Palestine Journal of Mathematics

Vol. 3(Spec 1) (2014) , 445–448

On $\alpha-^*$Centralizers of Semiprime Rings With Involution

Öznur Gölbaşı and Ahu Kadriye Gürcan

Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

Communicated by Ayman Badawi

MSC 2010 Classifications: 16W10, 16N60.

Keywords and phrases: Semiprime ring, centralizer, $\alpha-^*$centralizer.

Abstract. Let R be a semiprime ring equipped with an involution $*$ and α be an epimorphism of R. In this paper, we prove that an additive mapping $T : R \to R$ is a Jordan $\alpha-^*$centralizer if the following holds:

$$2T(xyx) = T(x)\alpha(y^*)\alpha(x^*) + \alpha(x^*)\alpha(y^*)T(x), \text{ for all } x, y \in R.$$

1 Introduction

Throughout, R will represent an associative ring with center Z. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$, and semiprime if $xRx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$ is called an involution and R is called a $*$-ring.

According B. Zalar [8], an additive mapping $T : R \to R$ is called a left (resp. right) centralizer of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. If T is both left as well right centralizer, then it is called a centralizer. This concept appears naturally C^*-algebras. In ring theory it is more common to work with module homorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of a ring module R into itself instead of a left centralizer. In case $T : R \to R$ is a centralizer, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ and $\lambda \in C$, where C is the extended centroid of R.

An additive mapping $T : R \to R$ is said to be a left (resp. right) Jordan centralizer if $T(x^2) = T(x)x$ (resp. $T(x^2) = xT(x)$) holds for all $x \in R$. Zalar proved in [8] that any left (right) Jordan centralizer on $2-$torsion free semiprime ring is a left (right) centralizer. Recently, in [1], E. Albaş introduced the definition of $\alpha-$centralizer of R, i. e. an additive mapping $T : R \to R$ is called a left (resp. right) $\alpha-$centralizer of R if $T(xy) = T(x)\alpha(y)$ (resp. $T(xy) = \alpha(x)T(y)$) holds for all $x, y \in R$, where α is an endomorphism of R. If T is left and right $\alpha-$centralizer then it is natural to call $\alpha-$centralizer. Clearly every centralizer is a special case of a $\alpha-$centralizer with $\alpha = id_R$. Also, an additive mapping $T : R \to R$ associated with a homomorphism $\alpha : R \to R$, if $L_a(x) = a\alpha(x)$ and $R_a(x) = \alpha(x)a$ for a fixed element $a \in R$ and for all $x \in R$, then L_a is a left $\alpha-$centralizer and R_a is a right $\alpha-$centralizer. Albaş showed Zalar's result holds for $\alpha-$centralizer. Considerable work has been done on this topic during the last couple of decades (see [1-8], where further references can be found).

On the other hand, it was proved that T is a centralizer if one of the following holds

$$2T(x^2) = T(x)x + xT(x),$$
$$2T(xyx) = T(x)yx + xyT(x), \text{ for all } x, y \in R,$$

where $T : R \to R$ is an additive mapping respectively in [5] and [7]. These results proved for $\alpha-$centralizer in [4] and [3].

Inspired by the definition centralizer, the notion of $^*-$centralizer was extended as follow:

Let R be a ring with involution $*$. An additive mapping $T : R \to R$ is called a left (resp. right) $^*-$centralizer of R if $T(xy) = T(x)y^*$ (resp. $T(xy) = x^*T(y)$) holds for all $x, y \in R$. An additive mapping $T : R \to R$ is said to be a left (resp. right) Jordan $^*-$centralizer if $T(x^2) = T(x)x^*$ (resp. $T(x^2) = x^*T(x)$) holds for all $x \in R$. In [2], the authors proved that if R is a $2-$torsion free semiprime ring and $T : R \to R$ is an additive mapping such that $2T(x^2) = T(x)\alpha(x^*) + \alpha(x^*)T(x)$, for all $x \in R$, then T is a Jordan $\alpha-^*$centralizer. Motivated this result, we will prove that an additive mapping $T : R \to R$ is a Jordan $\alpha-^*$centralizer if the following holds:

$$2T(xyx) = T(x)\alpha(y^*)\alpha(x^*) + \alpha(x^*)\alpha(y^*)T(x), \text{ for all } x, y \in R$$

This enables us a unified treatment (and extensions) of several results that can be found in the literature.

2 Results

Lemma 2.1. *[6, Lemma 1] Let R be a $2-$torsion free semiprime ring. Suppose that the identity $axb+bxc=0$ holds for all $x\in R$ and some $a,b,c\in R$. Then in this case $(a+c)xb=0$ satisfied for all $x\in R$.*

Lemma 2.2. *[2, Theorem 2.1] Let R be a a $2-$torsion free semiprime $^*-$ring. Suppose that α is an automorphism of R. If $T:R\to R$ is an additive mapping satisfying $2T(x^2)=T(x)\alpha(x^*)+\alpha(x^*)T(x)$ for all $x\in R$, then T is a Jordan $\alpha-^*$centralizer.*

Theorem 2.3. *Let R be a a $2-$torsion free semiprime $^*-$ring. Suppose that α is an automorphism of R. If $T:R\to R$ is an additive mapping satisfying $2T(xyx)=T(x)\alpha(y^*)\alpha(x^*)+\alpha(x^*)\alpha(y^*)T(x)$ for all $x,y\in R$, then T is a Jordan $\alpha-^*$centralizer.*

Proof. By the hyphotesis, we have

$$2T(xyx)=T(x)\alpha(y^*)\alpha(x^*)+\alpha(x^*)\alpha(y^*)T(x),\ \text{for all } x,y\in R. \tag{2.1}$$

Replacing x by $x+z$ in (2.1) and using this, we obtain that

$$\begin{aligned}2T(xyz+zyx)&=T(x)\alpha(y^*)\alpha(z^*)+T(z)\alpha(y^*)\alpha(x^*)\\&\quad+\alpha(x^*)\alpha(y^*)T(z)+\alpha(z^*)\alpha(y^*)T(x),\ \text{for all } x,y,z\in R.\end{aligned} \tag{2.2}$$

Taking x^2 instead of z, we arrive at

$$\begin{aligned}2T(xyx^2+x^2yx)&=T(x)\alpha(y^*)\alpha(x^*)\alpha(x^*)+T(x^2)\alpha(y^*)\alpha(x^*)\\&\quad+\alpha(x^*)\alpha(y^*)T(x^2)+\alpha(x^*)\alpha(x^*)\alpha(y^*)T(x),\quad \text{for all } x,y\in R.\end{aligned} \tag{2.3}$$

Substituting $xy+yx$ for x in (2.1), we have

$$\begin{aligned}2T(xyx^2+x^2yx)&=T(x)\alpha(y^*)\alpha(x^*)\alpha(x^*)+T(x)\alpha(x^*)\alpha(y^*)\alpha(x^*)\\&\quad+\alpha(x^*)\alpha(y^*)\alpha(x^*)T(x)+\alpha(x^*)\alpha(x^*)\alpha(y^*)T(x),\quad \text{for all } x,y\in R.\end{aligned} \tag{2.4}$$

By comparing (2.3) and (2.4), we get

$$T(x^2)\alpha(y^*)\alpha(x^*)+\alpha(x^*)\alpha(y^*)T(x^2)=T(x)\alpha(x^*)\alpha(y^*)\alpha(x^*)+\alpha(x^*)\alpha(y^*)\alpha(x^*)T(x),$$

and so

$$\begin{aligned}&(T(x^2)-T(x)\alpha(x^*))\alpha(y^*)\alpha(x^*)\\&+\alpha(x^*)\alpha(y^*)(T(x^2)-\alpha(x^*)T(x))=0,\ \text{for all } x,y\in R.\end{aligned} \tag{2.5}$$

Putting y^* for y in (2.4) yields that

$$(T(x^2)-T(x)\alpha(x^*))\alpha(y)\alpha(x^*)+\alpha(x^*)\alpha(y)(T(x^2)-\alpha(x^*)T(x))=0,\ \text{for all } x,y\in R.$$

Since α is an epimorphism of R, we have

$$(T(x^2)-T(x)\alpha(x^*))y\alpha(x^*)+\alpha(x^*)y(T(x^2)-\alpha(x^*)T(x))=0,\ \text{for all } x,y\in R.$$

By Lemma 2.1, we conclude that

$$(2T(x^2)-T(x)\alpha(x^*)-\alpha(x^*)T(x))y\alpha(x^*)=0,\ \text{for all } x,y\in R. \tag{2.6}$$

Define $A(x)=2T(x^2)-T(x)\alpha(x^*)-\alpha(x^*)T(x)$. Hence (2.6) can be rewritten as

$$A(x)y\alpha(x^*) = 0, \text{ for all } x, y \in R. \tag{2.7}$$

Substituting $\alpha(x^*)yA(x)$ for y in (2.7), we get

$$A(x)\alpha(x^*)yA(x)\alpha(x^*) = 0, \text{ for all } x, y \in R.$$

Using R is semiprime ring, we arrive at

$$A(x)\alpha(x^*) = 0, \text{ for all } x \in R. \tag{2.8}$$

Left multiplying (2.7) by $\alpha(x^*)$ and right multiplying (2.8) by $A(x)$, we have

$$\alpha(x^*)A(x)y\alpha(x^*)A(x) = 0, \text{ for all } x \in R.$$

By the semiprimeness of R, we find that

$$\alpha(x^*)A(x) = 0, \text{ for all } x \in R. \tag{2.9}$$

Replacing x by $x+y$ in (2.8) and using this, we obtain that

$$A(x+y)\alpha(x^*+y^*) = 0,$$

and so

$$(A(x)+A(y)+2T(xy+yx)-T(x)\alpha(y^*)-T(y)\alpha(x^*)-\alpha(x^*)T(y)-\alpha(y^*)T(x))\alpha(x^*+y^*) = 0.$$

That is

$$\begin{aligned} A(x)\alpha(x^*) + A(y)\alpha(x^*) + B(x,y)\alpha(x^*) + A(x)\alpha(y^*) \\ + A(y)\alpha(y^*) + B(x,y)\alpha(y^*) = 0, \text{ for all } x, y \in R, \end{aligned}$$

where $B(x,y) = 2T(xy+yx) - T(x)\alpha(y^*) - T(y)\alpha(x^*) - \alpha(x^*)T(y) - \alpha(y^*)T(x)$. Using (2.8) in the last equation, we see that

$$A(x)\alpha(y^*) + A(y)\alpha(x^*) + B(x,y)\alpha(x^*) + B(x,y)\alpha(y^*) = 0, \text{ for all } x, y \in R. \tag{2.10}$$

Substituting $-x$ for x in (2.10) and using $A(-x) = A(x), B(-x,y) = -B(x,y)$, we arrive at

$$A(x)\alpha(y^*) - A(y)\alpha(x^*) + B(x,y)\alpha(x^*) - B(x,y)\alpha(y^*) = 0, \text{ for all } x, y \in R. \tag{2.11}$$

Now, combining (2.10) and (2.11), we get

$$A(x)\alpha(y^*) + B(x,y)\alpha(x^*) = 0, \text{ for all } x, y \in R. \tag{2.12}$$

Multiplying (2.12) from the right by $A(x)$ and using (2.9), we obtain that

$$A(x)\alpha(y^*)A(x) = 0, \text{ for all } x, y \in R.$$

Writing y^* instead of y and α is an epimorphism of R, we have

$$A(x)yA(x) = 0, \text{ for all } x, y \in R.$$

By the semiprimeness of R, we find that $A(x) = 0$ for all $x \in R$, and so

$$2T(x^2) = T(x)\alpha(x^*) + \alpha(x^*)T(x), \text{ for all } x, y \in R.$$

Hence T is a Jordan $\alpha-^*$centralizer by Lemma 2.2. □

References

[1] Albaş, E.: On $\tau-$centralizers of semiprime rings, Siberian Math. J. 48 (2), (2007), 191-196.

[2] Ashraf, M., Mozumder, M. R.: On Jordan $\alpha-^{*}$centralizers in semiprime rings with involution, Int. J. Contemp. Math. Sciences, Vol. 7, no.23, (2012), 1103-1112.

[3] Huang, S., Haetinger, C.: On $\theta-$centralizers of semiprime rings, Demonstratio Mathematica, Vol. XLV, No.1, (2012), 29-34.

[4] Shakir, A., Haetinger, C.: Jordan $\alpha-$centralizers in rings and some applications, Bol. Soc. Paran. Mat. Vol. 26, 1-2, (2008), 71-80.

[5] Vukman, J.: An identity related to centralizers in semiprime rings, Comment. Math. Univ. Carolin., 40 (3), (1999), 447-456.

[6] Vukman, J.: Centralizers on semiprime rings, Comment. Math. Univ. Carolin., 42 (2), (2001), 237-245.

[7] Vukman, J., Kosi-Ulbl, I.: On centralizers of semiprime rings, , Aequationes Math. 66 (3), (2003), 277-283.

[8] Zalar, B., On centralizers of semiprime rings, Comment. Math. Univ. Carolin., 1991, 32(4), 609-614.

Author information

Öznur Gölbaşı and Ahu Kadriye Gürcan, Cumhuriyet University, Faculty of Science, Department of Mathematics, Sivas, Turkey.
E-mail: ogolbasi@cumhuriyet.edu.tr

Received: March 22, 2014.

Accepted: June 8, 2014.