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SOLUTION OF DIFFERENTIAL EQUATIONS BASED ON HAAR OPERATIONAL MATRIX

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Abstract.

In the present paper, the solution of lower and higher order differential equations based on Haar operational method is considered. Haar wavelet method is used because its computation is simple as it converts the problem into an algebraic matrix equation. The results and graphs show that the proposed way is quite reasonable when compared to with existing exact solution.

1 Introduction

The idea of operational matrix was established via the Walsh function [1]. Conventional methods of deriving the operational matrix are difficult and not uniform. In this paper, we present a unified approach to derive the operational matrices of orthogonal functions for finding the solution of lower and higher order differential equations. The Method is computer oriented and simple; therefore it is very useful in practice.

In recent years wavelet approach has become more popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used for the numerical solution of initial and boundary value problems. Chen and Hsiao [2] have gained popularity, due to their useful contribution in wavelet. Lepik [9, 10, 11, 12] applied Haar wavelet for solving differential equations and partial differential equations.

In this present paper, a computational method for solving lower and higher order ordinary differential equation is introduced. This method consists of reducing the problem to a set of algebraic equations by first expanding the terms, which have maximum derivatives, given in the equation as Haar function with unknown coefficients. The operational matrix of integration and product operational matrix are utilized to evaluate the coefficients of the Haar functions. The differentiation of Haar wavelets results in impulse functions which must be avoided, the integration of Haar wavelet is preferred. Since the integration of the Haar functions vector is continuous function, the solutions obtained are continuous. This method is simple, fast, flexible, convenient and of small computational cost because it is fully computer supported, we don't need to solve it manually.

2 Haar wavelet operational matrix of ordiuary differential eqnation

2.1 Haar wevelet

Haar wavelet is the simplest wavelet. The Haar wavelet transform, proposed in 1910 Alfred Haar [6], is the first known wavelet. Haar wavelet transform has been used as an earliest example for orthonormal wavelet transform with compact support. The Haar wavelet is defined as $t \in [0, 1]$ The orthogonal set of Haar functions are defined in the interval $[0\ 1)$ by $h_0(t) = 1$

$$h_i(t) = \begin{cases} 1 & \frac{k-1}{2j} \leq t < \frac{k-0.5}{2j} \\ -1 & \frac{k-0.5}{2j} \leq t < \frac{k}{2j} \\ 0 & otherwise \end{cases} \tag{2.1}$$

where $i = 1, 2, \ldots m-1, m = 2^M$ and M is a positive integer, j and k represent the integer decomposition of the index i, i.e. $i = 2^j + k - 1, 0 \leq j < i$ and $1 \leq k < 2^j + 1$.
Any function $f(t) \in L^2([0,1])$ can be expanded in Haar series

$$f(t) = \sum_{i=0}^{\infty} c_i h_i(t), \tag{2.2}$$

where $c_i, i = 0, 1, 2 \cdots$ is the Haar coefficient, which is given by

$$c_i = 2^j \int_0^1 f(t) h_i(t) dt \tag{2.3}$$

These coefficients are determined in such a way that the following square error integral ϵ is minimized

$$\epsilon \int_0^1 [f(t) - \sum_{i=0}^{m-1} c_i h_i(t)]^2 dt, \ m = 2^j, j \in \{0\} \cup N.$$

The series expansion of $f(t)$ contains an infinite number of terms. If $f(t)$ is piecewise constant, or may be approximated as piecewise constant during each subinterval, then f(t) will be terminated at finite terms, i.e.

$$f(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) = C_m^T H_m(t) = \hat{f}(t) \tag{2.4}$$

where $m = 2^j$, the superscript T indicates transposition, $\hat{f}(t)$ denotes the truncated sum.

The Haar coefficient vector C_m and Haar function vector $H_m(t)$ are defined as

$$C_m \triangleq [c_0, c_1, c_2, \cdots c_{m-1}]^T. \tag{2.5}$$

$$H_m(t) \triangleq [h_0, h_1, h_2, \cdots h_{m-1}]^T. \tag{2.6}$$

The collection points are taken as follows

$$t_k = \frac{(2k-1)}{2m}, k = 1, 2, \cdots m \tag{2.7}$$

We defined the m-square Haar matrix $\psi_{m\times m}$ as:

$$\psi_{m\times m} \triangleq \left[H_m\left(\frac{1}{2m}\right) H_m\left(\frac{3}{2m}\right) H_m\left(\frac{5}{2m}\right) \cdots H_m\left(\frac{2m-1}{2m}\right) \right]. \tag{2.8}$$

2.2 Operational matrix

The integration of the $H_m(t)$, were approximated by Chen and Hsiao [2] as :

$$\int_0^t H_m(\tau) d\tau \cong P^1_{m\times m} H_m(t) \tag{2.9}$$

where $P^1_{m\times m}$ is the Haar wavelet operational matrix of integration, which is a square matrix of dimension $m \times m$

$$\int_0^t P^{n-1}_{m\times m} H_m(\tau) d\tau \cong P^n_{m\times m} H_m(t) \quad n = 2, 3 \cdots \tag{2.10}$$

Also, we define an m-set of block pulse function [9] as:

$$b_i(t) = \begin{cases} 1 & \frac{i}{m} \leq t < \frac{i+1}{m} \\ \\ 0 & otherwise \end{cases} \tag{2.11}$$

where $i = 0, 1, 2, \cdots (m-1)$.

The function $b_i(t)$ is disjoint and orthogonal. That is,

$$b_i(t)b_l(t) = \begin{cases} 0, & i \neq l, \\ \\ b_i(t) & i = l \end{cases} \tag{2.12}$$

$$\int_0^1 b_i(\tau)b_l(\tau)d\tau = \begin{cases} 0, & i \neq l, \\ \\ \frac{1}{m}, & i = l, \end{cases} \tag{2.13}$$

Since Haar functions are piecewise constant, it may be expanded into an m-term block pulse functions

$$H_m(t) = \psi_{m\times m} B_m(t) \tag{2.14}$$

where $B_m(t) \triangleq [b_0(t)b_1(t)\cdots b_{m-1}(t)]^T$.
Kilicman and AL Zhour [8] have given the & Block pulse operational matrix F^n as follows

$$(I^n B_m)(t) \approx F^n B_m(t) \tag{2.15}$$

Where I^n shows that nth integration of function.
Where

$$F^n = \frac{1}{m^n}\frac{1}{(n+1)!}\begin{bmatrix} 1 & \eta_1 & \eta_2 & \cdots & \eta_{m-1} \\ 0 & 1 & \eta_1 & \cdots & \eta_{m-2} \\ 0 & 0 & 1 & \cdots & \eta_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} n \in I \tag{2.16}$$

with

$$\eta_k = (k+1)^{n+1} - 2k^{n+1} + (k-1)^{n+1}. \tag{2.17}$$

Next, we derive Haar wavelet operational matrix for general order integration.

Let

$$(I^n H_m)(t) \approx P^n_{m\times m} H_m(t). \tag{2.18}$$

Now from (14) and (15) we get

$$(I^n H_m)(t) \approx \psi_{m\times m} F^n B_m(t) \tag{2.19}$$

Now from (18), (19) and (16) we have

$$P^n_{m\times m} = \psi_{m\times m} F^n \psi^{-1}_{m\times m}. \tag{2.20}$$

3 Application of Haar wavelet method

In this section, we are using the operational matrix of Haar wavelet for finding the numerical solution of ordinary differential equations. All calculations have been done by Matlab programming.

Example: 2 Let us consider

$$y^{''} + 2y^{'} + 5y = f(t) \tag{3.1}$$

where $f(t) = 3e^{-t}sint$
and subject to $y(0) = 0, y^{'}(0) = 1$
Exact solution of equation (21) by homotopy petutbaton method (HPM) is $y = e^{-t}sint$.

Let

$$y'(t) = C_m^T H_m(t). \tag{3.2}$$

Integrating equation (22) with respect to t from 0 to t and using initial conditions

$$y'(t) = C_m^T P_{m\times m}^1 H_m(t) + 1, \tag{3.3}$$

$$y(t) = C_m^T P_{m\times m}^2 H_m(t) + [111\cdots 1]\psi_{m\times m}^{-1} P_{m\times m}^1 H_m(t), \tag{3.4}$$

$f(t)$ can be expanded by Haar function as

$$f(t) = f_m^T(t) H_m(t), \tag{3.5}$$

where $f_m^T(t)$ is a known constant vector

Now substituting equations (22), (23), (24) and (25) in equation (21), we get

$$\begin{aligned} C_m^T H_m(t) + 2[C_m^T P_{m\times m}^1 H_m(t) + [111\cdots 1] + 5[C_m^T P_{m\times m}^2 H_m(t) \\ +[111\cdots 1]\psi_{m\times m}^{-1} P_{m\times m}^1 H_m(t)] = 3f_m^T(t) H_m, \end{aligned} \tag{3.6}$$

$$\begin{aligned} C_m^T H_m(t) + 2P_{m\times m}^1 H_m(t) + 5P_{m\times m}^2 H_m(t)] = 3f_m^T(t) H_m \\ -2[111\cdots 1] - 5[111\cdots 1]\psi_{m\times m}^{-1} P_m^1 H_m(t). \end{aligned} \tag{3.7}$$

Equation (27) is algebraic form of equation (21). After solving the system of algebraic equations, we can obtain the Haar coefficient C_m^T . Then from equation (24), we can calculate values of $y(t)$, which are quite similar with those of the exact solution. The numerical result for $m = 8$ is shown in table 1 and figure 1.

Table - 1

t	Exact solution by HPM	Haar solution	Absolute Error
0.0625	0.0587	0.0576	0.0011
0.1875	0.1545	0.1539	0.0006
0.3125	0.2249	0.2245	0.0004
0.4375	0.2735	0.2735	0.0000
0.5625	0.3039	0.3042	0.0003
0.6875	0.3191	0.3192	0.0001
0.8125	0.3222	0.3225	0.0003
0.9375	0.3157	0.3160	0.0003

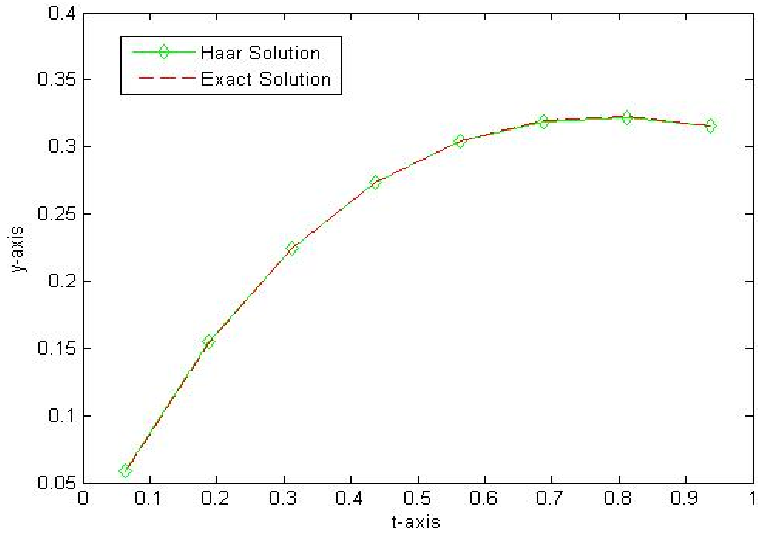

Figure-1

Example: 1 We consider an ordinary differential equation with variable coefficient

$$ty^{''} + (1 - 2t)y^{'} - 2y = f(t) \tag{3.8}$$

Subject to $y(0) = 1, y^{'}(0) = 2$ where $f(t) = 0$
Exact solution of equation (28) by homotopy petutbaton method (HPM) is $y = e^{2t}$.
Let

$$y''(t) = C_m^T H_m(t). \tag{3.9}$$

Integrating equation (22) twice with respect to t from 0 to t and using initial conditions

$$y'(t) = C_m^T P_{m\times m}^1 H_m(t) + 2 \tag{3.10}$$

$$y(t) = C_m^T P_{m\times m}^2 H_m(t) + 2[111\cdots 1]\psi_{m\times m}^{-1} P_{m\times m}^1 H_m(t) + 1, \tag{3.11}$$

$f(t)$ can be expanded by Haar function as

$$f(t) = f_m^T(t) H_m(t) \tag{3.12}$$

where $f_m^T(t)$ is a known constant vector

Now substituting equations (29), (30), (31) and (32) in equation (28), we get

$$tC_m^T H_m(t) + (1 - 2t)[C_m^T P_{m\times m}^1 H_m(t) + 2[111\cdots 1]$$

$$-2[C_m^T P_{m\times m}^2 H_m(t) + 2[111\cdots 1]\psi_{m\times m}^{-1} P_{m\times m}^1 H_m(t)] = [111\cdots 1] = 0, \tag{3.13}$$

$$C_m^T[tH_m(t) + (1 - 2t)P_{m\times m}^1 H_m(t) - 2P_{m\times m}^2 H_m(t)]$$

$$= 2[111\cdots 1] - (1 - 2t)2[111\cdots 1] + 4[111\cdots 1]\psi_{m\times m}^{-1} P_{m\times m}^1 H_m(t). \tag{3.14}$$

Equation (34) is algebraic form of equation (28). After solving the system of algebraic equations, we can obtain the Haar coefficient C_m^T . Then from equation (31), we can calculate value of $y(t)$, Which approximate the exact solution. The numerical result for $m = 8$ is shown in table 2 and figure 2.

Table - 2

t	Exact solution by HPM	Haar solution	Absolute Error
0.0625	1.1331	1.1364	0.0033
0.1875	1.4550	1.4596	0.0046
0.3125	1.8682	1.8758	0.0076
0.4375	2.3989	2.4088	0.0099
0.5625	3.0802	3.0931	0.0129
0.6875	3.9551	3.9756	0.0205
0.8125	5.0784	5.1078	0.0294
0.9375	6.5208	6.5637	0.0429

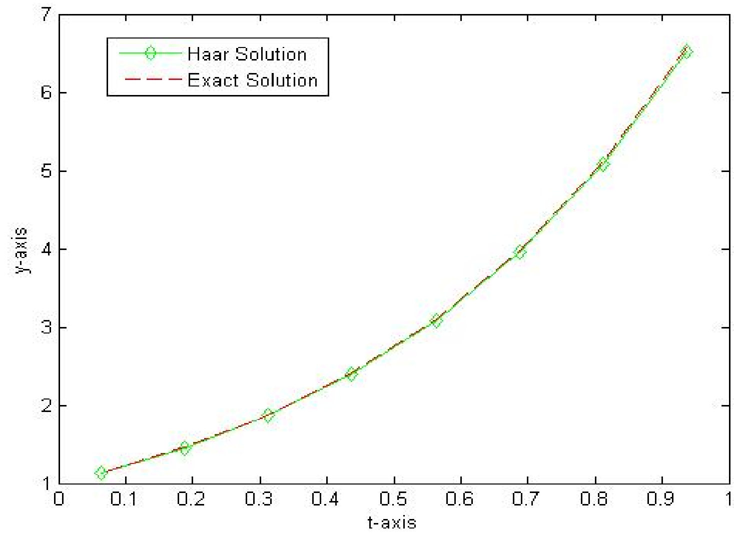

Figure-2

Example: 3 We consider the following eighth order differential equation:

$$y^{8}(t) = y(t) - 8e^{t}\text{where} \quad 0 \leq t \leq 1, \tag{3.15}$$

subjected to initial conditions

$$y(0) = 1,\ y^{'}(0) = 0,\ y^{''}(0) = -1, y^{'''}(0) = -2, y^{iv}(0) = -3,$$
$$y^{v}(0) = -4, y^{iv}(0) = -5, y^{vii}(0) = -6,$$

Now by homotopy petutbaton method (HPM)[5] exact solution of equation (35) is $y(t) = (1-t)e^{t}$
Let

$$y^{(8)}(t) = C_m^T H_m(t) \tag{3.16}$$

Now from successive integration of equation (36) with respect to t from 0 to t and using the initial conditions, we get

$$\begin{aligned} y(t) = C_m^T P^8_{m\times m} H_m(t) - 6[111\cdots 1]\psi^{-1}_{m\times m} P^7_{m\times m} H_m(t) \\ -5[111\cdots 1]\psi^{-1}_{m\times m} P^6_{m\times m} H_m(t) - 4[111\cdots 1]\psi^{-1}_{m\times m} P^5_{m\times m} H_m(t) \\ -3[111\cdots 1]\psi^{-1}_{m\times m} P^4_{m\times m} H_m(t) - 2[111\cdots 1]\psi^{-1}_{m\times m} P^3_{m\times m} H_m(t) \\ -[111\cdots 1]\psi^{-1}_{m\times m} P^2_{m\times m} H_m(t) + 1. \end{aligned} \tag{3.17}$$

Now from equation (35)

$$C_m^T[H_m(t) - P^8_{m\times m} H_m(t)] = -6[111\cdots 1]\psi^{-1}_{m\times m} P^7_{m\times m} H_m(t)$$
$$-5[111\cdots 1]\psi^{-1}_{m\times m} P^6_{m\times m} H_m(t) - 4[111\cdots 1]\psi^{-1}_{m\times m} P^5_{m\times m} H_m(t)$$
$$-3[111\cdots 1]\psi^{-1}_{m\times m} P^4_{m\times m} H_m(t) - 2[111\cdots 1]\psi^{-1}_{m\times m} P^3_{m\times m} H_m(t)$$

$$-[111\cdots 1]\psi^{-1}_{m\times m}P^2_{m\times m}H_m(t)+[111\cdots 1]-8f^T_m(t)H_m, \tag{3.18}$$

which is the algebraic form of equation (35), we can calculate the value of Haar coefficient C^T_m, after solving system of the algebraic equations for different values of t. Now substitute values of those coefficients in equation (36), we get numerical solution of equation (35), which is shown in following table 3 and figure 3.

Table - 3

t	Exact solution by HPM	Haar solution	Absolute Error
0.0625	0.9980	0.9970	0.0010
0.1875	0.9812	0.9790	0.0022
0.3125	0.9397	0.9386	0.0011
0.4375	0.8712	0.8693	0.0019
0.5625	0.7678	0.7661	0.0017
0.6875	0.6215	0.6193	0.0022
0.8125	0.4225	0.4200	0.0025
0.9375	0.1596	0.1565	0.0031

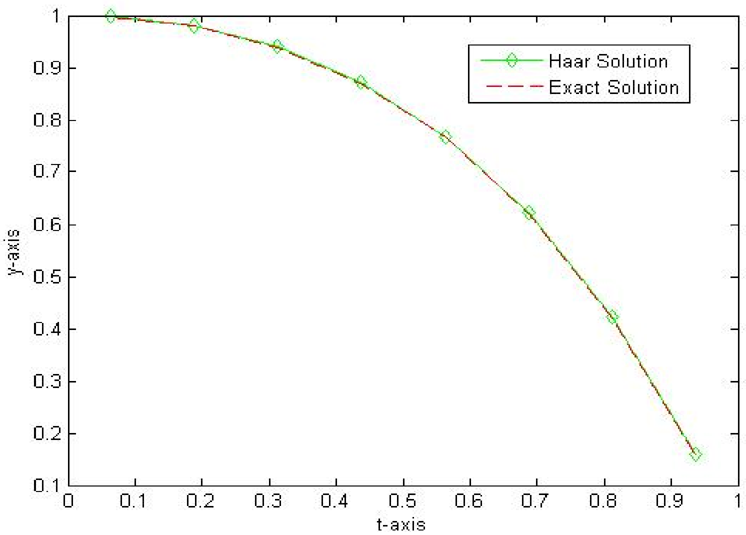

Figure-3

4 Conclusion

The main goal of this paper is to demonstrate that the Haar wavelet operational method is a powerful tool for solving lower and higher order differential equations. The result is compared with the exact solutions. It is worth mentioning that Haar solution provides excellent result even for small values of $m(m = 8)$. For large values of $m(m = 16, m = 32)$, we can also obtain the results closer to exact values.

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