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Generalized Fibonacci Sequences and Its Properties

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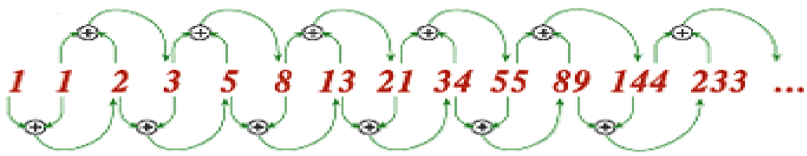
Abstract. In this paper, we present properties of Generalized Fibonacci sequences. Generalized Fibonacci sequence is defined by recurrence relation $F_k = pF_{k-1} + qF_{k-2}$, $k \geq 2$ *with* $F_0 = a$, $F_1 = b$. This was introduced by Gupta, Panwar and Sikhwal. We shall use the Induction method and Binet's formula for derivation.

1 Introduction

It is well-known that the Fibonacci sequence is most prominent examples of recursive sequence. The Fibonacci sequence is famous for possessing wonderful and amazing properties. Fibonacci numbers are a popular topic for mathematical enrichment and popularization. The Fibonacci appear in numerous mathematical problems. Fibonacci composed a number text in which he did important work in number theory and the solution of algebraic equations. The book for which he is most famous in the "Liber abaci" published in 1202. In the third section of the book, he posed the equation of rabbit problem which is known as the first mathematical model for population growth. From the statement of rabbit problem, the famous Fibonacci numbers can be derived,



This sequence in which each number is the sum of the two preceding numbers has proved extremely fruitful and appears in different areas in Mathematics and Science.

The Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence are most prominent examples of recursive sequences.

The Fibonacci sequence [8] is defined by the recurrence relation

$$F_k = F_{k-1} + F_{k-2}\ ,\ k \geq 2\ \ with\ F_0 = 0,\ F_1 = 1 \qquad (1.1)$$

The Lucas sequence [8] is defined by the recurrence relation

$$L_k = L_{k-1} + L_{k-2}\ ,\ k \geq 2\ \ with\ L_0 = 2,\ L_1 = 1 \tag{1.2}$$

The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation.

Kalman and Mena [4] generalize the Fibonacci sequence by

$$F_n = aF_{n-1} + bF_{n-2}\ ,\ n \geq 2 \quad with \quad F_0 = 0,\ F_1 = 1 \tag{1.3}$$

Horadam [2] defined generalized Fibonacci sequence { H_n } by

$$H_n = H_{n-1} + H_{n-2},\ n \geq 3\ with\ H_1 = p,\ H_2 = p + q, \tag{1.4}$$

where p and q are arbitrary integers**.**

Gupta, Panwar and Sikhwal [9], introduce generalized Fibonacci sequences. They focus only on two cases of sequences $\{V_k\}_{k\geq 0}$ and $\{U_k\}_{k\geq 0}$ which generated by generalized Fibonacci sequences. They defined related identities of generalized Fibonacci sequences consisting even and odd terms. Also they present connection formulas for generalized Fibonacci sequences, Jacobsthal sequence and Jacobsthal-Lucas sequence. In [10], authors have present identities involving common factors of generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers.

2. GENERALIZED FIBONACCI SEQUENCES

Generalized Fibonacci sequence [9], is defined as

$$F_k = pF_{k-1} + qF_{k-2}\ ,\ k \geq 2\,with\ \ F_0 = a,\ F_1 = b\ , \tag{2.1}$$

where $p,\ q,\ a\ \&\ b$ are positive integers.

For different values of $p,\ q,\ a\ \&\ b$ many sequences can be determined.

We will focus on two cases of sequences $\{V_k\}_{k\geq 0}$ and $\{U_k\}_{k\geq 0}$ which generated in (2.1).

If $p = 1\ ,\ q = a = b = 2$, then we get

$$V_k = V_{k-1} + 2V_{k-2} \quad for\ \ k \geq 2\ \ with \quad V_0 = 2\ ,\ V_1 = 2 \tag{2.2}$$

The first few terms of $\{V_k\}_{k\geq 0}$ are 2, 2, 6, 10, 22, 42 and so on.

If $p = 1\ ,\ q = a = 2\ ,\ b = 0$, then we get

$$U_k = U_{k-1} + 2U_{k-2} \quad for\ \ k \geq 2\ \ with \quad U_0 = 2\ ,\ U_1 = 0 \tag{2.3}$$

The first few terms of $\{U_k\}_{k\geq 0}$ are 2, 0, 4, 4, 12, 20 and so on.

In this paper, we present properties of Generalized Fibonacci sequences like Catalan's identity, Cassini's identity and d'ocagnes's Identity.

3. Properties of Generalized Fibonacci Sequences

3.1 Binet's formula

In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers [7]. In our case, Binet's formula allows us to express the generalized Fibonacci numbers in function of the roots $\Re_1$ & $\Re_2$ of the following characteristic equation, associated to the recurrence relation (2.2) and (2.3)

$$x^2 - x - 2 = 0 \quad (3.1)$$

Proposition 1: (*Binet's formula*). The *kth* generalized Fibonacci numbers V_k *and* U_k are given by

(i) $$V_k = 2\,\frac{\Re_1^{k+1} - \Re_2^{k+1}}{\Re_1 - \Re_2} \quad (3.2)$$

(ii) $$U_k = 4\,\frac{\Re_1^{k-1} - \Re_2^{k-1}}{\Re_1 - \Re_2} \quad (3.3)$$

Proof (i). We use the Principle of Mathematical Induction (PMI) on n. It is clear the result is true for $n = 0$ *and* $n = 1$by hypothesis. Assume that it is true for r such that$0 \le r \le s+1$, then

$$V_r = 2\,\frac{\Re_1^{r+1} - \Re_2^{r+1}}{\Re_1 - \Re_2}$$

It follows from definition of generalized Fibonacci numbers (2.2) and equation (3.2),

$$V_{r+2} = V_{r+1} + 2V_r = 2\,\frac{\Re_1^{r+3} - \Re_2^{r+3}}{\Re_1 - \Re_2}$$

Thus, the formula is true for any positive integer k.

Proof (ii). It can be proved same as **Proposition1: (i)**

3.2 Catalan's Identity

Catalan's identity for Fibonacci numbers was found in 1879 by Eugene Charles Catalan a Belgian mathematician who worked for the Belgian Academy of Science in the field of number theory.

Theorem 2: (*Catalan's identity*)

(i) $V_{n-r-1}\ V_{n+r-1} - V_{n-1}^2 = (-1)^{n-r+1}\ 2^{n-r}\ V_{r-1}^2$ (3.4)

(ii) $U_{n-r+1}\ U_{n+r+1} - U_{n+1}^2 = (-1)^{n-r+1}\ 2^{n-r}\ U_{r+1}^2$ (3.5)

Proof (i). *By Binet's formula (3.2), we have*

$$V_{n-r-1}\ V_{n+r+1} - V_{n-1}^2 = 2\left(\frac{\Re_1^{n-r} - \Re_2^{n-r}}{\Re_1 - \Re_2}\right)2\left(\frac{\Re_1^{n+r} - \Re_2^{n+r}}{\Re_1 - \Re_2}\right) - 4\left(\frac{\Re_1^{n} - \Re_2^{n}}{\Re_1 - \Re_2}\right)^2$$

$$= \frac{4}{9}\left[-\left(\Re_1\Re_2\right)^n\left(\frac{\Re_2}{\Re_1}\right)^r - \left(\Re_1\Re_2\right)^n\left(\frac{\Re_1}{\Re_2}\right)^r + 2\left(\Re_1\Re_2\right)^n\right]$$

$$= \frac{-4\left(-2\right)^n}{9}\left[\left(\frac{\Re_2}{\Re_1}\right)^r + \left(\frac{\Re_1}{\Re_2}\right)^r - 2\right]$$

$$= \frac{\left(-1\right)^{n+1-r}\ 2^{n+2}}{\left(-2\right)^r}\left(\frac{\Re_1^r - \Re_2^r}{\Re_1 - \Re_2}\right)^2$$

$$= \left(-1\right)^{n+1-r}\ 2^{n-2}\left(2\frac{\Re_1^r - \Re_2^r}{\Re_1 - \Re_2}\right)^2$$

$$V_{n-r-1}\ V_{n+r+1} - V_{n-1}^2 = (-1)^{n-r+1}\ 2^{n-r}\ V_{r-1}^2$$

Proof (ii). It can be proved same as **Theorem2: (i)**

3.3 Cassini's Identity

This is one of the oldest identities involving the Fibonacci numbers. It was discovered in 1680 by Jean-Dominique Cassini a French astronomer.

Theorem 3: (*Cassini's identity or Simpson's identity*)

(i) $V_{n-2}\ V_n - V_{n-1}^2 = (-1)^n\ 2^{n+1}$ (3.6)

(ii) $U_{n+2}\ U_n - U_{n+1}^2 = (-1)^n\ 2^{n+3}$ (3.7)

Proof: Taking $r = 1$ in Catalan's identity (3.4) and (3.5) the proof is completed.

3.4 d'Ocagne's Identity

Theorem 4: (*d'ocagnes's Identity*)

(i) $$U_{m+1}\ U_n - U_m\ U_{n+1} = \begin{cases} \frac{8}{3}\left(\Re_1^n + \Re_1^m\right);\ n \text{ is even, } m \text{ is odd} \\ -\frac{8}{3}\left(\Re_1^n + \Re_1^m\right);\ n \text{ is odd, } m \text{ is even} \end{cases} \quad (3.8)$$

(ii) $$V_m V_{n-1} - V_{m-1} V_n = \begin{cases} -\frac{4}{3}\left(\Re_1^n + \Re_1^m\right) ; n \text{ is even}, m \text{ is odd} \\ \frac{4}{3}\left(\Re_1^n + \Re_1^m\right) ; n \text{ is odd}, m \text{ is even} \end{cases} \tag{3.9}$$

Proof (i). *By Binet's formula (3.3), we have*

$$U_{m+1} U_n - U_m U_{n+1} = \frac{16}{9}\left[\left(\Re_1^m - \Re_2^m\right)\left(\Re_1^{n-1} - \Re_2^{n-1}\right) - \left(\Re_1^{m-1} - \Re_2^{m-1}\right)\left(\Re_1^n - \Re_2^n\right)\right]$$

$$= \frac{16}{9}\left[\Re_1^{m-1}\Re_2^n - \Re_1^m\Re_2^{n-1} + \Re_2^{m-1}\Re_1^n - \Re_2^m\Re_1^{n-1}\right]$$

$$= \frac{16}{9}\left(\frac{1}{\Re_1} - \frac{1}{\Re_2}\right)\left(\Re_1^m\Re_2^n - \Re_1^n\Re_2^m\right)$$

$$= \frac{8}{3}\left(\Re_1^m\Re_2^n - \Re_1^n\Re_2^m\right)$$

$$U_{m+1} U_n - U_m U_{n+1} = \begin{cases} \frac{8}{3}\left(\Re_1^n + \Re_1^m\right) ; n \text{ is even}, m \text{ is odd} \\ -\frac{8}{3}\left(\Re_1^n + \Re_1^m\right) ; n \text{ is odd}, m \text{ is even} \end{cases}$$

Proof (ii). It can be proved same as **Theorem4: (i)**

3.5 Limit of the quotient of two consecutive terms

A useful property in these sequences is that the limit of the quotient of two consecutive terms is equal to the positive root of the corresponding characteristic equation

Theorem 5:

(i) $$\lim_{k\to\infty}\left(\frac{V_{k-1}}{V_{k-2}}\right) = \Re_1 \tag{3.10}$$

(ii) $$\lim_{n\to\infty}\left(\frac{U_{k+1}}{U_k}\right) = \Re_1 \tag{3.11}$$

Proof (i). *By Binet's formula (3.2), we have*

$$\lim_{k\to\infty}\left(\frac{V_{k-1}}{V_{k-2}}\right) = \lim_{k\to\infty}\frac{\Re_1^k - \Re_2^k}{\Re_1^{k-1} - \Re_2^{k-1}} = \lim_{k\to\infty}\frac{1 - \left(\frac{\Re_2}{\Re_1}\right)^k}{\frac{1}{\Re_1} - \left(\frac{\Re_2}{\Re_1}\right)^k \frac{1}{\Re_2}}$$

and taking into account that $\lim_{k\to\infty}\left(\frac{\Re_2}{\Re_1}\right)^k = 0$, since $|\Re_2| < \Re_1$, Eq. (3.10) is obtained.

Proof (ii). It can be proved same as **Theorem5: (i)**

Theorem 6: *If* $\{V_k\}_{k\geq 0}$ *is the generalized Fibonacci sequence, then*

$$(i)\ V_{2k}^2 + V_{2k+2}^2 = \frac{8}{9}\left[34(16)^k + 10(4)^k + 1\right] \tag{3.12}$$

$$(ii)\ V_{2k}^2 - V_{2k+2}^2 = \frac{-16}{9}\left[12(16)^k + (4)^k\right] \tag{3.13}$$

Proof (i). *By Binet's formula (3.2), we have*

$$\begin{aligned} V_{2k}^2 + V_{2k+2}^2 &= \frac{4}{9}\left[\Re_1^{4k+2} + \Re_2^{4k+2} - 2(\Re_1\Re_2)^{2k+1} + \Re_1^{4k+6} + \Re_2^{4k+6} - 2(\Re_1\Re_2)^{2k+3}\right] \\ &= \frac{4}{9}\left[\Re_1^{4k+2}(1+\Re_1^{4k}) + \Re_2^{4k+2}(1+\Re_2^{4k}) - 2(\Re_1\Re_2)^{2k+1}\left\{1+(\Re_1\Re_2)^2\right\}\right] \\ &= \frac{4}{9}\left[17\Re_1^{4k+2} + 2\Re_2^{4k+2} - 10(\Re_1\Re_2)^{2k+1}\right] \\ &= \frac{4}{9}\left[68\Re_1^{4k} + 2\Re_2^{4k} + 20(\Re_1\Re_2)^{2k}\right] \\ &= \frac{4}{9}\left[68(16)^k + 2 + 20(4)^k\right] \\ &= \frac{8}{9}\left[34(16)^k + 1 + 10(4)^k\right] \end{aligned}$$

Proof (ii). It can be proved same as **Theorem6: (i)**

Theorem 7: *If* $\{U_k\}_{k\geq 0}$ *is the generalized Fibonacci sequence, then*

$$(i)\ U_{2k}^2 + U_{2k+2}^2 = \frac{16}{9}\left[\frac{17}{4}(16)^k + 5(4)^k\right] \tag{3.14}$$

$$(ii)\ U_{2k}^2 - U_{2k+2}^2 = \frac{-48}{9}\left[5(4)^{2k-1} + (4)^k\right] \tag{3.15}$$

Proof : It can be proved same as **Theorem6.**

4. Conclusion

In this paper we have stated and derived many properties of generalized Fibonacci sequences through Binet's formulas. Finally we present properties like Catalan's identity, Cassini's identity or Simpson's identity and d'ocagnes's identity for generalized Fibonacci sequences.

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