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Radical of Primary-like submodules satisfying the primeful property

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Abstract. In this article, we develop the tool of saturation in the context of primary-like submodules of modules. We are particularly interested in relationships among the saturation of a primary-like submodule satisfying the primeful property and its radical. Furthermore, we provide sufficient conditions involving saturation and torsion arguments under which the radical of such a submodule is prime.

1 Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. For a submodule N of an R-module M, we let $(N : M)$ denote the ideal $\{r \in R \mid rM \subseteq N\} = Ann(\frac{M}{N})$. A proper submodule P of M is said to be prime (resp. primary) or p-prime (resp. p-primary) if whenever $rm \in P$ for $r \in R$ and $m \in M$, then $m \in P$ or $r \in p = (P : M)$ (resp. $r \in p = \sqrt{(P : M)}$) [9, 10]. Note that for any ideal I of R, $\sqrt{I} = \{r \in R \mid r^n \in I$ for some positive integer $n\}$. For a submodule N of M the intersection of all prime submodules of M containing N is called the radical of N and denoted by rad N [7].

As a new generalization of a primary ideal on the one hand and a generalization of a prime submodule on the other hand, a proper submodule N of M is said to be primary-like if $rm \in N$ implies $r \in (N : M)$ or $m \in \text{rad}\, N$ [4]. An R-module M is said to be primary-like if the zero submodule of M is primary-like.

We say that a submodule N of an R-module M satisfies the primeful property if for each prime ideal p of R with $(N : M) \subseteq p$, there exists a prime submodule P containing N such that $(P : M) = p$. If N is a submodule of M satisfying the primeful property, then $(\text{rad}\, N : M) = \sqrt{(N : M)}$ [7, Proposition 5.3]. An R-module M is called primeful if $M = 0$ or the zero submodule of M satisfies the primeful property. For instance finitely generated modules, projective modules over domains and (finite and infinite dimensional) vector spaces are primeful [7]. In [4, Lemma 2.1], it has been shown that if N is a primary-like submodule of an R-module M satisfying the primeful property, then $p = \sqrt{(N : M)}$ is a prime ideal of R. By the p-primary-like submodule N, we mean the primary-like submodule N with $p = \sqrt{(N : M)}$.

The primary-like spectrum $Spec_L(M)$ (resp. p-primary-like spectrum $Spec^p_L(M)$) is defined to be the set of all primary-like (resp. p-primary-like) submodules of M satisfying the primeful property. If the submodule N of M satisfies the primeful property, then there exists a maximal ideal $\mathfrak{m}$ of R and a prime submodule P of M containing N such that $(P : M) = \mathfrak{m}$. In this case, rad $N \neq M$ and rad N satisfies the primeful property.

Let M be an R-module. We say that a submodule N of M has a primary-like decomposition if $N = N_1 \cap N_2 \cap \cdots \cap N_k$, where each N_i is a primary-like submodule of M. If $N_i \not\supseteq N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_k$ and the ideals $\sqrt{(N_i : M)}$ (resp. the submodules rad N_i) are distinct primes, then the above primary-like decomposition for N is said to be reduced (resp. module-reduced). In this paper, we investigate the behavior of the radical with respect to primary-like submodules satisfying the primeful property by using the saturation and torsion arguments. In particular, we provide conditions under which the radical of such submodules is prime; a necessary condition for the existence of the module-reduced primary-like decomposition for a submodule. However the radical of primary-like submodules satisfying the primeful property is prime in some certain classes of modules automatically. For example if N is a submodule of a multiplication R-module M (i. e., a module whose submodules have the form IM for some ideal I of R), then rad N is a prime submodule for every $N \in Spec_L(M)$ [4, Proposition 4.10].

For a prime ideal p of R, the submodule $S_p(N) = \{m \in M : cm \in N \text{ for some } c \in R\backslash p\}$ is called the saturation of N with respect to p [8]. The second section has been devoted to the relationship between the saturation and radical of primary-like submodules. It is easily verified that $S_p(N) \subseteq \operatorname{rad} N$ for every $p \in V((N : M))$. In particular, if $S_p(N)$ is a prime submodule of M for some $p \in V((N : M))$, then $S_p(N) = \operatorname{rad} N$. Thus for every primary-like submodule N with $p = \sqrt{(N : M)}$, it is more convenient to check that $S_p(N)$ is prime. We prove that if N is a p-primary-like submodule of M, then $p = (\operatorname{rad}(S_p(N)) : M) = (S_p(N + pM) : M)$ (Theorem 2.7). In particular, if $\mathfrak{m}$ is a maximal ideal of R and N is an $\mathfrak{m}$-primary-like submodule of M, then $\operatorname{rad} N = \operatorname{rad}(S_{\mathfrak{m}}(N)) = S_{\mathfrak{m}}(\operatorname{rad} N) = S_{\mathfrak{m}}(N + \mathfrak{m}M) = N + \mathfrak{m}M$ (Corollry 2.8).

In the third section, using a torsion argument, we give some conditions under which the radical of a primary-like submodule is prime. Specially it is shown that, if M is a module over a Noetherian ring R and the torsion submodule $T(M)$ satisfies the primeful property and is contained in only finitely many prime submodules of M, then the radical of each element of $Spec_L(M)$ is a prime submodule of M (Theorem 3.3). Also it is proved that, if M is a primary-like, primeful and torsion module over a one-dimensional domain R, then the R-module $\frac{M}{S_p(0)}$ is isomorphic to the R-module M_p, where $p = \sqrt{Ann(M)}$ and M_p is the localization of M at p (Theorem 3.8). Using this fact we conclude that $S_p(N)$ has a reduced primary-like decomposition if and only if the R-module N_p has a reduced primary-like decomposition (Corollary 3.9).

2 Radical and saturation

In this section, we investigate the behavior of primary-like submodules satisfying the primeful property under the tool of saturation of submodules. In particular, the interplay between saturation and radical of such modules are considered.

Lemma 2.1. *Let N be a primary-like submodule of an R-module M. Then $S_p(N) \subseteq \operatorname{rad} N$ for every $p \in V((N : M))$. In particular, if $S_p(N)$ is a prime submodule of M for some $p \in V((N : M))$, then $S_p(N) = \operatorname{rad} N$.*

Proof. Straightforward. □

Lemma 2.2. *Let M be an R-module and N be a primary-like submodule of M. If $p = (N : M)$ is a prime ideal of R, then $S_p(N) = M$ or $\operatorname{rad} N$ is a prime submodule of M.*

Proof. Suppose $S_p(N) \neq M$. By [8, Proposition 2.4], $S_p(N)$ is a prime submodule of M. It follows from Lemma 2.1 $\operatorname{rad} N$ is a prime submodule of M. □

Theorem 2.3. *Let M be an R-module and $N \in Spec_L^p(M)$. Then the following statements are equivalent.*

(i) $\operatorname{rad} N$ *is a p-prime submodule of M.*

(ii) $\operatorname{rad} N$ *is a p-primary submodule of M.*

(iii) $\operatorname{rad} N$ *is a p-primary-like submodule of M.*

Furthermore, if $(N : M) = p$, then the above statements are also equivalent to:

(iv) N is a p-primary-like submodule of M.

Proof. (iv)⇒(i) Since N is a primary-like submodule of M, we have $N \subseteq S_p(N) \subseteq \operatorname{rad} N$ by Lemma 2.1. Hence $(S_p(N) : M) = (N : M) = p$ and so $S_p(N) \neq M$. Thus $\operatorname{rad} N$ is a p-prime submodule of M by Lemma 2.2. The verification of the other implications is straightforward. □

Theorem 2.4. *Let M be an R-module and $N \in Spec_L^p(M)$. Then $S_p(N)$ is a p-primary and p-primary-like submodule of M.*

Proof. Using [4, Lemma 2.1] and Lemma 2.1, we have

$$p = \sqrt{(N : M)} \subseteq \sqrt{(S_p(N) : M)} \subseteq (\operatorname{rad} N : M) = \sqrt{(N : M)} = p.$$

It follows that $\sqrt{(S_p(N) : M)} = p$. We first show that $S_p(N)$ is a primary submodule. Suppose $rm \in S_p(N)$ and $m \notin S_p(N)$. Then there exists $c \in R\backslash p$ such that $crm \in N$ and $cm \notin N$. Therefore $cr \in p$ and so $r \in p$. Thus $S_p(N)$ is a p-primary submodule of M. Now, we show that $S_p(N)$ is a primary-like submodule of M. Let $rm \in S_p(N)$. Then there is $c \in R\backslash p$ such that

$crm \in N$. Since N is primary-like, we have $cr \in (N : M)$ or $m \in \text{rad}\, N \subseteq \text{rad}(S_p(N))$. Thus $r \in (N : M)$ or $m \in \text{rad}(S_p(N))$ because $(N : M)$ is a primary ideal of R. Therefore $S_p(N)$ is also a p-primary-like submodule of M. □

Corollary 2.5. *Let M be an R-module and $N \in Spec_L^p(M)$. If $(S_p(N) : M)$ is a radical ideal, then* rad N *is a prime submodule of M.*

Proof. By the proof of Theorem 2.4, $\sqrt{(S_p(N) : M)} = (\text{rad}\, N : M)$. Now, since $(S_p(N) : M)$ is a radical ideal, we have $(S_p(N) : M) = p$. It follows from [8, Theorem 2.3] and Lemma 2.1, rad N is a prime submodule of M. □

Theorem 2.6. *Let M be an R-module and $N \in Spec_L^p(M)$. Then $S_p((N : M)) = (S_p(N) : M)$. In particular, the following statements hold and are equivalent.*

(i) $S_p(N)$ is a p-primary submodule of M.

(ii) $\sqrt{S_p((N : M))} = p$.

(iii) p is a minimal prime ideal of $(N : M)$.

Proof. It is easy to verify that $S_p((N : M)) \subseteq (S_p(N) : M)$. For the reverse inclusion, let $r \in (S_p(N) : M)$ and $m \in M \setminus \text{rad}\, N$. Then there exists $c \in R \setminus p$ such that $crm \in N$. Since N is a primary-like submodule of M, we have $cr \in (N : M)$ and hence $r \in S_p((N : M))$. Thus $S_p((N : M)) = (S_p(N) : M)$. Since $N \in Spec_L^p(M)$, then $S_p(N)$ is a p-primary submodule of M by Theorem 2.4. Now, we show that the statements are equivalent. (i)⇒(ii) is clear.
(ii)⇔(iii) follows from [1, P. 55, Ex. 10, ii and P. 56, Ex. 11].
(iii)⇒(i) Suppose that $rm \in S_p(N)$ and $r \notin \sqrt{(S_p(N) : M)}$ for $r \in R$ and $m \in M$. Hence $m \in S_p(S_p(N)) = S_p(N)$ and so $S_p(N)$ is a p-primary submodule of M. □

Theorem 2.7. *Let M be an R-module and $N \in Spec_L^p(M)$. Then* $\text{rad}(S_p(N)) \subseteq S_p(N + pM) \subseteq S_p(\text{rad}\, N)$. *In particular,* $p = (\text{rad}(S_p(N)) : M) = (S_p(N + pM) : M)$.

Proof. Since N satisfies the primeful property, $S_p(N + pM)$ is a p-prime submodule of M by [7, Proposition 4.4] and so $\text{rad}(S_p(N)) \subseteq S_p(N + pM)$. Suppose $x \in S_p(N + pM)$. Then there exists $c \in R \setminus p$ such that $cx \in N + pM$. Since $\sqrt{(N : M)} = p$ and $cx \in \text{rad}\, N$, we conclude that $x \in S_p(\text{rad}\, N)$. Also we have $p = (\text{rad}\, N : M) \subseteq (\text{rad}(S_p(N)) : M) \subseteq (S_p(N + pM) : M) \subseteq \sqrt{(S_p(N + pM) : M)} = p$, as required. □

Corollary 2.8. *Let $\mathfrak{m}$ be a maximal ideal of R, M be an R-module and $N \in Spec_L^{\mathfrak{m}}(M)$. Then*

$$\text{rad}\, N = \text{rad}(S_{\mathfrak{m}}(N)) = S_{\mathfrak{m}}(\text{rad}\, N) = S_{\mathfrak{m}}(N + \mathfrak{m}M) = N + \mathfrak{m}M \quad (*).$$

Proof. It is easy to check that, $N + \mathfrak{m}M = \text{rad}\, N$. By Theorem 2.7, $\text{rad}\, N \subseteq \text{rad}(S_{\mathfrak{m}}(N)) \subseteq S_{\mathfrak{m}}(\text{rad}\, N)$. Since rad N is $\mathfrak{m}$-prime, then $S_{\mathfrak{m}}(\text{rad}\, N) = \text{rad}\, N$ and so the equality $(*)$ holds. □

Remark 2.9. Let R be an Artinian ring. In [2, Theorem 2.16], it has been shown that every R-module is primeful. Now, if N is a primary-like submodule of an R-module M, then N satisfies the primeful property and so rad N is an $\mathfrak{m}$-prime submodule of M, where $\mathfrak{m} = \sqrt{(N : M)}$. Furthermore, the equality $(*)$ in Corollary 2.8 holds again.

3 Radical and torsion

The torsion submodule of a module M over a domain R, denoted by $T(M)$, is the submodule $\{m \in M : Ann(m) \neq 0\}$ of M. An R-module M is said to be torsion (resp. torsion-free), if $T(M) = M$ (resp. $T(M) = 0$).

Proposition 3.1. *Let M be an R-module and $N \in Spec_L(M)$. Then* rad N *is a prime submodule of M if and only if $T(\frac{M}{\text{rad}\, N}) = 0$ as an $\frac{R}{\sqrt{(N:M)}}$-module.*

Proof. Suppose N is a primary-like submodule of M satisfying the primeful property. By [4, Lemma 2.1], $\sqrt{(N : M)} = (\text{rad}\, N : M)$ is a prime ideal of R and so the proof is completed by [5, Lemma 1]. □

Theorem 3.2. *Let M be a module over a Dedekind domain R and $N \in Spec_L(M)$. Then* rad N *is a prime submodule of M if and only if $M = \text{rad}\, N \bigoplus N'$ for some torsion-free submodule N' of M or* $(\text{rad}\, N : M) = \mathfrak{m}$ *for some maximal ideal $\mathfrak{m}$ of R.*

Proof. Suppose first that $\text{rad}\, N$ is a 0-prime submodule of M. It follows from Lemma 3.1 $\frac{M}{\text{rad}\, N}$ is a torsion-free R-module. Hence by [3, Exercise 19.6(a)] $\frac{M}{\text{rad}\, N}$ is projective and so $M = \text{rad}\, N \bigoplus N'$ for some submodule N' of M. Clearly N' is torsion-free. Now, let $\text{rad}\, N$ be a prime submodule of M with $(\text{rad}\, N : M) \neq 0$. Since R is Dedekind domain, $(\text{rad}\, N : M)$ is a maximal ideal of R. Conversely, suppose $M = \text{rad}\, N \bigoplus N'$ for some torsion-free submodule N' of M. Then $\frac{M}{\text{rad}\, N} \cong N'$ follows that $\frac{M}{\text{rad}\, N}$ is torsion-free and hence $\text{rad}\, N$ is a 0-prime submodule of M by [5, Lemma 1]. On the other hand, it is easy to verify that $\text{rad}\, N$ is prime when $(\text{rad}\, N : M)$ is a maximal ideal. □

Theorem 3.3. *Let R be a Noetherian domain and M be a non-torsion R-module such that $T(M)$ satisfies the primeful property and is contained in only finitely many prime submodules of M. Let $N \in Spec_L(M)$. Then* $\text{rad}\, N$ *is a prime submodule of M.*

Proof. By Theorem 2.3 we may assume that $(N : M) \neq 0$. If P is a prime submodule containing N, we have the chain $0 = (T(M) : M) \subset \sqrt{(N : M)} \subseteq (P : M)$ of prime ideals of R. If the later containment is proper, then by [6, P. 144] there are infinitely many prime ideals p with $(T(M) : M) \subset p \subset (P : M)$ and so we have infinitely prime submodules P containing $T(M)$, a contradiction. Hence we have $\sqrt{(N : M)} = (P : M)$, for all prime submodules P containing N. Now, if $rm \in \text{rad}\, N$ and $m \notin \text{rad}\, N$, there is a prime submodule P containing N such that $rm \in P$ and $m \notin P$ and therefore $r \in (P : M) = \sqrt{(N : M)} = (\text{rad}\, N : M)$. □

Let M be an R-module. The dimension of M is defined by $dimM = \underset{n}{Sup}\{P_0 \subset P_1 \subset \cdots \subset P_n \mid P_i \text{ is a prime submodule of } M\}$.

Theorem 3.4. *Let R be a one-dimensional domain and M be a one-dimensional torsion module over R such that every prime submodule of M is contained in $Spec_L(M)$. Then the following are equivalent.*

(i) 0 *is a prime submodule of M;*

(ii) $P_1 \cap P_2 = 0$ *for any distinct prime submodules P_1 and P_2;*

(iii) Every non-zero element N of $Spec_L(M)$ is contained in exactly one prime submodule;

(iv) Every non-zero prime submodule is maximal.

Proof. (i)⇒(ii). Since $T(M) = M$, for each $0 \neq m \in M$ there exists $0 \neq r$ such that $rm = 0$. Hence by (i) we have $r \in (0 : M)$ and so $(0 : M) \neq 0$. Now, if P is a non-zero prime submodule of M, then $(0 : M) = (P : M)$ since $dimR = 1$ and $0 \subset (0 : M) \subseteq (P : M)$ is a chain of prime ideals. In particular, for distinct non-zero prime submodules P_1 and P_2 we have $(0 : M) = (P_1 : M) = (P_2 : M)$ and so $P_1 \cap P_2$ is prime. We have the chain $0 \subseteq P_1 \cap P_2 \cap P_1$. Since $dim(M) = 1$, $P_1 \cap P_2 = 0$ or $P_1 \subset P_2$ which follows $P_1 = 0$.
(ii)⇒(iii) Since N satisfies the primeful property, there exists a prime submodule P such that $(P : M) = \sqrt{(N : M)}$. Now, if N is contained in more than one prime submodule, then it contradicts with (ii).
(iii)⇒(iv) is clear because $Spec_L(M)$ contains the set of all prime submodules of M.
(iv)⇒(i) Since $dimM = 1$, there must exist a chain of prime submodules $P_1 \subset P_2$ and so $P_1 = 0$ by (iv). □

Note that if the assumptions of Theorem 3.4 are satisfied, then $\text{rad}\, N$ is prime for all submodules $N \in Spec_L(M)$.
For an R-module M and $m \in M$, we mean that $(N : m)$ is the set $\{r \in R : rm \in N\}$. Now, we have the following elementary lemma.

Lemma 3.5. *Let M be an R-module. Then N is a primary-like submodule of M if and only if $(N : M) = (N : m)$ for all $m \in M \setminus \text{rad}\, N$.*

Theorem 3.6. *Let M be a primary-like and primeful module over a one-dimensional domain R. Then either $\sqrt{Ann(M)} = 0$ or $\sqrt{Ann(M)} = \sqrt{(N : M)}$ for all proper submodules N of M. In particular, if M is a non-cyclic torsion module, then $\sqrt{(Rm : M)} = \sqrt{Ann(m)}$ for all $m \in M \setminus \text{rad}\, 0$.*

Proof. Suppose $\sqrt{Ann(M)} \neq 0$. Since R is a one-dimensional domain, $\sqrt{Ann(M)}$ is a maximal ideal of R. It follows that $\sqrt{Ann(M)} = \sqrt{(N : M)}$ for all proper submodules N. Since 0 is a primary-like submodule satisfying the primeful property, $\text{rad}\, 0 \neq M$. Now, if M is a torsion module, then $\sqrt{Ann(M)} \neq 0$. Again since 0 is primary-like, $Ann(M) = Ann(m)$ for all $m \in M \setminus \text{rad}\, 0$ by Lemma 3.5. Since Rm is a proper submodule for all $m \in M$, by the first part $\sqrt{(Rm : M)} = \sqrt{Ann(M)} = \sqrt{Ann(m)}$ □

Theorem 3.7. *Let M be a primary-like, primeful and torsion module over a one-dimensional domain R. Then there exists a prime ideal p of R such that $r \notin p$ implies $rM = M$.*

Proof. Use Theorem 3.6. □

Theorem 3.8. *Let M be a primary-like, primeful and torsion module over a one-dimensional domain R. If $p = \sqrt{Ann(M)}$ and M_p is the localization of M at p, then the R-module $\frac{M}{S_p(0)}$ is isomorphic to the R-module M_p.*

Proof. Consider the R-module homomorphism $\psi : M \longrightarrow M_p$ given by $m \mapsto \frac{m}{1}$. To show that ψ is an epimorphism, take any $\frac{m}{s} \in M_p$. Since $s \notin p$, $sM = M$ by Theorem 3.7 and so there exists $m' \in M$ such that $m = sm'$. Thus $\frac{m}{s} = \frac{sm'}{s} = \frac{m'}{1} = \psi(m')$. Also it is easy to verified that the kernel of ψ is $S_p(0)$. Hence $\frac{M}{S_p(0)} \cong M_p$. □

Corollary 3.9. *Let M be a primary-like, primeful and torsion module over a one-dimensional domain R and N be a submodule of M. If $p = \sqrt{Ann(M)}$, then $S_p(N)$ has a reduced primary-like decomposition if and only if the R-module N_p has a reduced primary-like decomposition. In particular, $S_p(N) = S_p(N_1) \cap \cdots \cap S_p(N_k)$ is a reduced primary-like decomposition of $S_p(N)$ if and only if $N_p = (N_1)_p \cap \cdots \cap (N_k)_p$ is a reduced primary-like decomposition of R-module N_p.*

Proof. Suppose $\phi : \frac{M}{S_p(0)} \longrightarrow M_p$ is the natural isomorphism in Theorem 3.8. We show that $\phi(\frac{S_p(N)}{S_p(0)}) = N_p$. Let $\frac{n}{s} \in N_p$. Then $\frac{n}{s} = \phi(m + S_p(0)) = \frac{m}{1}$ for some $m \in M$. Thus there exists $u \in R \backslash p$ such that $um \in N$ and so $m \in S_p(N)$. Therefore $N_p \subseteq \phi(\frac{S_p(N)}{S_p(0)})$. For the reverse inclusion, let $m + S_p(0) \in \frac{S_p(N)}{S_p(0)}$. Then $um \in N$ for some $u \in R \backslash p$. It follows that $\phi(m + S_p(0)) = \frac{m}{1} = \frac{um}{u} \in N_p$. Thus $\frac{S_p(N)}{S_p(0)} \cong N_p$. Now, the assertion holds by [4, Corollary 3.6]. □

References

[1] Atiyah, M. F., Macdonald, I. G. (1969). Introduction to commutative algebra. Addison-Wesley.

[2] Behboodi, M., Sabzevari, M. (submitted). Modules satisfying the prime radical condition and a sheaf construction for modules I.

[3] Eisenbud, D. (1994). Commutative algebra with a view toward algebraic geometry. Springer-Verlag.

[4] Fazaeli Moghimi, H., Rashedi, F. (to appear). Primary-like submodules satisfying the primeful property.

[5] Jenkins, J., Smith, P. F. (1992). On the prime radical of a module over a commutative ring. Comm. Algebra. 20: 3593-3602.

[6] Kaplansky, I. (1974). Commutative rings. University of Chicago press, Chicago.

[7] Lu, C. P. (2007). A module whose prime spectrum has the surjective natural map. Houston J. Math. 33(1): 127-143.

[8] Lu, C. P. (2003). Saturations of submodules. Comm. Algebra. 31(6): 2655-2673.

[9] McCasland, R. L., Moore, M. E. (1992). Prime submodules. Comm. Algebra. 20: 1803-1817.

[10] Smith, P. F. (2001). Primary modules over commutative rings. Glasg. Math. J. 43(1): 103-111.

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