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The stability of approximately mappings in the intuitionistic fuzzy normed algebras

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Abstract. In this paper we prove the stability of approximate linear maps in the intuitionistic fuzzy normed spaces and then we define the intuitionistic fuzzy normed algebras and consider the stability problem for approximately mappings in the intuitionistic fuzzy normed algebras.

1 Introduction

It seems that the stability problem of functional equations had been first raised by Ulam [10]. Moreover the approximately mappings have been studied extensively in several papers. (See for instance [8], [3], [1], and [2]).

Fuzzy notion introduced firstly by Zadeh [11] that has been widely involved in different subjects of mathematics. Later in 1984, Katsaras [4] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space.

There are many situations where the norm of a vector is not possible to find and the concept of intuitionistic fuzzy norm ([7], [9] and [6]) seems to be more suitable in such cases, that is, we can deal with such situations by modelling the inexactness through the intuitionistic fuzzy norm.

Defining the class of approximate solutions of a given functional equation one can ask whether every mapping from this class can be somehow approximated by an exact solution of the considered equation in the intuitionistic fuzzy normed algebras. To answer this question, we use here the definition of intuitionistic fuzzy normed algebras to exhibit some reasonable notions of fuzzy approximately additive and approximately ring homomorphisms in the intuitionistic fuzzy normed algebras and we will prove that under some suitable conditions an approximately ring homomorphism f from an algebra X into the intuitionistic fuzzy normed algebra Y can be approximated in a fuzzy sense by a homomorphism T from X to Y.

2 Preliminaries

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

Definition 2.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,

(N1) $N(x, c) = 0$ for $c \leq 0$;

(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;

(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N4) $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$;

(N5) $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty} N(x, t) = 1$;

(N6) for $x \neq 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space.

Definition 2.2. Let (X, N) be a fuzzy normed linear space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - lim_{n\to\infty} x_n = x$.

Definition 2.3. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a fuzzy Banach space.

Definition 2.4. Let X be an algebra and (X, N) be complete fuzzy normed space. The pair (X, N) is said to be a fuzzy Banach algebra if for every $x, y \in X$ and $s, t \in \mathbb{R}$: $N(xy, st) \geq min\{N(x, s), N(y, t)\}$.

Theorem 2.5. *Let X be a linear space and let (Z, N') be a fuzzy normed space. Let $\psi : X \times X \rightarrow Z$ be a function such that for some $0 < \alpha < 2$,*

$$N'(\psi(2x, 2y), t) \geq N'(\alpha\psi(x, y), t)$$

for all $x, y \in X$ and $t > 0$. Let (Y, N) be a fuzzy Banach space and let $f : X \rightarrow Y$ be a mapping in the sense that

$$N(f(x + y) - f(x) - f(y), t) \geq N'(\psi(x, y), t)$$

for each $t > 0$ and $x, y \in X$. Then there exists unique additive mapping $T : X \rightarrow Y$ such that

$$N(f(x) - T(x), t) \geq N'(\tfrac{2\psi(x,x)}{2-\alpha}, t),$$

where $x \in X$ and $t > 0$.

Proof. [5] □

Definition 2.6. A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t-norm if it satisfies the following conditions:
(∗1) $*$ is associative,
(∗2) $*$ is continuous,
(∗3) $a * 1 = a$ for all $a \in [0, 1]$ and
(∗4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 2.7. A binary operation $\diamond : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t-conorm if it satisfies the following conditions:
(⋄1) $\diamond$ is associative and commutative,
(⋄2) $\diamond$ is continuous,
(⋄3) $a \diamond 0 = a$ for all $a \in [0, 1]$ and
(⋄4) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 2.8. The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ such that for all $x, y \in X$ and $s, t > 0$ satisfying the following conditions:
(1) $\mu(x, t) + \nu(x, t) \leq 1$,
(2) $\mu(x, t) > 0$,
(3) $\mu(x, t) = 1$ if and only if $x = 0$,
(4) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(5) $\mu(x, t) * \mu(y, s) \geq \mu(x + y, t + s)$,
(6) $\mu(x, .) : (0, \infty) \rightarrow [0, 1]$ is continuous,
(7) $lim_{t\rightarrow\infty}\mu(x, t) = 1$ and $lim_{t\rightarrow 0}\mu(x, t) = 0$,
(8) $\nu(x, t) < 1$,
(9) $\nu(x, t) = 0$ if and only if $x = 0$,
(10) $\nu(\alpha x, t) = \nu((x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(11) $\nu(x, t) \diamond \nu(y, s) \leq \nu(x + y, t + s)$,
(12) $\nu(x, .) : (0, \infty) \rightarrow [0, 1]$ is continuous,
(13) $lim_{t\rightarrow\infty}\nu(x, t) = 0$ and $lim_{t\rightarrow 0}\nu(x, t) = 1$,
In this case (μ, ν) is called an intuitionistic fuzzy norm.

Definition 2.9. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $\{x_n\}$ is said to be intuitionistic fuzzy convergent to $L \in X$ if $lim_{n\rightarrow\infty}\mu(x_n - L, t) = 1$ and $lim_{n\rightarrow\infty}\nu(x_n - L, t) = 0$ for all $t > 0$.

Definition 2.10. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $\{x_n\}$ is said to be intuitionistic fuzzy Cauchy sequence if $lim_{n\rightarrow\infty}\mu(x_{n+p} - x_n, t) = 1$ and $lim_{n\rightarrow\infty}\nu(x_{n+p} - x_n, t) = 0$ for all $t > 0$ and $p = 1, 2, ...$

Definition 2.11. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then $(X, \mu, \nu, *, \diamond)$ is said to be Banach space if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamond)$.

3 Intuitionistic fuzzy stability of approximately homomorphisms

Troughs this paper, we show the $a_1 * a_2 * ... * a_n$ by $\prod_{j=1}^{n} a_j$ and $a_1 \diamond a_2 \diamond ... \diamond a_n$ by $\coprod_{j=1}^{n} a_j$.

Theorem 3.1. *Let X be a linear space and f be a mapping from X to an intuitionistic fuzzy Banach space (Y, μ, ν) such that $f(0) = 0$. Suppose that φ is a function from X to an intuitionistic fuzzy normed space (Z, μ', ν') such that*

$$\mu(f(x+y) - f(x) - f(y), t+s) \geq \mu'(\varphi(x), t) * \mu'(\varphi(y), s) \quad \textit{and} \tag{3.1}$$
$$\nu(f(x+y) - f(x) - f(y), t+s) \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(y), s)$$

for all $x, y \in X - \{0\}$ and $t, s > 0$. If $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $0 < \alpha < 2$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\mu(T(x) - f(x), t) \geq M(x, \tfrac{(2-\alpha)t}{4}) \textit{ and}$$
$$\nu(T(x) - f(x), t) \leq N(x, \tfrac{(2-\alpha)t}{4})$$

where

$$M(x,t) = \mu'(\varphi(x), t) * \mu'(\varphi(x), t) \textit{ and}$$
$$N(x,t) = \nu'(\varphi(x), t) \diamond \nu'(\varphi(x), t).$$

Proof. Putting $y = x$ and $s = t$ in (3.1) we obtain

$$\mu(f(2x) - 2f(x), 2t) \geq \mu'(\varphi(x), t) * \mu'(\varphi(x), t) \text{ and}$$
$$\nu(f(2x) - 2f(x), 2t) \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(x), t)$$

for all $x \in X$ and $t > 0$. Thus

$$\mu(f(x) - 2^{-1}f(2x), t) \geq \mu'(\varphi(x), t) * \mu'(\varphi(x), t) \quad \textit{and} \tag{3.2}$$
$$\nu(f(x) - 2^{-1}f(2x), t) \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(x), t)$$

for all $x \in X$ and $t > 0$.

Using $\varphi(2x) = \alpha\varphi(x)$ and induction on n, we can verify that $\varphi(2^n x) = \alpha^n \varphi(x)$ for all $x \in X$. Define

$$M(x,t) = \mu'(\varphi(x), t) * \mu'(\varphi(x), t) \text{ and}$$
$$N(x,t) = \nu'(\varphi(x), t) \diamond \nu'(\varphi(x), t).$$

Replacing x by $2^n x$ in (3.2), we get

$$\mu(\tfrac{f(2^n x)}{2^n} - \tfrac{f(2^{n+1}x)}{2^{n+1}}, \tfrac{\alpha^n t}{2^n}) \geq M(x,t) \text{ and}$$
$$\nu(\tfrac{f(2^n x)}{2^n} - \tfrac{f(2^{n+1}x)}{2^{n+1}}, \tfrac{\alpha^n t}{2^n}) \leq N(x,t).$$

Thus for each $n > m$, we have

$$\mu(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{2^k}) = \mu(\sum_{k=m}^{n-1} \frac{f(2^k x)}{2^k} - \frac{f(2^{k+1}x)}{2^{k+1}}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{2^k}) \tag{3.3}$$
$$\geq \prod_{k=m}^{n-1} \mu(\tfrac{f(2^k x)}{2^k} - \tfrac{f(2^{k+1}x)}{2^{k+1}}, \tfrac{\alpha^k t}{2^k}) \geq M(x,t) \text{ and}$$

$$\nu(\tfrac{f(2^m x)}{2^m} - \tfrac{f(2^n x)}{2^n}, \textstyle\sum_{k=m}^{n-1} \tfrac{\alpha^k t}{2^k}) = \nu(\textstyle\sum_{k=m}^{n-1} \tfrac{f(2^k x)}{2^k} - \tfrac{f(2^{k+1}x)}{2^{k+1}}, \textstyle\sum_{k=m}^{n-1} \tfrac{\alpha^k t}{2^k}) \leq$$
$$\coprod_{k=m}^{n-1} \nu(\tfrac{f(2^k x)}{2^k} - \tfrac{f(2^{k+1}x)}{2^{k+1}}, \tfrac{\alpha^k t}{2^k}) \leq N(x,t).$$

Let $\varepsilon > 0$ and $\delta > 0$ be given. Thanks to the fact that $lim_{t\to\infty} M(x,t) = 1$ and $lim_{t\to\infty} N(x,t) = 0$ there exists some $t_0 > 0$ such that $M(x, t_0) > 1 - \varepsilon$ and $N(x, t_0) < \varepsilon$. Since $\sum_{k=0}^{\infty} \frac{\alpha^k t_0}{2^k}$ is convergent, there exists some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1} \frac{\alpha^k t_0}{2^k} < \delta$ for all $n > m \geq n_0$. It follows that

$$\mu(\tfrac{f(2^m x)}{2^m} - \tfrac{f(2^n x)}{2^n}, \delta) \geq \mu(\tfrac{f(2^m x)}{2^m} - \tfrac{f(2^n x)}{2^n}, \textstyle\sum_{k=m}^{n-1} \tfrac{\alpha^k t_0}{2^k}) \geq M(x, t_0) \geq 1 - \varepsilon \text{ and}$$
$$\nu(\tfrac{f(2^m x)}{2^m} - \tfrac{f(2^n x)}{2^n}, \delta) \leq \nu(\tfrac{f(2^m x)}{2^m} - \tfrac{f(2^n x)}{2^n}, \textstyle\sum_{k=m}^{n-1} \tfrac{\alpha^k t_0}{2^k}) \leq N(x, t_0) \leq \varepsilon.$$

This shows that $\{\frac{f(2^n x)}{2^n}\}$ is Cauchy sequence in (Y, μ, ν). Since (Y, μ, ν) is complete, $\{\frac{f(2^n x)}{2^n}\}$ converges to some point $T(x) \in Y$. Hence, we define $T : X \to Y$ such that $T(x) = (\mu, \nu) - lim_{n\to\infty} \frac{f(2^n x)}{2^n}$. Moreover, if we put $m = 0$ in (3.3), we get

$$\mu(\frac{f(2^n x)}{2^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{2^k}) \geq M(x, t) \quad and \tag{3.4}$$

$$\nu(\frac{f(2^n x)}{2^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{2^k}) \leq N(x, t).$$

Now, we will show that T is additive. Let $x, y \in X$. Then

$$\mu(T(x+y)-T(x)-T(y), t) \geq \mu(T(x+y)-\frac{f(2^n(x+y))}{2^n}, t/4)*\mu(\frac{f(2^n x)}{2^n}-T(x), t/4)* \tag{3.5}$$

$$\mu(\frac{f(2^n y)}{2^n} - T(y), t/4) * \mu(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}, t/4).$$

On the other hand by using (3.1) we have

$$\mu(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}, t/4) \geq \mu'(\varphi(2^n x), \frac{2^n t}{8}) * \mu'(\varphi(2^n y), \frac{2^n t}{8}) = \tag{3.6}$$

$$\mu'(\varphi(x), (\tfrac{2}{\alpha})^n t/8) * \mu'(\varphi(y), (\tfrac{2}{\alpha})^n t/8).$$

Letting n to infinity in (3.5) and by (3.6), we get

$$\mu(T(x+y) - T(x) - T(y), t) = 1$$
and
$$\nu(T(x+y) - T(x) - T(y), t) = 0$$

for all $x, y \in X$ and $t > 0$. This means that T is additive. Now, by (3.4) we have

$$\mu(T(x) - f(x), t) \geq \mu(T(x) - \tfrac{f(2^n x)}{2^n}, t/2) * \mu(\tfrac{f(2^n x)}{2^n} - f(x), t/2) \geq M(x, \tfrac{t}{2\sum_{k=0}^{\infty}(\alpha/2)^k}) = M(x, \tfrac{(2-\alpha)t}{4})$$
and
$$\nu(T(x) - f(x), t) \leq \nu(T(x) - \tfrac{f(2^n x)}{2^n}, t/2) \diamond \nu(\tfrac{f(2^n x)}{2^n} - f(x), t/2) \leq N(x, \tfrac{t}{2\sum_{k=0}^{\infty}(\alpha/2)^k}) = N(x, \tfrac{(2-\alpha)t}{4}),$$

for every $x \in X$ and $t > 0$. To prove the uniqueness property, assume that T' be another additive mapping from X into Y, which satisfies the required properties. Then

$$\mu(T(x) - T'(x), t) \geq \mu(T(x) - f(x), t/2) * \mu(T'(x) - f(x), t/2) \geq M(x, \tfrac{(2-\alpha)t}{8}) \text{ and}$$
$$\nu(T(x) - T'(x), t) \leq \nu(T(x) - f(x), t/2) \diamond \nu(T'(x) - f(x), t/2) \leq N(x, \tfrac{(2-\alpha)t}{8})$$

for all $x \in X, t > 0$ and $n \in \mathbb{N}$. Therefore $\mu(T(x) - T'(x), t) = 1$ and $\nu(T(x) - T'(x), t) = 0$ for all $x \in X$ and $t > 0$. Hence $T(x) = T'(x)$ for all $x \in X$.

□

Definition 3.2. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. X is said to be an intuitionistic fuzzy normed algebra if it is satisfying the conditions $\mu(x, t) * \mu(y, s) \geq \mu(xy, ts)$ and $\nu(x, t) \diamond \nu(y, s) \leq \nu(xy, ts)$.

Theorem 3.3. *Let X be a linear space and f be a mapping from X to an intuitionistic fuzzy Banach algebra (Y, μ, ν) such that $f(0) = 0$. Suppose that φ is a function from X to an intuitionistic fuzzy normed algebra (Z, μ', ν') such that*

$$\mu(f(x+y) - f(x) - f(y), t+s) \geq \mu'(\varphi(x), t) * \mu'(\varphi(y), s), \tag{3.7}$$

$$\nu(f(x+y) - f(x) - f(y), t+s) \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(y), s) \text{ and}$$

$$\mu(f(xy) - f(x)f(y), ts) \geq \mu'(\varphi(x), t) * \mu'(\varphi(y), s),$$

$$\nu(f(xy) - f(x)f(y), ts) \leq \nu'(\varphi(x), t) \diamond \mu'(\varphi(y), s)$$

for all $x, y \in X - \{0\}$ and $t, s > 0$. If $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $0 < \alpha < 2$, then there exists a unique ring homomorphism $T : X \to Y$ such that

$$\mu(T(x) - f(x), t) \geq M(x, \frac{(2-\alpha)t}{4}) \quad and \tag{3.8}$$

$$\nu(T(x)-f(x),t)\leq N(x,\tfrac{(2-\alpha)t}{4})$$

where

$$M(x,t)=\mu'(\varphi(x),t)*\mu'(\varphi(x),t) \text{ and}$$
$$N(x,t)=\nu'(\varphi(x),t)\diamond\nu'(\varphi(x),t).$$

Proof. Theorem 3.1 shows that there exists an additive function $T:X\to Y$ such that

$$\mu(f(x)-T(x),t)\geq M(x,\tfrac{(2-\alpha)t}{4}) \text{ and}$$
$$\nu(f(x)-T(x),t)\leq N(x,\tfrac{(2-\alpha)t}{4})$$

where $x\in X$ and $t>0$. Now we only need to show that T is a multiplicative function. Our inequality follows that

$$\mu(f(nx)-T(nx),t)\geq M(nx,\tfrac{(2-\alpha)t}{4}) \text{ and}$$
$$\nu(f(nx)-T(nx),t)\leq N(nx,\tfrac{(2-\alpha)t}{4})$$

for all $x\in X$ and all $t>0$. Thus

$$\mu(n^{-1}f(nx)-n^{-1}T(nx),n^{-1}t)\geq M(nx,\tfrac{(2-\alpha)t}{4}) \text{ and}$$
$$\nu(n^{-1}f(nx)-n^{-1}T(nx),n^{-1}t)\leq N(nx,\tfrac{(2-\alpha)t}{4})$$

for all $x\in X$ and all $t>0$. By the additivity of T it is easy to see that

$$\mu(n^{-1}f(nx)-T(x),t)\geq M(nx,\frac{(2-\alpha)nt}{4})and \tag{3.9}$$

$$\nu(n^{-1}f(nx)-T(x),t)\leq N(nx,\tfrac{(2-\alpha)nt}{4})$$

for all $x\in X$ and all $t>0$. By taking n tend to infinity in (3.9) we see that

$$T(x)=(\mu,\nu)-lim_{n\to\infty}n^{-1}f(nx) \tag{3.10}$$

for all $x\in X$. Using inequality (3.7), we get

$$\mu(f((nx)y)-f(nx)f(y),ts)\geq\mu'(\varphi(nx),t)*\mu'(\varphi(y),s) \text{ and}$$
$$\nu(f((nx)y)-f(nx)f(y),ts)\leq\nu'(\varphi(nx),t)\diamond\nu'(\varphi(y),s)$$

for all $x,y\in X$ and all $t,s>0$. Thus

$$\mu(n^{-1}f((nx)y)-n^{-1}f(nx)f(y),ts)\geq\mu'(\varphi(nx),nt)*\mu'(\varphi(y),ns)\quad and \tag{3.11}$$

$$\nu(n^{-1}f((nx)y)-n^{-1}f(nx)f(y),ts)\leq\nu'(\varphi(nx),nt)\diamond\nu'(\varphi(y),ns)$$

for all $x,y\in X$ and all $t,s>0$. By taking n tend to infinity in (3.11) we see that

$$(\mu,\nu)-lim_{n\to\infty}n^{-1}f((nx)y)=(\mu,\nu)lim_{n\to\infty}n^{-1}f(nx)f(y). \tag{3.12}$$

Applying (3.10) and (3.12) we have

$$T(xy)=(\mu,\nu)-lim_{n\to\infty}n^{-1}f(n(xy))=(\mu,\nu)lim_{n\to\infty}n^{-1}f((nx)y)= \tag{3.13}$$

$$(\mu,\nu)-lim_{n\to\infty}n^{-1}f(nx)f(y)=T(x)f(y)$$

for all $x,y\in X$. From this equation by the additivity of T we have

$$T(x)f(ny)=T(x(ny))=T((nx)y)=T(nx)f(y)=nT(x)f(y)$$

for all $x,y\in X$. Therefore,

$$T(x)n^{-1}f(ny)=T(x)f(y)$$

for all $x,y\in X$. Again by taking n tend to infinity and by (3.10) we see that

$$T(x)T(y)=T(x)f(y)$$

for all $x,y\in X$. Combining this formula with equation (3.13) we have that T is a ring homomorphism.

To prove the uniqueness property of T, assume that T' is another ring homomorphism satisfying (3.8). Since both T and T' are additive, we deduce that

$$\mu(T(x)-T'(x),t)=\mu(T(nx)-T'(nx),nt)\geq$$
$$\mu(T(nx)-f(nx),nt/2)*\mu(f(nx)-T'(nx),nt/2)\geq M(nx,\tfrac{(2-\alpha)nt}{8})*M(nx,\tfrac{(2-\alpha)nt}{8}) \text{ and}$$
$$\nu(T(x)-T'(x),t)=\nu(T(nx)-T'(nx),nt)\leq$$
$$\nu(T(nx)-f(nx),nt/2)\diamond\nu(f(nx)-T'(nx),nt/2)\leq N(nx,\tfrac{(2-\alpha)nt}{8})\diamond N(nx,\tfrac{(2-\alpha)nt}{8})$$

for all $x\in X$ and all $t>0$. Taking n tend to infinity we find that

$$\mu(T(x)-T'(x),t)=1 \text{ and}$$
$$\nu(T(x)-T'(x),t)=0$$

for all $x\in X$ and all $t>0$. Hence $T(x)=T'(x)$ for all $x\in X$.

□

Theorem 3.4. *Let X be a linear space and f be a mapping from X to an intuitionistic fuzzy Banach algebra (Y,μ,ν) such that $f(0)=0$. Suppose that φ is a function from X to an intuitionistic fuzzy normed algebra (Z,μ',ν') such that*

$$\mu(f(x+y)-f(x)-f(y),t+s)\geq\mu'(\varphi(x),t)*\mu'(\varphi(y),s),$$

$$\nu(f(x+y)-f(x)-f(y),t+s)\leq\nu'(\varphi(x),t)\diamond\nu'(\varphi(y),s) \text{ and}$$

$$\mu(f(x^2)-f(x)^2,t^2)\geq\mu'(\varphi(x),t)*\mu'(\varphi(x),t),$$

$$\nu(f(x^2)-f(x)^2,t^2)\leq\nu'(\varphi(x),t)\diamond\mu'(\varphi(x),t)$$

for all $x\in X-\{0\}$ and $t,s>0$. If $\varphi(2x)=\alpha\varphi(x)$ for some real number α with $0<\alpha<2$, then there exists a unique additive Jordan map $T:X\to Y$ such that

$$\mu(T(x)-f(x),t)\geq M(x,\tfrac{(2-\alpha)t}{4}) \text{ and}$$

$$\nu(T(x)-f(x),t)\leq N(x,\tfrac{(2-\alpha)t}{4})$$

where

$$M(x,t)=\mu'(\varphi(x),t)*\mu'(\varphi(x),t) \text{ and}$$
$$N(x,t)=\nu'(\varphi(x),t)\diamond\nu'(\varphi(x),t).$$

Proof. Theorem 3.1 shows that there exists an additive function $T:X\to Y$ such that

$$\mu(f(x)-T(x),t)\geq M(x,\tfrac{(2-\alpha)t}{4}) \text{ and}$$
$$\nu(f(x)-T(x),t)\leq N(x,\tfrac{(2-\alpha)t}{4})$$

where $x\in X$ and $t>0$. Now we only need to show that T is a Jordan map. If $a=0$; since $T(0)=0$, it is obvious. In the other case,

$$\mu(n^{-2}f(n^2a^2)-n^{-2}T(n^2a^2),n^{-2}t)\geq M(n^2a^2,\tfrac{(2-\alpha)t}{4}) \text{ and}$$
$$\nu(n^{-2}f(n^2a^2)-n^{-2}T(n^2a^2),n^{-2}t)\leq N(n^2a^2,\tfrac{(2-\alpha)t}{4}),$$

for all $a\in X$, $t>0$ and $n\in\mathbb{N}$. By the additivity of T it is easy to see that

$$\mu(n^{-2}f(n^2a^2)-T(a^2),t)\geq M(n^2a^2,\frac{(2-\alpha)tn^2}{4}) \quad \text{and} \tag{3.14}$$

$$\nu(n^{-2}f(n^2a^2)-T(a^2),t)\leq N(n^2a^2,\tfrac{(2-\alpha)tn^2}{4})$$

for all $a\in X$, $t>0$ and $n\in\mathbb{N}$. Letting n tend to infinity in (3.14) we see that

$$T(a^2)=(\mu,\nu)-lim_{n\to\infty}n^{-2}f(n^2a^2). \tag{3.15}$$

Also a similar argument represented above, shows that:

$$\mu(n^{-1}f(na)-T(a),t)\geq M(na,\tfrac{(2-\alpha)tn}{4}) \text{ and}$$
$$\nu(n^{-1}f(na)-T(a),t)\leq N(na,\tfrac{(2-\alpha)tn}{4})$$

for all $a\in X$, $t>0$ and $n\in\mathbb{N}$. Hence we have

$$T(a)=(\mu,\nu)-lim_{n\to\infty}n^{-1}f(na). \tag{3.16}$$

On the other hand

$$\mu(f(n^2a^2) - f(na)^2, t^2) \geq \mu'(\varphi(na), t) * \mu'(\varphi(na), t) \text{ and}$$
$$\nu(f(n^2a^2) - f(na)^2, t^2) \leq \nu'(\varphi(na), t) \diamond \mu'(\varphi(na), t)$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. We observe that

$$\mu(n^{-2}f(n^2a^2) - n^{-2}f(na)^2, t^2) \geq \mu'(\varphi(na), n^2t) * \mu'(\varphi(na), n^2t) \text{ and}$$
$$\nu(n^{-2}f(n^2a^2) - n^{-2}f(na)^2, t^2) \leq \nu'(\varphi(na), n^2t) \diamond \mu'(\varphi(na), n^2t)$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. So again by taking n tend to infinity we have

$$(\mu, \nu) - lim_{n\to\infty} n^{-2}f(n^2a^2) = (\mu, \nu) - lim_{n\to\infty} n^{-2}f(na)^2. \tag{3.17}$$

Applying (3.15), (3.16) and (3.17) we have

$$T(a^2) = (\mu, \nu) - lim_{n\to\infty} n^{-2}f(n^2a^2) = (\mu, \nu) - lim_{n\to\infty} n^{-2}f(na)^2 =$$
$$((\mu, \nu) - lim_{n\to\infty} n^{-1}f(na))^2 = T(a)^2.$$

To prove the uniqueness property of T, assume that T' is another additive Jordan mapping satisfying

$$\mu(f(x) - T(x), t) \geq M(x, \tfrac{(2-\alpha)t}{4}) \text{ and}$$
$$\nu(f(x) - T(x), t) \leq N(x, \tfrac{(2-\alpha)t}{4}).$$

Since both T and T' are additive, we deduce that

$$\mu(T(a) - T'(a), t) \geq \mu(T(a) - n^{-1}f(na), t/2) * \mu(n^{-1}f(na) - T'(a), t/2) \text{ and}$$
$$\nu(T(a) - T'(a), t) \leq \nu(T(a) - n^{-1}f(na), t/2) \diamond \nu(n^{-1}f(na) - T'(a), t/2)$$

for all $a \in X$ and all $t > 0$. Letting n tend to infinity we find that $T(a) = T'(a)$ for all $a \in X$. □

Now we consider the stability of ring derivations in the intuitionistic fuzzy normed algebras.

Theorem 3.5. *Let X be a linear space and f be a mapping from X to an intuitionistic fuzzy Banach algebra (Y, μ, ν) such that $f(0) = 0$. Suppose that φ is a function from X to an intuitionistic fuzzy normed algebra (Z, μ', ν') such that*

$$\mu(f(x + y) - f(x) - f(y), t + s) \geq \mu'(\varphi(x), t) * \mu'(\varphi(y), s), \tag{3.18}$$
$$\nu(f(x + y) - f(x) - f(y), t + s) \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(y), s) \text{ and}$$
$$\mu(f(xy) - xf(y) - f(x)y, ts) \geq \mu'(\varphi(x), t) * \mu'(\varphi(y), s),$$
$$\nu(f(xy) - xf(y) - f(x)y, ts) \leq \nu'(\varphi(x), t) \diamond \mu'(\varphi(y), s)$$

for all $x, y \in X - \{0\}$ and $t, s > 0$. If $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $0 < \alpha < 2$, then there exists a unique ring derivation $T : X \to Y$ such that

$$\mu(T(x) - f(x), t) \geq M(x, \tfrac{(2-\alpha)t}{4}) \text{ and}$$
$$\nu(T(x) - f(x), t) \leq N(x, \tfrac{(2-\alpha)t}{4})$$

where

$$M(x, t) = \mu'(\varphi(x), t) * \mu'(\varphi(x), t) \text{ and}$$
$$N(x, t) = \nu'(\varphi(x), t) \diamond \nu'(\varphi(x), t).$$

Proof. Theorem 3.1 shows that there exists an additive function $T : X \to Y$ such that

$$\mu(f(x) - T(x), t) \geq M(x, \tfrac{(2-\alpha)t}{4}) \text{ and}$$
$$\nu(f(x) - T(x), t) \leq N(x, \tfrac{(2-\alpha)t}{4})$$

where $x \in X$ and $t > 0$. Now we only need to show that T is a ring derivation. Our inequality implies that

$$\mu(f(na) - T(na), t) \geq M(na, \tfrac{(2-\alpha)t}{4}), \text{ and}$$
$$\nu(f(na) - T(na), t) \leq N(na, \tfrac{(2-\alpha)t}{4}),$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. By the additivity of T, it is easy to see that

$$\mu(n^{-1}f(na) - T(a), t) \geq M(na, \frac{(2-\alpha)tn}{4}) \quad and \tag{3.19}$$

$$\nu(n^{-1}f(na) - T(a), t) \leq N(na, \tfrac{(2-\alpha)tn}{4})$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. Letting n tend to infinity in (3.19) we see that

$$T(a) = (\mu, \nu) - lim_{n\to\infty} n^{-1} f(na). \tag{3.20}$$

Using inequality (3.18), we get

$$\mu(f((na)b) - (na)f(b) - f(na)b, ts) \geq \mu'(\varphi(na), t) * \mu'(\varphi(b), s) \text{ and}$$
$$\nu(f((na)b) - (na)f(b) - f(na)b, ts) \leq \nu'(\varphi(na), t) \diamond \nu'(\varphi(b), s),$$

for all $a \in X$, $t, s > 0$ and $n \in \mathbb{N}$. Hence we have

$$(\mu, \nu) - lim_{n\to\infty} n^{-1} f((na)b) - n^{-1} naf(b) - n^{-1} f(na)b = 0. \tag{3.21}$$

Now applying (3.20) in (3.21), we get

$$T(ab) = af(b) + T(a)b \tag{3.22}$$

for all $a, b \in X$. Indeed,

$$T(ab) = (\mu, \nu) - lim_{n\to\infty} n^{-1} f(n(ab)) = (\mu, \nu) - lim_{n\to\infty} n^{-1} f((na)b) =$$
$$(\mu, \nu) - lim_{n\to\infty} n^{-1}[naf(b) + f(na)b + f((na)b) - naf(b) - f(na)b] =$$
$$(\mu, \nu) - lim_{n\to\infty}[af(b) + n^{-1}f(na)b + n^{-1}f((na)b) - n^{-1}naf(b) - n^{-1}f(na)b] = af(b) + T(a)b$$

for all $a, b \in X$. Let $a, b \in X$ and $n \in \mathbb{N}$ be fixed. Then using (3.22) and the additivity of T, we have

$$af(nb) + nT(a)b = af(nb) + T(a)nb = T(a(nb)) = T((na)b) = naf(b) + T(na)b =$$
$$naf(b) + nT(a)b.$$

Therefore,

$$af(b) = an^{-1} f(nb)$$

for all $a, b \in X$. By taking n tend to infinity and (3.20), we see that

$$af(b) = aT(b)$$

for all $a, b \in X$. Combining this formula with equation (3.22) we have

$$T(ab) = aT(b) + T(a)b$$

for all $a, b \in X$.

The proof of the uniqueness property of T is similar with the proof of the uniqueness in Theorem 3.5.

□

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