Palestine Journal of Mathematics

Vol. 5(1) (2016) , 127–130

A Connection Between the Modular $j-$Invariant and the Ramanujan's Cubic Continued Fraction

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Communicated by Ayman Badawi

MSC 2010 Classifications: 30B70; 33D15; 33D90; 11F20.

Keywords and phrases: Modular $j-$invariant; Ramanujan's cubic continued fraction.

Abstract. On page 393 of his third notebook Ramanujan defined two functions t_n and J_n closely connected to the modular j-invariant and listed some explicit values or simple polynomials satisfied by them. In order to establish Ramanujan's assertions, Berndt and Chan established a connection between the modular $j-$invariant and Ramanujan's cubic theory of elliptic functions to alternative bases. They also established that for certain values of n, t_n generates the Hilbert class field of $\mathbb{Q}(\sqrt{-n})$. In this paper, we establish a connection between the modular j-invariant and the Ramanujan's cubic continued fraction.

1 Introduction

On page 393 of his third notebook [11] (also see [4, p. 313, Entry 11.17]) Ramanujan defined the parameter t_n as

$$t_n = \frac{\sqrt{3}q^{1/18}f(q^{1/3})f(q^3)}{f^2(q)}, \qquad q = e^{-\pi\sqrt{n}}, \tag{1.1}$$

where, for $q = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$,

$$f(-q) := (q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n). \tag{1.2}$$

Ramanujan then asserted that

$$t_n = \left(2\sqrt{64J_n^2 - 24J_n + 9} - (16J_n - 3)\right)^{1/6}, \tag{1.3}$$

where

$$J_n = \frac{1-16\alpha_n(1-\alpha_n)}{8(4\alpha_n(1-\alpha_n))^{1/3}}, \tag{1.4}$$

is defined by Ramanujan [11, p. 392] for any natural number n and $\sqrt{\alpha_n} := \sqrt{\alpha(e^{-\pi\sqrt{n}})}$, $0 < \sqrt{\alpha_n} < 1$, is the singular modulus in the usual theory of elliptic functions. Ramanujan also considered some extremely simple polynomials satisfied by t_n for n =11, 35, 59, 83, and 107 from which the explicit values of t_n can be easily computed. From [5], we also note that

$$J_n = -\frac{1}{32}\sqrt[3]{j\left(\frac{3+\sqrt{-n}}{2}\right)}, \tag{1.5}$$

where $j(\tau)$, for $\tau \in \mathbb{H} = \{\tau : Im(\tau) > 0\}$, is the modular j-invariant. From (1.3) and (1.5) it is easily seen that the parameter t_n, J_n, and the modular j-invariant are closely connected. For 15 values of n, $n \equiv 3(mod\, 4)$, Ramanujan indicated the values of J_n, although not all values are given explicitly by him. There are 13 cases when the class number of the order in an imaginary quadratic fields equals 1 and the value of $j-$ invariant is known to be an integer. See [8, p. 260] for details. In these cases, Ramanujan explicitly recorded the values of J_n for n =3, 27, 11, 19, 35, 43, 67, 163, 51, 75, 91, 99, and 115. An account of these can be found in [4, p. 310-312]. Yi [12] also evaluated J_n for n =1, 2, 3, 4, 5, 6, 7, 8, 9 , and 10. More recently, Paek and Yi [9] evaluated new values of J_n for n =16, 32, 64, 128, and 256.

The motivation behind the study of modular $j-$invariant by Ramanujan is not clear. In order to establish Ramanujan's assertions, Berndt and Chan [5] established a connection between the modular $j-$invariant and Ramanujan's cubic theory of elliptic functions to alternative bases. They also established that for certain values of n, t_n generates the Hilbert class field of $\mathbb{Q}(\sqrt{-n})$. In this paper, we establish a connection between the modular $j-$invariant and Ramanujan's cubic continued fraction $G(q)$, where $G(q)$ is defined by

$$G(q) := \frac{q^{1/3}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{+\cdots}{}, \qquad |q|<1. \tag{1.6}$$

We prove general theorems for the explicit evaluation of $G(q)$ in terms of the functions t_n and J_n in Theorems 2.1 and 2.3, respectively and give examples. Over the years, many authors contributed to Ramanujan's cubic continued fraction $G(q)$ and its explicit evaluations (for example, see [1, 2, 6, 7, 10]).

2 Explicit Evaluations of $G(q)$

In this section we prove new general theorems for the explicit evaluation of Ramanujan's cubic continued fraction $G(q)$ in terms of the parameters t_n and J_n and give examples.

Theorem 2.1. *For any positive real number n, we have*

$$\frac{1}{G(-e^{-\pi\sqrt{n}})} + 4G^2(-e^{-\pi\sqrt{n}}) = \frac{3\left(-9 - t_n^6 - \sqrt{3}\sqrt{27+18t_n^6-t_n^{12}}\right)}{2t_n^6}.$$

Proof. From [3, p. 345, Entry 1(iv)], we note that

$$3 + \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)} = \left(27 + \frac{f^{12}(-q)}{qf^{12}(-q^3)}\right)^{1/3} = \frac{1}{G(q)} + 4G^2(q). \tag{2.1}$$

From (2.1) it is obvious that

$$\frac{1}{G(q)} + 4G^2(q) - 3 = \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)} \tag{2.2}$$

and

$$\left(\frac{1}{G(q)} + 4G^2(q)\right)^3 - 27 = \frac{f^{12}(-q)}{qf^{12}(-q^3)}. \tag{2.3}$$

Replacing $q^{1/3}$ by $-q^{1/3}$ in (2.2) and simplifying, we obtain

$$3 - \left(\frac{1}{G(-q)} + 4G^2(-q)\right) = \frac{f^3(q^{1/3})}{q^{1/3}f^3(q^3)}. \tag{2.4}$$

Again, replacing q by $-q$ in (2.3), we obtain

$$27 - \left(\frac{1}{G(-q)} + 4G^2(-q)\right)^3 = \frac{f^{12}(q)}{qf^{12}(q^3)}. \tag{2.5}$$

Set

$$X := \frac{1}{G(-q)} + 4G^2(-q). \tag{2.6}$$

Then combining (2.4) and (2.5), we obtain

$$\frac{(3-X)^2}{27-X^3} = \frac{q^{1/3}f^6(q^{1/3})f^6(q^3)}{f^{12}(q)}. \tag{2.7}$$

Equivalently,

$$\frac{3-X}{X^2+3X+9}=\left(\frac{q^{1/18}f(q^{1/3})f(q^3)}{f^2(q)}\right)^6. \tag{2.8}$$

Setting $q=e^{-\pi\sqrt{n}}$ in (2.8), employing the definition of t_n from (1.1), and simplifying, we obtain

$$t_n^6X^2+\left(3t_n^6+27\right)X+\left(9t_n^6-81\right)=0. \tag{2.9}$$

Solving (2.9) and noting the fact that $X<0$, we obtain

$$X=\frac{3\left(-9-t_n^6-\sqrt{3}\sqrt{27+18t_n^6-t_n^{12}}\right)}{2t_n^6}. \tag{2.10}$$

This completes the proof. □

Remark 2.2. From Theorem 2.1 it is obvious that the explicit value of $G(-e^{-\pi\sqrt{n}})$ can be evaluated if the explicit value of the function t_n is known for the corresponding value of n. For example, setting $n=1$ in Theorem 2.1, employing the value $t_{11}=1$ from [4, p. 314] and solving the resulting equation, we obtain

$$G(-e^{-\pi\sqrt{11}})=\frac{-5-\sqrt{33}+\left(-1+3\sqrt{3(23+4\sqrt{33})}\right)^{2/3}}{2\left(-1+\sqrt{621+108\sqrt{33}}\right)}.$$

Theorem 2.3. *For any positive real number n, we have*

$$\frac{1}{G(-e^{-\pi\sqrt{n}})}+4G^2(-e^{-\pi\sqrt{n}})=\frac{-3\left(6-8J_n+M+\sqrt{3}\sqrt{(3+8J_n)(3-16J_n+2M)}\right)}{3-16J_n+2M},$$

where $M=\sqrt{9-24J_n+64J_n^2}$.

Proof. Follows easily from (1.3) and Theorem 2.1. □

Remark 2.4. From Theorem 2.3 it is clear that if we know the explicit values of J_n then $G(-e^{-\pi\sqrt{n}})$ can be evaluated for the corresponding values of n. For example, setting $n=3$ in Theorem 2.3, employing the value $J_3=0$ from [4, p. 310, Entry 11.1] and solving the resulting equation, we obtain

$$G(-e^{-\pi\sqrt{3}})=\frac{1-2^{1/3}}{2^{2/3}}.$$

Similarly, other values of J_n can be used to evaluate the explicit values of $G(-e^{-\pi\sqrt{n}})$.

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Received: November 22, 2014.

Accepted: May 7, 2015