrow R$ is an additive mapping satisfying $2T(xyx) = T(x)\alpha(y^*)\alpha(x^*) + \alpha(x^*)\alpha(y^*)T(x)$ for all $x, y \in R$, then T is a Jordan α - * centralizer.*

Proof. By the hypothesis, we have

$$2T(xyx) = T(x)\alpha(y^*)\alpha(x^*) + \alpha(x^*)\alpha(y^*)T(x), \text{ for all } x, y \in R. \quad (2.1)$$

Replacing x by $x + z$ in (2.1) and using this, we obtain that

$$\begin{aligned} 2T(xyz + zyx) &= T(x)\alpha(y^*)\alpha(z^*) + T(z)\alpha(y^*)\alpha(x^*) \\ &+ \alpha(x^*)\alpha(y^*)T(z) + \alpha(z^*)\alpha(y^*)T(x), \text{ for all } x, y, z \in R. \end{aligned} \quad (2.2)$$

Taking x^2 instead of z , we arrive at

$$\begin{aligned} 2T(xyx^2 + x^2yx) &= T(x)\alpha(y^*)\alpha(x^*)\alpha(x^*) + T(x^2)\alpha(y^*)\alpha(x^*) \\ &+ \alpha(x^*)\alpha(y^*)T(x^2) + \alpha(x^*)\alpha(x^*)\alpha(y^*)T(x), \text{ for all } x, y \in R. \end{aligned} \quad (2.3)$$

Substituting $xy + yx$ for x in (2.1), we have

$$\begin{aligned} 2T(xyx^2 + x^2yx) &= T(x)\alpha(y^*)\alpha(x^*)\alpha(x^*) + T(x)\alpha(x^*)\alpha(y^*)\alpha(x^*) \\ &+ \alpha(x^*)\alpha(y^*)\alpha(x^*)T(x) + \alpha(x^*)\alpha(x^*)\alpha(y^*)T(x), \text{ for all } x, y \in R. \end{aligned} \quad (2.4)$$

By comparing (2.3) and (2.4), we get

$$T(x^2)\alpha(y^*)\alpha(x^*) + \alpha(x^*)\alpha(y^*)T(x^2) = T(x)\alpha(x^*)\alpha(y^*)\alpha(x^*) + \alpha(x^*)\alpha(y^*)\alpha(x^*)T(x),$$

and so

$$\begin{aligned} &(T(x^2) - T(x)\alpha(x^*))\alpha(y^*)\alpha(x^*) \\ &+ \alpha(x^*)\alpha(y^*)(T(x^2) - \alpha(x^*)T(x)) = 0, \text{ for all } x, y \in R. \end{aligned} \quad (2.5)$$

Putting y^* for y in (2.4) yields that

$$(T(x^2) - T(x)\alpha(x^*))\alpha(y)\alpha(x^*) + \alpha(x^*)\alpha(y)(T(x^2) - \alpha(x^*)T(x)) = 0, \text{ for all } x, y \in R.$$

Since α is an epimorphism of R , we have

$$(T(x^2) - T(x)\alpha(x^*))y\alpha(x^*) + \alpha(x^*)y(T(x^2) - \alpha(x^*)T(x)) = 0, \text{ for all } x, y \in R.$$

By Lemma 2.1, we conclude that

$$(2T(x^2) - T(x)\alpha(x^*) - \alpha(x^*)T(x))y\alpha(x^*) = 0, \text{ for all } x, y \in R. \quad (2.6)$$

Define $A(x) = 2T(x^2) - T(x)\alpha(x^*) - \alpha(x^*)T(x)$. Hence (2.6) can be rewritten as

$$A(x)y\alpha(x^*) = 0, \text{ for all } x, y \in R. \quad (2.7)$$

Substituting $\alpha(x^*)yA(x)$ for y in (2.7), we get

$$A(x)\alpha(x^*)yA(x)\alpha(x^*) = 0, \text{ for all } x, y \in R.$$

Using R is semiprime ring, we arrive at

$$A(x)\alpha(x^*) = 0, \text{ for all } x \in R. \quad (2.8)$$

Left multiplying (2.7) by $\alpha(x^*)$ and right multiplying (2.8) by $A(x)$, we have

$$\alpha(x^*)A(x)y\alpha(x^*)A(x) = 0, \text{ for all } x \in R.$$

By the semiprimeness of R , we find that

$$\alpha(x^*)A(x) = 0, \text{ for all } x \in R. \quad (2.9)$$

Replacing x by $x + y$ in (2.8) and using this, we obtain that

$$A(x + y)\alpha(x^* + y^*) = 0,$$

and so

$$(A(x) + A(y) + 2T(xy + yx) - T(x)\alpha(y^*) - T(y)\alpha(x^*) - \alpha(x^*)T(y) - \alpha(y^*)T(x))\alpha(x^* + y^*) = 0.$$

That is

$$\begin{aligned} &A(x)\alpha(x^*) + A(y)\alpha(x^*) + B(x, y)\alpha(x^*) + A(x)\alpha(y^*) \\ &+ A(y)\alpha(y^*) + B(x, y)\alpha(y^*) = 0, \text{ for all } x, y \in R, \end{aligned}$$

where $B(x, y) = 2T(xy + yx) - T(x)\alpha(y^*) - T(y)\alpha(x^*) - \alpha(x^*)T(y) - \alpha(y^*)T(x)$. Using (2.8) in the last equation, we see that

$$A(x)\alpha(y^*) + A(y)\alpha(x^*) + B(x, y)\alpha(x^*) + B(x, y)\alpha(y^*) = 0, \text{ for all } x, y \in R. \quad (2.10)$$

Substituting $-x$ for x in (2.10) and using $A(-x) = A(x)$, $B(-x, y) = -B(x, y)$, we arrive at

$$A(x)\alpha(y^*) - A(y)\alpha(x^*) + B(x, y)\alpha(x^*) - B(x, y)\alpha(y^*) = 0, \text{ for all } x, y \in R. \quad (2.11)$$

Now, combining (2.10) and (2.11), we get

$$A(x)\alpha(y^*) + B(x, y)\alpha(x^*) = 0, \text{ for all } x, y \in R. \quad (2.12)$$

Multiplying (2.12) from the right by $A(x)$ and using (2.9), we obtain that

$$A(x)\alpha(y^*)A(x) = 0, \text{ for all } x, y \in R.$$

Writing y^* instead of y and α is an epimorphism of R , we have

$$A(x)yA(x) = 0, \text{ for all } x, y \in R.$$

By the semiprimeness of R , we find that $A(x) = 0$ for all $x \in R$, and so

$$2T(x^2) = T(x)\alpha(x^*) + \alpha(x^*)T(x), \text{ for all } x, y \in R.$$

Hence T is a Jordan α -*centralizer by Lemma 2.2. \square

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