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ON IDEALS AND CONGRUENCES OF GENERALIZED ALMOST DISTRIBUTIVE LATTICES

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Abstract Necessary and sufficient conditions for a GADL to become an ADL are obtained interms of congruence relations and proved that every associative GADL is an ADL.

1 Introduction

The concept of a Generalized Almost Distributive Lattice(GADL) was introduced by Rao, Ravi Kumar and Rafi[3] as a generalization of an Almost Distributive Lattice(ADL)[6]. The class of GADLs inherit almost all the properties of a distributive lattice except possibly the commutativity of $\wedge, \vee$, the right distributivity of either of the operations $\vee$ or $\wedge$ over the other. The class of GADLs include the class of ADLs properly and retain many important properties of ADLs. In section 3, we give equivalent conditions for a GADL to become an ADL in terms of ideals, filters and congruences. Finally we proved that every associative GADL is an ADL.

2 Preliminaries

First, we recall certain definitions and properties of ADLs and GADLs from [3, 5] and [6] that are required in the paper.

Definition 2.1. *An Almost Distributive Lattice (ADL) is an algebra $(L, \vee, \wedge)$ of type (2, 2) satisfying*

1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3) $(x \vee y) \wedge y = y$
4) $(x \vee y) \wedge x = x$
5) $x \vee (x \wedge y) = x$

If there is an element $0 \in L$ such that $0 \wedge a = 0$ for all $a \in L$, then $(L, \vee, \wedge, 0)$ is called an ADL with 0.

Definition 2.2. *Let X be a non-empty set. Fix some element $x_0 \in X$. Then, for any $x, y \in X$ define $\vee$ and $\wedge$ on X by,*

$$x \vee y = \begin{cases} x, & \text{if } x \neq x_0 \\ y, & \text{if } x = x_0. \end{cases} \qquad x \wedge y = \begin{cases} y, & \text{if } x \neq x_0 \\ x_0, & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL, with x_0 as its zero element. This ADL is called a ***Discrete ADL****.*

Definition 2.3. *An algebra $(L, \vee, \wedge)$ of type $(2, 2)$ is called a Generalized Almost Distributive Lattice if it satisfies the following axioms*:

(As $\wedge$) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
(LD $\wedge$) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(LD $\vee$) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(A_1) $x \wedge (x \vee y) = x$
(A_2) $(x \vee y) \wedge x = x$
(A_3) $(x \wedge y) \vee y = y$.

Example 2.4. *Let $L = \{a, b, c\}$. Define two binary operations $\vee$ and $\wedge$ on L as follows:*

$\vee$	a	b	c
a	a	b	a
b	b	b	b
c	c	c	c

$\wedge$	a	b	c
a	a	a	c
b	a	b	c
c	a	a	c

Hence the algebra $(L, \vee, \wedge)$ is a Generalized Almost Distributive Lattice.

Example 2.5. *Let $L = \{a, b, c\}$. Define two binary operations $\vee$ and $\wedge$ on L as follows:*

$\vee$	a	b	c
a	a	a	a
b	a	b	b
c	c	c	c

$\wedge$	a	b	c
a	a	b	c
b	b	b	c
c	b	b	c

Hence the algebra $(L, \vee, \wedge)$ is a Generalized Almost Distributive Lattice.

For brevity, we will refer to this Generalized Almost Distributive Lattice as GADL. The GADL $(L, \vee, \wedge)$ in example 2.4 is not an ADL for $(c \vee b) \wedge b \neq b$.
Let $(L, \vee, \wedge)$ be a GADL. For any $a, b \in L$ define $a \leq b$ if and only if $a \wedge b = a$ or, equivalently, $a \vee b = b$. Then $\leq$ is a partial ordering on L. In this section, L stands for a GADL unless otherwise mentioned.

Lemma 2.6. For any $a, b \in L$, we have the following:
(1) $a \vee a = a$
(2) $a \wedge a = a$
(3) $a \vee (a \wedge b) = a$
(4) $a \vee (b \wedge a) = a$
(5) $a \wedge b = b \Rightarrow a \vee b = a$
(6) $a \vee b = b \Leftrightarrow a \wedge b = a$
(7) $a \vee (a \vee b) = a \vee b$
(8) $b \wedge (a \wedge b) = a \wedge b$
(9) $a \wedge (b \wedge a) = b \wedge a.$
(10) $a \leq c, b \leq c$ if and only if $a \wedge b = b \wedge a$ and $a \vee b = b \vee a.$

Regarding the remaining absorption laws we have the following theorem:

Theorem 2.7. For any $a, b \in L$, the following are equivalent:
(1) $(a \wedge b) \vee a = a$
(2) $a \wedge (b \vee a) = a$
(3) $(b \wedge a) \vee b = b$
(4) $b \wedge (a \vee b) = b$
(5) $a \wedge b = b \wedge a$
(6) $a \vee b = b \vee a.$

The following lemma is very useful in the GADL.

Lemma 2.8. For any $a, b, c \in L, \quad a \wedge b \wedge c = b \wedge a \wedge c.$

In the following, we give the equivalent conditions for a GADL to become an ADL.

Theorem 2.9. Let $(L, \vee, \wedge)$ be a GADL. Then the following are equivalent:
(1) L is an Almost Distributive Lattice
(2) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all $a, b, c \in L.$
(3) $(a \vee b) \wedge b = b$ for all $a, b \in L.$
(4) $(a \vee b) \wedge c = (b \vee a) \wedge c$ for all $a, b, c \in L.$

Definition 2.10. *Let $(L, \vee, \wedge)$ be a GADL. An element $0 \in L$ is called a zero element of L if*
(0_1) $0 \wedge a = 0$ *for all* $a \in L$.

We always denote the zero element of L, if it exists, by '0'. If L has 0, then the algebra $(L, \vee, \wedge, 0)$ is called a GADL with 0. Now we have the following

Lemma 2.11. Let $(L, \vee, \wedge, 0)$ be a GADL with 0. Then, for any $a \in L$, the following hold:

(0_2) $a \vee 0 = a$
(0_3) $0 \vee a = a$
(0_4) $a \wedge 0 = 0.$

Definition 2.12. *A GADL $(L, \vee, \wedge)$ is said to be associative if the operation $\vee$ in L is associative.*

Proposition 2.13. For any $m \in L$, the following are equivalent:
(1) m is maximal
(2) $m \vee x = m$ for all $x \in L$.

Definition 2.14. *Let $(L, \vee, \wedge)$ be a GADL. An element $e \in L$ is said to be left identity element in L if $e \wedge x = x$ for all $x \in L$.*

Note that every left identity element is maximal element but converse need not be true. In Example 2.4, we observe that c is maximal but not left identity element.

3 On Ideals and Congruences

In this section, we give necessary and sufficient conditions for a GADL L to become an ADL interms of ideals and congruence relations on L.

Definition 3.1. [4] A non-empty subset I of L is said to be an ideal of L, if it satisfies the following:

(i) $a, b \in I \Rightarrow a \vee b \in I$
(ii) $a \in I, x \in L \Rightarrow a \wedge x \in I.$

Lemma 3.2. *If I is an ideal of L, then $x \wedge a \in I$ for any $a \in I$ and $x \in L$.*

Proof. Let I be an ideal of $L, a \in I$ and $x \in L$. Then $x \wedge a = x \wedge a \wedge a = a \wedge (x \wedge a) \in I.$ □

Therefore, in this case, any right ideal in the usual sense is a left ideal too and hence a two sided ideal in the usual sense. However, a left ideal may not be a right ideal; for, consider the following

Example 3.3. Let D be a discrete ADL. For any $x \neq 0$, the set $\{0, x\}$ is a left ideal but not a right ideal of D.

Definition 3.4. A non-empty subset F of L is said to be an $R-$filter ($L-$filter) of L, if it satisfies the following:

(i) $a, b \in F \Rightarrow a \wedge b \in F$
(ii) $a \in F, x \in L \Rightarrow a \vee x \in F (x \vee a \in F).$

A non-empty subset F of L is said to be a filter of L if it is both $L-$filter and $R-$filter. It can be easily seen that, if L is an associative GADL then every $L-$filter is an $R-$filter but an $R-$filter need not be an $L-$ filter. For consider the following example.

Example 3.5. Let D be a discrete ADL. For any $x \neq 0$, the set $\{x\}$ is $R-$filter but not an $L-$filter of D.

Theorem 3.6. *Let L be a GADL and $a \in L$. Define $S = \{a \wedge x \mid x \in L\}$. Then S is the smallest ideal of L containing a.*

This smallest ideal S of L containing a is denoted by $(a]$. Note that $\{x \wedge a \mid x \in L\}$ is not an ideal of L. Similarly, we can prove that $[a) = \{a \vee x \mid x \in L\}$ is the smallest $R-$filter of L containing a. But $\{x \vee a \mid x \in L\}$ need not be an $L-$filter of L. For, from example 2.4, we observe that $\{x \vee b \mid x \in L\} = \{b, c\}$ is not an $L-$filter.
Note that if L is a GADL with 0 then the intersection of any family of ideals of L is again an ideal of L. Also the set of all ideals of L forms a complete lattice under the ordering set inclusion. Now we discuss some important properties of the principal ideals ($R-$filters) of L.

Lemma 3.7. *Let L be a GADL and $a, b \in L$. Then*
(i) $a \in (b]$ *if and only if* $b \wedge a = a$.
(ii) $a \in [b)$ *if and only if* $b \vee a = a$.
(iii) $a \leq b \Rightarrow (a] \subseteq (b]$
(iv) $a \in (b] \Rightarrow (a] \subseteq (b]$

Proof. (i) Suppose $a \in (b]$. Then $a = b \wedge x$ for some $x \in L$. Now, $b \wedge a = b \wedge b \wedge x = b \wedge x = a$. Therefore $b \wedge a = a$. Conversely assume that $b \wedge a = a$. Then $a \in (b]$. Therefore $a \in (b]$ if and only if $b \wedge a = a$.
Similarly, we can prove (ii).
(iii) Let $a \leq b$ and $x \leq (a]$. Then $b \wedge a = a$ and $a \wedge x = x$. Now, $b \wedge x = b \wedge a \wedge x = a \wedge x = x$. Therefore $x \in (b]$. Hence $(a] \subseteq (b]$.
(iv) Let $a \in (b]$ and $x \in (a]$. Then $b \wedge a = a$ and $a \wedge x = x$. Now, $b \wedge x = b \wedge a \wedge x = a \wedge x = x$. Therefore $x \in (b]$. Hence $(a] \subseteq (b]$. □

Lemma 3.8. *Let L be a GADL and $a, b \in L$. Then $(a] \cap (b] = (a \wedge b]$.*

But regarding $R-$filters, we donot get that $[a) \wedge [b) = [a \vee b)$. For, in example 2.4, $[a) = \{a, b\}, [c) = \{c\}$ and $[a) \cap [c) \neq [a \vee c)$.

Theorem 3.9. *Let L be a GADL and I be an ideal of L. Then, for any $a, b \in L$, the following hold:*

(i) $a \wedge b \in I$ *if and only if* $b \wedge a \in I$

(ii) $(a \wedge b] = (b \wedge a]$

(iii) *if* $a \in I$ *and* $x \in L$ *such that* $x \leq a$ *then* $x \in I$.

If L is an ADL and $a, b \in L$, then $(a] \vee (b] = (a \vee b]$. But if this condition holds in a GADL then the GADL becomes an ADL. We prove this in the following:

Theorem 3.10. *Let L be a GADL with 0. Then the following are equivalent:*

(i) *L is an Almost Distributive Lattice*

(ii) *For any $a, b \in L$, $(a \vee b]$ is the supremum of $(a]$ and $(b]$ in $(I(L), \subseteq)$, where $I(L)$ is the set of all ideals of L*

(iii) *$(a \vee b] = (b \vee a]$ for all $a, b \in L$.*

Proof. $(1) \Rightarrow (2)$: Assume (1). Let $a, b \in L$. Then
$a \leq a \vee b \Rightarrow (a] \subseteq (a \vee b]$ and $b \leq b \vee a \Rightarrow (b] \subseteq (b \vee a] = (a \vee b]$ since L is an ADL. Therefore $(a \vee b]$ is an upperbound of $(a]$ and $(b]$.
Let J be any ideal of L such that $(a] \subseteq J$ and $(b] \subseteq J$.
Clearly $a \in J$ and $b \in J$. Therefore $a \vee b \in J$ and hence $(a \vee b] \subseteq J$.
Thus $(a \vee b]$ is the supremum of $(a]$ and $(b]$ in $(I(L), \subseteq)$.
$(2) \Rightarrow (3)$: Assume (2). Then $(a \vee b]$ and $(b \vee a]$ both are supremums of $(a]$ and $(b]$ in the poset $(I(L), \subseteq)$. Therefore $(a \vee b] = (b \vee a]$.
$(3) \Rightarrow (1)$: Assume (3). Let $a, b \in L$. Since $(b \vee a) \wedge b = b$, we have $b \in (b \vee a] = (a \vee b]$. Therefore $(a \vee b) \wedge b = b$ and hence, by Theorem 2.9, L is an Almost Distributive Lattice. □

Theorem 3.11. *Let F be an $R-$filter of a GADL L. Then the relation*
$\varphi^F = \{(x, y) \in L \times L \mid a \wedge x = a \wedge y,$ for some $a \in F\}$ is a congruence relation on L.

Proof. Clearly φ^F is an equivalence relation on L. Let $(x, y), (u, v) \in \varphi^F$. Then $a \wedge x = a \wedge y$ and $b \wedge u = b \wedge v$ for some $a, b \in F$. Now $a, b \in F$ implies that $a \wedge b \in F$ and $a \wedge b \wedge x \wedge u = a \wedge x \wedge b \wedge u = a \wedge y \wedge b \wedge v = a \wedge b \wedge y \wedge v$. Therefore $(x \wedge u, y \wedge v) \in \varphi^F$. Also

$$\begin{aligned}
a \wedge b \wedge (x \vee u) &= a \wedge b \wedge (x \vee u) \\
&= (a \wedge b \wedge x) \vee (a \wedge b \wedge u) \\
&= (b \wedge a \wedge x) \vee (a \wedge b \wedge u) \\
&= (b \wedge a \wedge y) \vee (a \wedge b \wedge v) \\
&= (a \wedge b \wedge y) \vee (a \wedge b \wedge v) \\
&= (a \wedge b) \wedge (y \vee v)
\end{aligned}$$

Therefore $(x \vee u, y \vee v) \in \varphi^F$ and hence φ^F is a congruence relation on L. □

Lemma 3.12. *Let L be a GADL. Then for any $a \in L$. $\varphi^{[a)} = \varphi^a$.*

Proof. Clearly $\varphi^a \subseteq \varphi^{[a)}$. Let $(x, y) \in \varphi^{[a)}$. Then $t \wedge x = t \wedge y$ for some $t \in [a)$. Then $a \vee t = t$ and hence $a \wedge t = a$. Now $a \wedge x = a \wedge t \wedge x = a \wedge t \wedge y = a \wedge y$. Therefore $(x, y) \in \varphi^a$. Thus $\varphi^{[a)} \subseteq \varphi^a$. Hence $\varphi^a = \varphi^{[a)}$. □

In general, for any $a \in L$, $\psi_a = \{(x,y) \in L \times L \mid x \wedge a = y \wedge a\}$ is an equivalence relation but not a congruence relation on L. For, in example 2.4, $\psi_b = \Delta_L \cup \{(a,c)(c,a)\}$ is not a congruence relation on L because $(a \vee b, c \vee b) = (b,c) \notin \psi_b$. Also, for any $R-$filter F of L, $\psi_F = \{(x,y) \in L \times L \mid x \wedge a = y \wedge a$ for some $a \in F\}$ is an equivalence relation on L and $F \times F \subseteq \psi_F$. ψ_F is also not a congruence relation on L.

Lemma 3.13. *Let L be a GADL. Then for any $a \in L$,*
$\psi_a = \{(x,y) \in L \times L \mid x \wedge a = y \wedge a\} = \psi_{[a)}$

Proof. Clearly $\psi_a \subseteq \psi_{[a)}$. Let $(x,y) \in \psi_{[a)}$. Then $x \wedge t = y \wedge t$ for some $t \in [a)$. Now $t \in [a)$ implies that $a \vee t = t$ and hence $a \wedge t = a$. Also,
$x \wedge a = x \wedge a \wedge t = a \wedge x \wedge t = a \wedge y \wedge t = y \wedge a \wedge t = y \wedge a$. Therefore $(x,y) \in \psi_a$.
Thus $\psi_{[a)} \subseteq \psi_a$. Hence $\psi_a = \psi_{[a)}$. □

Theorem 3.14. *[4] Let L be an associative GADL with 0. Then for any ideal I of L, the relation $\theta_I = \{(x,y) \in L \times L \mid a \vee x = a \vee y$ for some $a \in I\}$ is the smallest congruence relation on L containing $I \times I$.*

Theorem 3.15. *Let L be an associative GADL. Then for any $a \in L$,*
$\theta_a = \{(x,y) \in L \times L \mid a \vee x = a \vee y\} = \theta_{(a]}$.

Proof. Clearly $\theta_a \subseteq \theta_{(a]}$.
Let $(x,y) \in \theta_{(a]}$. Then $b \vee x = b \vee y$ for some $b \in (a]$.
Then $b = a \wedge c$, for some $c \in L$. Now,

$$\begin{aligned} a \vee x &= [a \vee (a \wedge c)] \vee x \\ &= a \vee [(a \wedge c) \vee x] \\ &= a \vee [(a \wedge c) \vee y] \ \ (\text{ since } (a \wedge c) \vee x = (a \wedge c) \vee y) \\ &= [a \vee (a \wedge c)] \vee y \\ &= a \vee y \end{aligned}$$

Therefore $(x,y) \in \theta_a$. Hence $\theta_{(a]} = \theta_a$. □

Lemma 3.16. *For any $a \in L$, $\varphi^a := \{(x,y) \in L \times L \mid a \wedge x = a \wedge y\}$ is a congruence relation on L. Further, $\varphi^a = \Delta_L$ if and only if a is a left identity element of L and $\varphi^a = L \times L$ if and only if $a = 0$.*

Proof. Clearly φ^a is an equivalence relation on L. Let $(u,v) \in \varphi^a$ and $(c,d) \in \varphi^a$. Then $a \wedge u = a \wedge v$ and $a \wedge c = a \wedge d$. Now $a \wedge (u \wedge c) = (a \wedge u) \wedge c = (a \wedge v) \wedge c = a \wedge v \wedge c = v \wedge a \wedge c = v \wedge a \wedge d = a \wedge v \wedge d = a \wedge (v \wedge d)$. Therefore $(u \wedge c, v \wedge d) \in \varphi^a$. Also, $a \wedge (u \vee c) = (a \wedge u) \vee (a \wedge c) = (a \wedge v) \vee (a \wedge d) = a \wedge (v \vee d)$. Therefore $(u \vee c, v \vee d) \in \varphi^a$. Hence φ^a is a congruence relation on L. Suppose $\varphi^a = \Delta_L$. Let $x \in L$. Then $a \wedge (a \wedge x) = a \wedge x$. So that $(a \wedge x, x) \in \varphi^a = \Delta_L$ and hence $a \wedge x = x$. Thus a is left identity element. Conversely suppose a is left identity element and $(x,y) \in \varphi^a$. Then $a \wedge x = a \wedge y$; i.e., $x = y$. Hence $\varphi^a = \Delta_L$. Also if $\varphi^a = L \times L$, then $(a,0) \in \varphi^a$ and hence $a = 0$ and the converse is trivial. □

Theorem 3.17. *Let L be a GADL and $a, b \in L$. Then the following hold:*
(1). $\varphi^{a \wedge b} = \varphi^a \vee \varphi^b = \varphi^b \circ \varphi^a \circ \varphi^b = \varphi^a \circ \varphi^b \circ \varphi^a$
(2). $\varphi^{a \vee b} \subseteq \varphi^a$.

Proof. Let $(x,y) \in \varphi^a$. Then $a \wedge x = a \wedge y$. Now

$$(a \wedge b) \wedge x = (b \wedge a) \wedge x = b \wedge a \wedge y = a \wedge b \wedge y.$$

Therefore $(x,y) \in \varphi^{a \wedge b}$. Hence $\varphi^a \subseteq \varphi^{a \vee b}$. Clearly $\varphi^b \subseteq \varphi^{a \wedge b}$. Therefore $\varphi^a \vee \varphi^b = \varphi^{a \wedge b}$.
Also

$$\begin{aligned} (x,y) \in \varphi^{a \wedge b} &\Rightarrow a \wedge b \wedge x = a \wedge b \wedge y \\ &\Rightarrow (x, b \wedge x) \in \varphi^b, (b \wedge x, b \wedge y) \in \varphi^a, (b \wedge y, y) \in \varphi^b \\ &\Rightarrow (x,y) \in \varphi^b \circ \varphi^a \circ \varphi^b \end{aligned}$$

Hence $\varphi^{a \wedge b} \subseteq \varphi^b \circ \varphi^a \circ \varphi^b$. Also clearly $\varphi^b \circ \varphi^a \circ \varphi^b \subseteq \varphi^a \vee \varphi^b$.
Therefore $\varphi^a \vee \varphi^b \subseteq \varphi^{a \wedge b} \subseteq \varphi^b \circ \varphi^a \circ \varphi^b \subseteq \varphi^a \vee \varphi^b$ and hence $\varphi^{a \wedge b} = \varphi^a \vee \varphi^b = \varphi^b \circ \varphi^a \circ \varphi^b$.
(2) Let $(x,y) \wedge \varphi^{a \vee b}$. Then $(a \vee b) \wedge x = (a \vee b) \wedge y$. Now

$$\begin{aligned} a \wedge x &= (a \vee b) \wedge a \wedge x \\ &= a \wedge (a \vee b) \wedge x \\ &= a \wedge (a \vee b) \wedge y \\ &= a \wedge y \end{aligned}$$

Therefore $(x,y) \in \varphi^a$ and hence $\varphi^{a \vee b} \subseteq \varphi^a$. □

Note that, in general, $\varphi^{a\vee b}$ may not be contained in φ^b. For, in Example 2.5, we observe that $\varphi^{c\vee a} = \varphi^c = \Delta \cup \{(a,b),(b,a)\}$ and $\varphi^a = \Delta$.

Theorem 3.18. *Let L be a GADL. Then L is an Almost Distributive Lattice if and only if $\varphi^{a\vee b} \subseteq \varphi^b$ for all $a,b \in L$.*

Proof. Suppose L is an Almost Distributive Lattice. Let $a,b \in L$ and $(x,y) \in \varphi^{a\vee b}$. Then $(a\vee b)\wedge x = (a\vee b)\wedge y$. Now,
$b\wedge x = (a\vee b)\wedge b\wedge x = b\wedge(a\vee b)\wedge x = b\wedge(a\vee b)\wedge y = (a\vee b)\wedge b\wedge y = b\wedge y$.
Therefore $(x,y)\in\varphi^b$ and hence $\varphi^{a\vee b}\subseteq\varphi^b$.
Conversely, assume that $\varphi^{a\vee b}\subseteq\varphi^b$ for all $a,b\in L$. Let $a,b\in L$. Then $((a\vee b)\wedge b, b)\in\varphi^{a\vee b}$ and hence $((a\vee b)\wedge b,b)\in\varphi^b$. Therefore $b\wedge(a\vee b)\wedge b = b\wedge b$. Thus $(a\vee b)\wedge b = b$. Hence, by Theorem 2.9, L is an Almost Distributive Lattice. □

The following result can be easily verified

Lemma 3.19. *For any $a\in L$, $\theta_a = \{(x,y)\in L\times L \mid a\vee x = a\vee y\}$ is an equivalence relation on L.*

In general, θ_a is not a congruence relation on L. For, in Example 2.4, $\theta_a = \Delta_L\cup\{(a,c),(c,a)\}$ is not a congruence relation on L because $(a\vee b, c\vee b)\notin\theta_a$. But, if $\vee$ is associative in L, then θ_a is a congruence relation on L. In fact, we prove the following

Theorem 3.20. *Let L be any GADL. Then θ_a is a congruence on L if and only if $\vee$ is associative. Further, $\theta_a = \Delta_L$ if and only if a is the zero(least) element of L.*

In the following theorem we characterize a subdirectly irreducible associative GADL.

Theorem 3.21. *Let L be an associative GADL. Then L is subdirectly irreducible if and only if every nonzero element of L is left identity and L contains atmost two nonzero elements.*

The following can be obtained directly, using Birkhoff's subdirect representation theorem[1] and Theorem 3.21.

Theorem 3.22. *The following are equivalent:*

(1) *L is associative*
(2) *θ_a is a congruence relation for all $a\in L$*
(3) *L is a subdirect product of GADLs in each of which there are atmost two nonzero elements and every nonzero element is left identity element.*

We conclude our paper with the following important result.

Corollary 3.23. *Every associative GADL is an ADL.*

Open Problem 1. Describe the nature of the supremum of the complete lattice $(I(L),\subseteq)$ where $I(L)$ is the set of all ideals of a GADL L.
Open Problem 2. Describe the ideal generated by any nonempty subset S of a GADL L.

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