A Connection Between the Modular $j$--Invariant and the Ramanujan’s Cubic Continued Fraction

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Abstract. On page 393 of his third notebook Ramanujan defined two functions $t_n$ and $J_n$ closely connected to the modular $j$--invariant and listed some explicit values or simple polynomials satisfied by them. In order to establish Ramanujan’s assertions, Berndt and Chan established a connection between the modular $j$--invariant and Ramanujan’s cubic theory of elliptic functions to alternative bases. They also established that for certain values of $n$, $t_n$ generates the Hilbert class field of $\mathbb{Q}(\sqrt{-n})$. In this paper, we establish a connection between the modular $j$--invariant and the Ramanujan’s cubic continued fraction.

1 Introduction

On page 393 of his third notebook [11] (also see [4, p. 313, Entry 11.17]) Ramanujan defined the parameter $t_n$ as

\[ t_n = \frac{\sqrt[18]{3}q^{1/18}f(q^{1/3})f(q^3)}{f^2(q)}, \quad q = e^{-\pi \sqrt{n}}, \tag{1.1} \]

where, for $q = e^{2\pi iz}$ and $\text{Im}(z) > 0$,

\[ f(-q) := (q; q)_\infty = \prod_{n=1}^{\infty} (1 - q^n). \tag{1.2} \]

Ramanujan then asserted that

\[ t_n = \left(2\sqrt{64J_n^2 - 24J_n + 9 - (16J_n - 3)}\right)^{1/6}, \tag{1.3} \]

where

\[ J_n = \frac{1 - 16\alpha_n(1 - \alpha_n)}{8(4\alpha_n(1 - \alpha_n))^{1/3}}, \tag{1.4} \]

is defined by Ramanujan [11, p. 392] for any natural number $n$ and $\sqrt{\alpha_n} := \sqrt{\alpha(e^{-\pi \sqrt{n}})}<1$, is the singular modulus in the usual theory of elliptic functions. Ramanujan also considered some extremely simple polynomials satisfied by $t_n$ for $n = 11, 35, 59, 83, \text{ and } 107$ from which the explicit values of $t_n$ can be easily computed. From [5], we also note that

\[ J_n = -\frac{1}{32} \sqrt{j \left(\frac{3 + \sqrt{-n}}{2}\right)}, \tag{1.5} \]

where $j(\tau)$, for $\tau \in \mathbb{H} = \{\tau : \text{Im}(\tau) > 0\}$, is the modular $j$--invariant. From (1.3) and (1.5) it is easily seen that the parameter $t_n$, $J_n$, and the modular $j$--invariant are closely connected. For 15 values of $n$, $n \equiv 3(\text{mod } 4)$, Ramanujan indicated the values of $J_n$, although not all values are given explicitly by him. There are 13 cases when the class number of the order in an imaginary quadratic fields equals 1 and the value of $j$--invariant is known to be an integer. See [8, p. 260] for details. In these cases, Ramanujan explicitly recorded the values of $J_n$ for $n = 3, 27, 11, 19,$
Then combining (2.4) and (2.5), we obtain

Replacing $q$ from (2.1) it is obvious that

$$J_Y[12]$$ also evaluated $35, 43, 67, 163, 51, 75, 91, 99, \text{ and } 115.$ An account of these can be found in [4, p. 310-312].

The motivation behind the study of modular $j$–invariant by Ramanujan is not clear. In order to establish Ramanujan’s assertions, Berndt and Chan [5] established a connection between the modular $j$–invariant and Ramanujan’s cubic theory of elliptic functions to alternative bases. They also established that for certain values of $n$, $t_n$ generates the Hilbert class field of $\mathbb{Q}(\sqrt{-n})$. In this paper, we establish a connection between the modular $j$–invariant and Ramanujan’s cubic continued fraction $G(q)$, where $G(q)$ is defined by

$$G(q) := \frac{q^{1/3}}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \frac{q^3 + q^6}{1 + \cdots}}}, |q| < 1. \quad (1.6)$$

We prove general theorems for the explicit evaluation of $G(q)$ in terms of the functions $t_n$ and $J_n$ in Theorems 2.1 and 2.3, respectively and give examples. Over the years, many authors contributed to Ramanujan’s cubic continued fraction $G(q)$ and its explicit evaluations (for example, see [1, 2, 6, 7, 10]).

## 2 Explicit Evaluations of $G(q)$

In this section we prove new general theorems for the explicit evaluation of Ramanujan’s cubic continued fraction $G(q)$ in terms of the parameters $t_n$ and $J_n$ and give examples.

**Theorem 2.1.** For any positive real number $n$, we have

$$\frac{1}{G(-e^{-\pi\sqrt{n}})} + 4G^2(-e^{-\pi\sqrt{n}}) = \frac{3}{2t_n^6} \left(-9 - t_n^6 - \sqrt{3} \sqrt{27 + 18t_n^6 - t_n^{12}}\right).$$

**Proof.** From [3, p. 345, Entry 1(iv)], we note that

$$3 + \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)} = \left(27 + \frac{f^{12}(-q)}{q^{4}f^{12}(-q^3)}\right)^{1/3} = \frac{1}{G(q)} + 4G^2(q). \quad (2.1)$$

From (2.1) it is obvious that

$$\frac{1}{G(q)} + 4G^2(q) - 3 = \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)} \quad (2.2)$$

and

$$\left(\frac{1}{G(q)} + 4G^2(q)\right)^3 - 27 = \frac{f^{12}(-q)}{q^{4}f^{12}(-q^3)}. \quad (2.3)$$

Replacing $q^{1/3}$ by $-q^{1/3}$ in (2.2) and simplifying, we obtain

$$3 - \left(\frac{1}{G(-q)} + 4G(-q)\right) = \frac{f^{3}(q^{1/3})}{q^{1/3}f^{3}(q^3)}. \quad (2.4)$$

Again, replacing $q$ by $-q$ in (2.3), we obtain

$$27 - \left(\frac{1}{G(-q)} + 4G(-q)\right)^3 = \frac{f^{12}(q)}{q^{4}f^{12}(q^3)}. \quad (2.5)$$

Set

$$X := \frac{1}{G(-q)} + 4G(-q). \quad (2.6)$$

Then combining (2.4) and (2.5), we obtain

$$\frac{(3 - X)^2}{27 - X^3} = \frac{q^{1/3}f^6(q^{1/3})f^6(q^3)}{f^{12}(q)}. \quad (2.7)$$
Equivalently,
\[
\frac{3 - X}{X^2 + 3X + 9} = \left( \frac{q^{1/18} f(q^{1/3}) f(q^3)}{f^2(q)} \right)^6. \tag{2.8}
\]
Setting \( q = e^{-\pi \sqrt{27}} \) in (2.8), employing the definition of \( t_n \) from (1.1), and simplifying, we obtain
\[
t_n^6 X^2 + (3t_n^6 + 27) X + (9t_n^6 - 81) = 0. \tag{2.9}
\]
Solving (2.9) and noting the fact that \( X < 0 \), we obtain
\[
X = \frac{3 \left( -9 - t_n^6 - \sqrt{3} \sqrt{27 + 18t_n^6 - t_n^{12}} \right)}{2t_n^6}. \tag{2.10}
\]
This completes the proof. \( \Box \)

**Remark 2.2.** From Theorem 2.1 it is obvious that the explicit value of \( G(-e^{-\pi \sqrt{27}}) \) can be evaluated if the explicit value of the function \( t_n \) is known for the corresponding value of \( n \). For example, setting \( n = 1 \) in Theorem 2.1, employing the value \( t_{11} = 1 \) from [4, p. 314] and solving the resulting equation, we obtain
\[
G(-e^{-\pi \sqrt{27}}) = \frac{-5 - \sqrt{33} + \left( -1 + 3 \sqrt{3(23 + 4\sqrt{33})} \right)^{2/3}}{2 \left( -1 + \sqrt{621 + 108\sqrt{33}} \right)}.
\]

**Theorem 2.3.** For any positive real number \( n \), we have
\[
\frac{1}{G(-e^{-\pi \sqrt{27}})} + 4G^2(-e^{-\pi \sqrt{27}}) = \frac{-3 \left( 6 - 8J_n + M + \sqrt{3(3 + 8J_n)} (3 - 16J_n + 2M) \right)}{3 - 16J_n + 2M},
\]
where \( M = \sqrt{9 - 24J_n + 64J_n^2} \).

**Proof.** Follows easily from (1.3) and Theorem 2.1. \( \Box \)

**Remark 2.4.** From Theorem 2.3 it is clear that if we know the explicit values of \( J_n \) then \( G(-e^{-\pi \sqrt{27}}) \) can be evaluated for the corresponding values of \( n \). For example, setting \( n = 3 \) in Theorem 2.3, employing the value \( J_3 = 0 \) from [4, p. 310, Entry 11.1] and solving the resulting equation, we obtain
\[
G(-e^{-\pi \sqrt{27}}) = \frac{1 - 2^{1/3}}{2^{2/3}}.
\]
Similarly, other values of \( J_n \) can be used to evaluate the explicit values of \( G(-e^{-\pi \sqrt{27}}) \).

**References**


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