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A criterion on multiples of generalized repunits

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Abstract. We prove that the sum of n powers of b is divisible by $(b^n - 1)/(b - 1)$ if and only if the n exponents are all distinct modulo n. If $b = 10$, this result is already known, and we present an alternate proof with this generalization.

1 Introduction

The generalized repunit $R_n(b)$ was introduced by Snyder [1] and is given by

$$R_n(b) = \frac{b^n - 1}{b - 1},$$

where $n \geq 1$ and $b \geq 2$, both integers. The name repunit refers to the fact that $R_n(b)$ is represented by a string of n ones when the base-b number system is considered. In particular, when $b = 10$, the number $R_n = R_n(10)$ is called the n-th repunit.

This article is a short note on a criterion involving multiples of $R_n(b)$ which has been given recently for the case $b = 10$ as part of a construction of Smith numbers [2, Theorem 2.3]. We state the result as follows.

Theorem 1.1. Let $m = b^{e_1} + b^{e_2} + \cdots + b^{e_n}$ with non-negative integers $e_1, e_2, \ldots, e_n$, not assumed distinct. Then m is divisible by $R_n(b)$ if and only if the set $\{e_1, e_2, \ldots, e_n\}$ is a complete residue system modulo n.

For example with $n = 2$, we have that $b + 1$ divides $b^{e_1} + b^{e_2}$ if and only if $e_1 + e_2$ is an odd number. This special case is obvious if we observe that

$$b^{e_1} + b^{e_2} \equiv (-1)^{e_1} + (-1)^{e_2} \pmod{b + 1},$$

which is congruent to zero if and only if e_1 and e_2 are of opposite parity.

2 Proof

Let $m = b^{e_1} + b^{e_2} + \cdots + b^{e_n}$. By the definition of $R_n(b)$, the following congruence holds:

$$b^n \equiv 1 \pmod{R_n(b)}, \tag{2.1}$$

which then implies the congruence

$$b^{e_k} \equiv b^{e_k \text{ mod } n} \pmod{R_n(b)}. \tag{2.2}$$

Hence, if $\{e_1, e_2, \ldots, e_n\}$ is a complete residue system modulo n, then

$$m = \sum_{k=1}^{n} b^{e_k} \equiv b^0 + b^1 + b^2 + \cdots + b^{n-1} = R_n(b) \equiv 0 \pmod{R_n(b)}.$$

This establishes the sufficiency in the theorem.

To prove necessity, we will now assume without loss of generality, in view of Congruence (2.2), that $e_k \leq n-1$ for all k in the range $1 \leq k \leq n$. Furthermore, let us agree that by the notation $(a_1, a_2, \ldots, a_n)$ we mean the quantity given by

$$(a_1, a_2, \ldots, a_n) = a_1 + a_2 b + \cdots + a_n b^{n-1}.$$

So by collecting identical terms among the n powers $b^{e_1}, b^{e_2}, \ldots, b^{e_n}$, we will be able to write $m = (a_1, a_2, \ldots, a_n)$ with non-negative integers $a_1, a_2, \ldots, a_n$, where each a_k is determined by the number of exponents among $e_1, e_2, \ldots, e_n$, which are equal to $k-1$. Note that $a_1 + a_2 + \cdots + a_n = n$.

Because of Congruence (2.1), we now have

$$\begin{aligned}(a_1, a_2, \ldots, a_n) &\equiv b(a_2, a_3, \ldots, a_n, a_1)\\ &\equiv b^2(a_3, a_4, \ldots, a_n, a_1, a_2)\\ &\equiv \ldots\\ &\equiv b^{n-1}(a_n, a_1, a_2, \ldots, a_{n-1}) \qquad (\text{mod } R_n(b)).\end{aligned}$$

This chain of congruences, together with the fact that $\gcd(R_n(b), b) = 1$, implies that the number $R_n(b)$ divides $(a_1, a_2, \ldots, a_n)$ if and only if $R_n(b)$ also divides each one of the quantities

$$(a_2, a_3, \ldots, a_n, a_1), (a_3, a_4, \ldots, a_n, a_1, a_2), \ldots, (a_n, a_1, a_2, \ldots, a_{n-1}).$$

However, we observe that since $a_1 + a_2 + \cdots + a_n = n$,

$$(a_1, a_2, \ldots, a_n) + (a_2, a_3, \ldots, a_n, a_1) + \cdots + (a_n, a_1, a_2, \ldots, a_{n-1}) = nR_n(b).$$

And the only way we can have n positive multiples of $R_n(b)$ that add up to $nR_n(b)$ is when each multiple actually equals $R_n(b)$. In particular, we cannot have $a_k \geq 2$ for any of $k = 1, 2, \ldots, n$; otherwise we would have a contradiction:

$$R_n(b) = (a_{(k \text{ mod } n)+1}, \ldots, a_n, a_1, \ldots, a_k) \geq 2b^{n-1} > R_n(b).$$

So we must have $a_1, a_2, \ldots a_n \leq 1$, and to have their sum equals n, we conclude that $a_1 = a_2 = \cdots = a_n = 1$.

Thus we have proved that the number m is a multiple of $R_n(b)$ if and only if $\{e_1, e_2, \ldots, e_n\} = \{0, 1, 2, \ldots n-1\}$. That is, if we omit the assumption that $e_k \leq n-1$, then we have in general that $R_n(b)$ divides m if and only if $\{e_1, e_2, \ldots, e_n\}$ is a complete residue system modulo n.

3 Remarks

(i) Congruence (2.2) gives a divisibility test by $R_n(b)$ for any number m, where we are allowed to replace m by the sum of successive digital strings of length n truncated from m, when written in base b. For example, consider the decimal number 9959585640719, which is supposedly a multiple of $R_4(10)$. We may state that $m = 9, 9595, 8564, 0719$ is divisible by 1111 if and only if the sum

$$9 + 9595 + 8564 + 0719 = 18887$$

is also divisible by 1111. In turn, 18887 is divisible by 1111 if and only if $1 + 8887 = 8888$ is too. At this point it is clear that 8888 is a multiple of 1111, so we conclude that m is in fact divisible by R_4.

(ii) Another fact concerning multiples of $R_n(b)$ which is already known states that if $R_n(b)$ divides a positive number m, then at least n of the base-b digits in m must be non-zero. Theorem 1.1 supplements this result by dealing with numbers m which are composed of n ones as the only non-zero digits. However, the theorem does not generalize to any number m having exactly n non-zero digits. For example, in base 10 the number $m = 3060805$ is a multiple of R_4, since $306 + 0805 = 1111$. Nevertheless, note that m has exactly 4 non-zero digits and that

$$3060805 = 3 \cdot 10^6 + 6 \cdot 10^4 + 8 \cdot 10^2 + 5 \cdot 10^0,$$

where the four exponents 6, 4, 2, 0, do not form a complete residue system modulo 4.

References

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