Palestine Journal of Mathematics

Vol. 4(2) (2015), 360–366

Some coupled fixed point theorems for mappings satisfying a generalized contractive condition of rational type

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 47H10, 54H25; Secondary 46T99.

Keywords and phrases: Coupled fixed point, mixed monotone property, Ordered metric spaces.

Abstract The purpose of this paper is to establish some coupled fixed point theorems for mappings having a strict mixed monotone property and satisfying a generalized contractive condition of rational type in the framework of partially ordered metric spaces. Also, we present a result on the existence and uniqueness of coupled fixed points. The results presented in the paper generalize and extend several well-known results in the literature.

1 Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a large number of applications. The Banach contraction mapping is one of the most important results of analysis. It is a popular tool for solving existence problems in different fields of mathematics. There are a lot of generalizations of the Banach contraction principle in the literature. Ran and Reurings [17] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and Rodŕiguez-López [16] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [2] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results on a first order differential equation with periodic boundary conditions. Many researchers have obtained fixed point, common fixed point, coupled fixed point and coupled common fixed point results in cone metric spaces, partially ordered metric spaces and others (see [1]-[18]).

The purpose of this paper is to establish some coupled fixed point results in partially ordered metric spaces for a pair of mappings having strict mixed monotone property and satisfying a generalized contractive condition of rational type. Also, we present a result on the existence and uniqueness of coupled fixed points.

Definition 1.1. Let $(X, \leq)$ be a partially ordered set and $F : X \to X$. The mapping F is said to be *strictly increasing* if for $x, y \in X$, $x < y$ implies $F(x) < F(y)$ and *strictly decreasing* if for $x, y \in X$, $x < y$ implies $F(x) > F(y)$.

Definition 1.2. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the *strict mixed monotone property* if $F(x, y)$ is strictly increasing in x and strictly decreasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 < x_2 \Rightarrow F(x_1, y) < F(x_2, y),$$

and

$$y_1, y_2 \in X, y_1 < y_2 \Rightarrow F(x, y_1) > F(x, y_2).$$

Definition 1.3. An element $(x, y) \in X \times X$ is called a *coupled fixed point* of the mapping $F : X \times X \to X$ if $F(x, y) = x$, and $F(y, x) = y$.

2 Main Results

2.1 Coupled fixed point theorems

In this section, we prove some coupled fixed point theorems in the context of ordered metric spaces.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X \times X \to X$ is a self mapping on X such that F has the strict mixed monotone property on X. Suppose that there exists $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $\alpha + \beta + 2\gamma + 2\delta < 1$ such that*

$$\begin{aligned} & d(F(x,y), F(u,v)) \\ \leq\ & \alpha\left(\frac{d(x, F(x,y))d(u, F(u,v))}{d(x,u)}\right) + \beta(d(x,u)) + \gamma[d(x, F(x,y)) + d(u, F(u,v))] \\ & +\delta[d(x, F(u,v)) + d(u, F(x,y))], \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.

Also suppose that either

a) F is continuous or

b) X has the following properties:

(i) if a sequence $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$ in X, then $x = \sup\{x_n\}$, for every n;

(ii) if a sequence $\{y_n\} \subset X$ is a non-increasing sequence with $y_n \to y$ in X, then $y = \inf\{y_n\}$, for every n.

If there exists two elements $x_0, y_0 \in X$ with $x_0 < F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$, then there exists $x, y \in X$ such that $F(x, y) = x$ and $y = F(y, x)$, that is, F has a coupled fixed point $(x, y) \in X \times X$.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 < F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$. We can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n), \forall n \geq 0. \tag{2.2}$$

We claim that for all $n \geq 0$,

$$x_n < x_{n+1}, \tag{2.3}$$

and

$$y_n > y_{n+1}. \tag{2.4}$$

We shall use the mathematical induction. Let $n = 0$. Since $x_0 < F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$, in view of $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, we have $x_0 < x_1$ and $y_0 > y_1$, that is, (2.3) and (2.4) hold for $n = 0$. Suppose that (2.3) and (2.4) hold for some $n > 0$. As F has the strict mixed monotone property and $x_n < x_{n+1}$ and $y_n > y_{n+1}$, from (2.2), we get

$$x_{n+1} = F(x_n, y_n) < F(x_{n+1}, y_n) < F(x_{n+1}, y_{n+1}) = x_{n+2}, \tag{2.5}$$

and

$$y_{n+1} = F(y_n, x_n) > F(y_{n+1}, x_n) > F(y_{n+1}, x_{n+1}) = y_{n+2}. \tag{2.6}$$

Now from (2.5) and (2.6), we obtain that $x_{n+1} < x_{n+2}$ and $y_{n+1} > y_{n+2}$. Thus by the mathematical induction, we conclude that (2.3) and (2.4) hold for all $n \geq 0$. Therefore

$$x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots, \tag{2.7}$$

and

$$y_0 > y_1 > y_2 > \ldots > y_n > y_{n+1} > \ldots. \tag{2.8}$$

Since $x_n > x_{n-1}$ and $y_n < y_{n-1}$, from (2.1) and (2.2), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq \alpha\left(\frac{d(x_n, F(x_n, y_n))d(x_{n-1}, F(x_{n-1}, y_{n-1}))}{d(x_n, x_{n-1})}\right) + \beta(d(x_n, x_{n-1})) + \\ &\quad \gamma[d(x_n, F(x_n, y_n)) + d(x_{n-1}, F(x_{n-1}, y_{n-1}))] + \\ &\quad \delta[d(x_n, F(x_{n-1}, y_{n-1})) + d(x_{n-1}, F(x_n, y_n))] \\ &= \alpha\left(\frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_n, x_{n-1})}\right) + \beta(d(x_n, x_{n-1})) + \\ &\quad \gamma[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + \delta[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] \\ &\leq \alpha\left(d(x_n, x_{n+1})\right) + \beta(d(x_n, x_{n-1})) + \gamma[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + \\ &\quad \delta[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &= [\alpha + \gamma + \delta]d(x_n, x_{n+1}) + [\beta + \gamma + \delta]d(x_n, x_{n-1}), \end{aligned} \tag{2.9}$$

which implies that $d(x_{n+1},x_n) \le \frac{\beta+\gamma+\delta}{1-[\alpha+\gamma+\delta]} d(x_n,x_{n-1})$.

Similarly, we have $d(y_{n+1},y_n) \le \frac{\beta+\gamma+\delta}{1-[\alpha+\gamma+\delta]} d(y_n,y_{n-1})$.

Hence $d(x_{n+1},x_n) + d(y_{n+1},y_n) \le \frac{\beta+\gamma+\delta}{1-[\alpha+\gamma+\delta]}(d(x_n,x_{n-1}) + d(y_n,y_{n-1}))$. Set $\{\varrho_n := d(x_{n+1},x_n) + d(y_{n+1},y_n)\}$ and $k = \frac{\beta+\gamma+\delta}{1-[\alpha+\gamma+\delta]} < 1$, we have

$$0 \le \varrho_n \le k\varrho_{n-1} \le k^2\varrho_{n-2} \le \ldots \le k^n\varrho_0$$

which implies that

$$\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty}[d(x_{n+1},x_n) + d(y_{n+1},y_n)] = 0. \tag{2.10}$$

Thus $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$ and $\lim_{n\to\infty} d(y_{n+1},y_n) = 0$.

Now, we shall prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. For each $m \ge n$, we have

$$d(x_m,x_n) \le d(x_m,x_{m-1}) + d(x_{m-1},x_{m-2}) + \ldots + d(x_{n+1},x_n)$$

and

$$d(y_m,y_n) \le d(y_m,y_{m-1}) + d(y_{m-1},y_{m-2}) + \ldots + d(y_{n+1},y_n).$$

Therefore

$$\begin{aligned} d(x_m,x_n) + d(y_m,y_n) &\le \varrho_{m-1} + \varrho_{m-2} + \ldots + \varrho_n \\ &\le (k^{m-1} + k^{m-2} + \ldots + k^n)\varrho_0 \\ &\le \frac{k^n}{1-k}\varrho_0 \end{aligned}$$

which implies that $\lim_{m,n\to\infty}[d(x_m,x_n)+d(y_m,y_n)] = 0$. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a complete metric space X. Therefore, there exists $(x,y) \in X\times X$ such that $x_n \to x$ and $y_n \to y$.

First, suppose that F is continuous, we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n,y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(x,y)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n,x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(y,x).$$

Therefore, $x = F(x,y)$. Similarly, we can show that $y = F(y,x)$. Therefore, $x = F(x,y)$ and $y = F(y,x)$.

Second, suppose that (b) holds. Since $\{x_n\}$ is an increasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$. Particularly, $x_n \le x$ for all $n \in \mathbb{N}$. We claim that $x_n < x$ for all n. If it is not the case, there is a n_0 such that $x_{n_0} = x$. So, $x = x_{n_0} < x_{n_0+1} \le x$ which is impossible. Thus, $x_n < x$ for all n. Now, F is a strict monotone increasing mapping for the first variable implies

$$F(x_n,y_n) < F(x,y_n). \tag{2.11}$$

Further, $\{y_n\}$ is a decreasing sequence in X such that $y \to y_n$, then $y = \sup\{y_n\}$. Particularly, $y \le y_n$ for all $n \in \mathbb{N}$. Similar reasoning as above yields $y < y_n$ for all n. Since F is a strict monotone decreasing mapping for the second variable then

$$F(x,y_n) < F(x,y). \tag{2.12}$$

Linking (2.11) and (2.12) we get

$$F(x_n,y_n) < F(x,y), \tag{2.13}$$

for all $n \in \mathbb{N}$ or, equivalently,

$$x_{n+1} < F(x,y), \tag{2.14}$$

for all $n \in \mathbb{N}$. Moreover, as $x_n < x_{n+1} < F(x,y)$ and $x = \sup\{x_n\}$, we get $x \le F(x,y)$.

Construct a sequence $\{z_n\}$ as $z_0 = x$, $z_{n+1} = F(z_n,y_n)$, for all $n \ge 0$. Since $z_0 \le F(z_0,y_0)$, arguing as above, we obtain that $\{z_n\}$ is a non-decreasing sequence and $\lim_{n\to\infty} z_n = z$ for certain $z \in X$, so we have $z = \sup\{z_n\}$. Since $x_n \le x = z_0 \le F(z_0,y_0) \le z_n \le z$, for all n, using (2.1), we have

$$\begin{aligned}
d(x_{n+1}, z_{n+1}) &= d(F(x_n, y_n), F(z_n, y_n)) \\
&\leq \alpha\left(\frac{d(x_n, F(x_n, y_n))d(z_n, F(z_n, y_n))}{d(x_n, z_n)}\right) + \beta(d(x_n, z_n)) + \\
&\quad \gamma\left(d(x_n, F(x_n, y_n)) + d(z_n, F(z_n, y_n))\right) + \delta\left(d(x_n, F(z_n, y_n)) + d(z_n, F(x_n, y_n))\right) \\
&= \alpha\left(\frac{d(x_n, x_{n+1})d(z_n, z_{n+1})}{d(x_n, z_n)}\right) + \beta(d(x_n, z_n)) + \qquad (2.15)\\
&\quad \gamma\left(d(x_n, x_{n+1}) + d(z_n, z_{n+1})\right) + \delta\left(d(x_n, z_{n+1}) + d(z_n, x_{n+1})\right),
\end{aligned}$$

letting $n \to \infty$, we have $d(x, z) \leq (\beta + 2\delta)d(x, z)$. As $(\beta + 2\delta) < 1$, we have $d(x, z) = 0$. Particularly, $x = z = \sup\{x_n\}$, and consequently, $x \leq F(x, y) \leq x$ i.e. $x = F(x, y)$.

Similarly, we can prove that $y = F(y, x)$.

□

Now, we shall prove the existence and uniqueness of a coupled common fixed point. Note that, if $(X, \leq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order relation:

$$\text{for } (x, y), (u, v) \in X \times X, (u, v) \leq (x, y) \Leftrightarrow x \leq u, y \geq v.$$

Theorem 2.2. *In addition to hypotheses of Theorem 2.1, suppose that for every $(x, y), (z, t) \in X \times X$, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Then F has a unique coupled fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$.*

Proof. From Theorem 2.1, the set of coupled fixed points of F is non-empty. Suppose that (x, y) and (z, t) are coupled fixed points of F, that is, $x = F(x, y)$, $y = F(y, x)$, $z = F(z, t)$ and $t = F(t, z)$. We shall show that $x = z$ and $y = t$. By the assumption, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $u_1 = F(u_0, v_0)$ and $v_1 = F(v_0, u_0)$. Then similarly as in the proof of Theorem 2.1, we can inductively define sequences $\{u_n\}$, $\{v_n\}$ as $u_{n+1} = F(u_n, v_n)$ and $v_{n+1} = F(v_n, u_n)$ for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$, $t_0 = t$ and on the same way define the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$, $\{t_n\}$. Then as in Theorem 2.1, we can show that $x_n \to x = F(x, y)$, $y_n \to y = F(y, x)$, $z_n \to z = F(z, t)$, $t_n \to t = F(t, z)$, for all $n \geq 1$. Since $(F(x, y), F(y, x)) = (x, y)$ and $(F(u, v), F(v, u)) = (u_1, v_1)$ are comparable, then $x \geq u_1$ and $y \leq v_1$. Now, we shall show that (x, y) and (u_n, v_n) are comparable, that is, $x \geq u_n$ and $y \leq v_n$ for all n. Suppose that it holds for some $n \geq 0$, then by the strict strict mixed monotone property of F, we have $u_{n+1} = F(u_n, v_n) \leq F(x, y) = x$ and $v_{n+1} = F(v_n, u_n) \geq F(y, x) = y$. Hence $x \geq u_n$ and $y \leq v_n$ hold for all n. Thus from (2.1), we have

$$\begin{aligned}
d(x, u_{n+1}) &= d(F(x, y), F(u_n, v_n)) \\
&\leq \alpha\left(\frac{d(x, F(x, y))d(u_n, F(u_n, v_n))}{d(x, u_n)}\right) + \beta(d(x, u_n)) + \\
&\quad \gamma[d(x, F(x, y)) + d(u_n, F(u_n, v_n))] + \delta[d(x, F(u_n, v_n)) + d(u_n, F(x, y))] \\
&\leq \alpha\left(\frac{d(x, F(x, y))d(u_n, F(u_n, v_n))}{d(x, u_n)}\right) + \beta(d(x, u_n)) + \\
&\quad \gamma[d(x, u_n) + d(u_{n+1}, x))] + \delta[d(x, u_{n+1}) + d(u_n, x)] \\
&= (\beta + \gamma + \delta)d(x, u_n) + (\gamma + \delta)d(x, u_{n+1}), \qquad (2.16)
\end{aligned}$$

which implies that $d(x, u_{n+1}) \leq \frac{\beta+\gamma+\delta}{1-(\gamma+\delta)}d(x, u_n)$. Similarly, we can prove that $d(y, v_{n+1}) \leq \frac{\beta+\gamma+\delta}{1-(\gamma+\delta)}d(y, v_n)$. Assume $L = \frac{\beta+\gamma+\delta}{1-(\gamma+\delta)} < 1$. Hence

$$\begin{aligned}
d(x, u_{n+1}) + d(y, v_{n+1}) &\leq L[d(x, u_n) + d(y, v_n)] \\
&\leq (L)^2[d(x, u_{n-1}) + d(y, v_{n-1})] \\
&\quad \ldots \\
&\leq (L)^{n+1}[d(x, u_0) + d(y, v_0)].
\end{aligned}$$

On taking limit, $n \to \infty$, we get $\lim_{n\to\infty}[d(x, u_{n+1}) + d(y, v_{n+1})] = 0$. Thus $\lim_{n\to\infty} d(x, u_{n+1}) = 0$ and $\lim_{n\to\infty} d(y, v_{n+1}) = 0$.

Similarly, we can prove that $\lim d(z, u_n) = 0 = \lim d(t, v_n)$. Finally, we have $d(x, z) \leq d(x, u_n) + d(u_n, z)$ and $d(y, t) \leq d(y, v_n) + d(v_n, t)$. Taking $n \to \infty$ in these inequalities, we get $d(x, z) = 0 = d(y, t)$, that is $x = z$ and $y = t$. Hence we get the result. □

Theorem 2.3. *In addition to hypotheses of Theorem 2.1, if x_0 and y_0 are comparable. Then F has a coupled fixed point, that is, there exists a $(x, y) \in X \times X$ such that $x = F(x, y) = F(y, x) = y$.*

Proof. By Theorem 2.1, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_n \to x$ and $y_n \to y$, where (x, y) is a coupled fixed point of F. Suppose $x_0 \leq y_0$. We shall show that $x_n \leq y_n$, where $x_n = F(x_{n-1}, y_{n-1})$, $y_n = F(y_{n-1}, x_{n-1})$, for all n. Suppose it holds for some $n \geq 0$. Then by strict mixed monotone property of F, we have $x_{n+1} = F(x_n, y_n) \leq F(y_n, x_n) = y_{n+1}$. From (2.1), we have

$$\begin{aligned}
d(x_{n+1}, y_{n+1}) &= d(F(x_n, y_n), F(y_n, x_n)) \\
&\leq \alpha\left(\frac{d(x_n, F(x_n, y_n))d(y_n, F(y_n, x_n))}{d(x_n, y_n)}\right) + \beta(d(x_n, y_n)) + \\
&\quad \gamma[d(x_n, F(x_n, y_n)) + d(y_n, F(y_n, x_n))] + \delta[d(x_n, F(y_n, x_n)) + d(y_n, F(x_n, y_n))] \\
&= \alpha\left(\frac{d(x_n, x_{n+1})d(y_n, y_{n+1})}{d(x_n, y_n)}\right) + \beta(d(x_n, y_n)) + \\
&\quad \gamma[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + \delta[d(x_n, y_{n+1}) + d(y_n, x_{n+1})].
\end{aligned}$$

On taking $n \to \infty$, we obtain $d(y, x) \leq (\beta + 2\delta)d(y, x)$. Since $\beta + 2\delta < 1$, $d(y, x) = 0$. Hence $F(x, y) = x = y = F(y, x)$.

A similar arguments can be used if $y_0 \leq x_0$. □

Remark 2.4. If $\gamma = 0 = \delta$ in above Theorems, then we have Theorems 2.1 and 2.2 of Ciric, Olatinwo, Gopal and Akinbo [12].

3 Applications

The aim of the section is to apply our new results to mappings involving contractions of integral type. For this purpose, denote by Λ the set of functions $\mu\colon [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

(h1) μ is a Lebesgue-integrable mapping on each compact subset of $[0, \infty)$;

(h2) for any $\epsilon > 0$, we have $\int_0^\epsilon \mu(t) > 0$.

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X \times X \to X$ is a self mapping on X such that F has the strict mixed monotone property on X. Suppose that there exists $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $\alpha + \beta + 2\gamma + 2\delta < 1$ such that*

$$\begin{aligned}
&\int_0^{d(F(x,y),F(u,v))} \psi(t)dt \\
\leq\ & \alpha \int_0^{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)}} \psi(t)dt + \beta \int_0^{d(x,y)} \psi(t)dt \\
&\gamma \int_0^{d(x,F(x,y))+d(u,F(u,v))} \psi(t)dt + \delta \int_0^{d(x,F(u,v))+d(u,F(x,y))} \psi(t)dt, \qquad (3.1)
\end{aligned}$$

for all $x, y, u, v \in X$, with $x \geq u$ and $y \leq v$, where $\psi \in \Lambda$.

Also suppose that either

a) F is continuous or

b) X has the following properties:

(i) if a sequence $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$ in X, then $x = \sup\{x_n\}$, for every n;

(ii) if a sequence $\{y_n\} \subset X$ is a non-increasing sequence with $y_n \to y$ in X, then $y = \inf\{y_n\}$, for every n.

If there exists two elements $x_0, y_0 \in X$ with $x_0 < F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$, then there exists $x, y \in X$ such that $F(x, y) = x$ and $y = F(y, x)$, that is, F has a coupled fixed point $(x, y) \in X \times X$.

If $\gamma = \delta = 0$, we have the following result.

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X \times X \to X$ is a self mapping on X such that F has the strict mixed monotone property on X. Suppose that there exists $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that*

$$\int_0^{d(F(x,y),F(u,v))} \psi(t)dt \quad \leq \quad \alpha \int_0^{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)}} \psi(t)dt + \beta \int_0^{d(x,y)} \psi(t)dt, \qquad (3.2)$$

for all $x, y, u, v \in X$, with $x \geq u$ and $y \leq v$, where $\psi \in \Lambda$.

Also suppose that either

a) F is continuous or

b) X has the following properties:

(i) if a sequence $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$ in X, then $x = \sup\{x_n\}$, for every n;

(ii) if a sequence $\{y_n\} \subset X$ is a non-increasing sequence with $y_n \to y$ in X, then $y = \inf\{y_n\}$, for every n.

If there exists two elements $x_0, y_0 \in X$ with $x_0 < F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$, then there exists $x, y \in X$ such that $F(x, y) = x$ and $y = F(y, x)$, that is, F has a coupled fixed point $(x, y) \in X \times X$.

If $\alpha = \gamma = \delta = 0$, we have the following result.

Theorem 3.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X \times X \to X$ is a self mapping on X such that F has the mixed monotone property on X. Suppose that there exists $\beta \in [0, 1)$ such that*

$$\int_0^{d(F(x,y),F(u,v))} \psi(t)dt \quad \leq \quad \beta \int_0^{d(x,y)} \psi(t)dt, \qquad (3.3)$$

for all $x, y, u, v \in X$, with $x \geq u$ and $y \leq v$, where $\psi \in \Lambda$.

Also suppose that either

a) F is continuous or

b) X has the following properties:

(i) if a sequence $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$ in X, then $x_n \leq x$, for every n;

(ii) if a sequence $\{y_n\} \subset X$ is a non-increasing sequence with $y_n \to y$ in X, then $y_n \geq y$, for every n.

If there exists two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exists $x, y \in X$ such that $F(x, y) = x$ and $y = F(y, x)$, that is, F has a coupled fixed point $(x, y) \in X \times X$.

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Received: June 5, 2014.

Accepted: October 29, 2014.