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ON AN OPTIMAL CONTROL CONSTRAINED PROBLEM GOVERNED BY PARABOLIC TYPE EQUATIONS

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Abstract. This paper presents the convergence of the difference approximations of an optimal control problem for a quasilinear parabolic equation with controls in the heat conductivity coefficient, boundary conditions, additional restrictions and the right side of the equation. The difference approximations problem (DAP) associated to the problem is constructed. The estimations of stability for the difference approximations problem are established. The convergence of the difference approximations problem for the discrete optimal control problem is studied.

1 Introduction

Owing to its importance for engineering applications, the field of partial differential equations (PDE) constrained optimization has become increasingly popular [1-4]. In them, the control can occur both in the equations and in the boundary and initial conditions. The question arises of the convergence of a solution of an approximate problem to a solution of the differential problem; the possibility of finding a good approximation to an optimal control depends on the properties of the approximation and the original problem, and the Tikhonov correctness of optimal control problems [6-8]. In this paper, we focus on the convergence of the difference approximations problem for an optimal control problem governed by a quasilinear parabolic equation with controls in the heat conductivity coefficient, boundary conditions, additional restrictions and the right side of the equation. The difference approximations problem associated to the problem is constructed. The estimations of stability for the difference approximations problem are established. The convergence of the difference approximations problem for the discrete optimal control problem is studied.

2 Problem Formulation

Let D be a bounded domain of the N-dimensional Euclidean space E_N, l,T be given positive numbers and let $\Omega = \{(x,t) : x \in D = (0,l), t \in (0,T)\}$. We consider the following optimal control problem: minimize

$$J_\alpha(v) = \beta_0 \int_0^T [u(0,t) - y_0(t)]^2 dt + \beta_1 \int_0^T [u(l,t) - y_1(t)]^2 dt + \alpha \|v - \omega\|^2_{E_N} \tag{2.1}$$

subject to

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(\lambda(u,v)\frac{\partial u}{\partial x}) = f(x,t,u,v), (x,t) \in \Omega \tag{2.2}$$

with initial and boundary conditions

$$u(x,0) = \phi(x), x \in D \tag{2.3}$$

$$\lambda(u,v)\frac{\partial u}{\partial x}|_{x=0} = Y_0(t), \lambda(u,v)\frac{\partial u}{\partial x}|_{x=l} = Y_1(t), 0 \le t \le T \tag{2.4}$$

and to the constraints

$$\nu_0 \le \lambda(u,v) \le \mu_0, \qquad r_1 \le u(x,t) \le r_2 \tag{2.5}$$

on the set

$$V = \{v : v = (v_1, v_2, ..., v_N) \in E_N, \|v\|_{E_N} \le R\}$$

where $R > 0$, r_1, r_2 ,$\alpha \ge 0$, $\nu_0, \mu_0 > 0$, $\beta_0 \ge 0$, $\beta_1 \ge 0$, $\beta_0 + \beta_1 \ne 0$ be given positive numbers. $\omega \in E_N$ is also given : $\omega = (\omega_1, \omega_2, ..., \omega_N)$.

Besides, $\phi(x) \in L_2(D)$, $Y_0(t), Y_1(t) \in L_2(0,T)$ and $y_0(t)$,$y_1(t) \in L_2(0,T)$ are real-valued given functions. Moreover, the functions $\lambda(u,v), f(x,t,u,v)$ are continuous for $(u,v) \in [r_1, r_2]$ x E_N, have continuous derivatives in u and $\forall (u,v) \in [r_1,r_2]$ x E_N, the derivatives $\frac{\partial \lambda(u,v)}{\partial u}$, $\frac{\partial f(u,v)}{\partial u}$ are bounded.

The state-function $u = u(x,t) \in V_2^{1,0}(\Omega)$ is defined as the solution of (2.1)-(2.4). On the basis of adopted assumptions and the results of [9] it follows that for every $v \in V$ the solution of the problem (2.1)-(2.4) is existed, unique and $|u_x| \leq C_0$,$\forall (x,t) \in \Omega$, $\forall v \in V$, where C_0 is a certain constant.

Definition 2.1. For given $v \in V$, the problem of finding a function $u = u(x,t;v) \in V_2^{1,0}(\Omega)$ from conditions (2.1)-(2.4) is called the reduced problem.

Definition 2.2. The solution of the reduced problem (2.1)-(2.4) corresponding to the $v \in V$ is a function $u(x,t) \in V_2^{1,0}(\Omega)$ and satisfies the integral identity

$$\begin{aligned} &\int_0^l \int_0^T [u\tfrac{\partial \eta}{\partial t} - \lambda(u,v)\tfrac{\partial u}{\partial x}\tfrac{\partial \eta}{\partial x} + \eta f(x,t,u,v)]dxdt = \\ &- \int_0^l \phi(x)\eta(x,0)dx - \int_0^T \eta(0,t)Y_0(t)dt + \int_0^T \eta(l,t)Y_1(t)dt, \end{aligned} \tag{2.6}$$

$\forall \quad \eta = \eta(x,t) \in W_2^{1,1}(\Omega) \quad and \quad \eta(x,T) = 0.$

Optimal control problems of for solutions of partial differential equations do not always have a solution [8]. The existence and uniquness of a solution of optimal control problem (2.1)-(2.5) can be found in Farag [10].

The inequality constrained optimal control problem (2.1)-(2.5) is converted to an unconstrained control problem by adding a penalty function [11] to the cost functional (2.1), yielding the modified function $\Phi_{\alpha,n}(v, A_n)$

$$\Phi_{\alpha,n}(v, A_n) \equiv \Phi(v) = J_\alpha(v) + P_n(v), \tag{2.7}$$

where

$$F(u,v) = [\max\{\nu_0 - \lambda(u,v); 0\}]^2 + [\max\{\lambda(u,v) - \mu_0; 0\}]^2$$
$$Q(u) = [\max\{r_1 - u(x,t;v); 0\}]^2, B(u) = [\max\{u(x,t;v) - r_2; 0\}]^2$$
$$P_n(v) = A_n \int_0^l \int_0^T [F(u,v) + Q(u) + B(u)]dxdt$$

and A_n , n=1,2,... are positive numbers, $\lim_{n\to\infty} A_n = +\infty$.

The sufficient differentiability conditions of function (2.7) and its gradient formulae are investigated by Farag [12]. Also the necessary conditions for optimization for the optimal control problem (2.1)-(1.4),(2.7) are proved by Farag [13].

3 The Discrete Optimal Control Problem

3.1 The Difference Approximations problem (DAP)

In this section, we will find the difference approximations problem for the optimal control problem (2.1)-(2.4) and (2.7). For this purpose, we must discrete the optimal control problem.

Here and further for arbitrary net functions $u = u_i^j = u(x,t) = u(x_i,t_j), x = x_i \in \overline{\omega}_h, t = t_j \in \overline{\omega}_\tau$ adopt denotations [14]:

$$\hat{u} = u(x_i, t_{j+1}) = u_i^{j+1}, \qquad u^* = u(x_i, t_{j-1}) = u_i^{j-1}$$
$$u^- = u(x_{i-1}, t_j) = u_{i-1}^j, \qquad u^+ = u(x_{i+1}, t_j) = u_{i+1}^j$$
$$u_x = \frac{u^+ - u}{h}, u_{\overline{x}} = \frac{u - u^-}{h}, u_t = \frac{\hat{u} - u}{\tau}, u_{\overline{t}} = \frac{u - u^*}{\tau}.$$

The functions $\lambda(u(x,t),v), f(u(x,t),v), \phi(x), Y_1(t), Y_2(t)$ approximate as follows:

$$\lambda_i^j = \frac{1}{h\tau}\int_{t_{j-1}}^{t_j}\int_{x_i}^{x_{i+1}} \lambda(u(x,t),v))dxdt, \qquad i = \overline{0, N-1}, j = \overline{1, M},$$

$$f_i^j = \frac{1}{h\tau}\int_{t_{j-1}}^{t_j}\int_{x_i}^{x_{i+1}} f(u(x,t),v))dxdt, \quad i=\overline{0,N-1}, j=\overline{1,M},$$

$$\phi_i = \frac{1}{h}\int_{x_i}^{x_{i+1}}\phi(x)dx, \quad i=\overline{0,N-1},$$

$$(Y_0)^j = \frac{1}{\tau}\int_{t_j-\frac{\tau}{2}}^{t_j+\frac{\tau}{2}} Y_0(t)dt, \quad (Y_1)^j = \frac{1}{\tau}\int_{t_j-\frac{\tau}{2}}^{t_j+\frac{\tau}{2}} Y_1(t)dt, \quad j=\overline{1,M-1}.$$

The discrete analogy of the integral identity (2.6) writes in the form

$$h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M-1} u_i^j(\eta_i^j)_t - h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M}[-\lambda_i^j(u_i^j)_x(\eta_i^j)_x + f_i^j\eta_i^j] =$$
$$= -h\sum_{i=0}^{N-1}\phi_i\eta_i^0 - \tau\sum_{j=1}^{M}\eta_0^j(Y_0)^j - \tau\sum_{j=1}^{M}\eta_N^j(Y_1)^j, \tag{3.1}$$

for any network function $\eta_i^j, \eta_i^M = 0$.

From [14], we have

$$h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M-1} u_i^j(\eta_i^j)_t = -h\tau\sum_{i=1}^{N-1}\sum_{j=1}^{M}(u_i^j)_{\bar{t}}\eta_i^j + h\sum_{i=0}^{N-1} u_i^M\eta_i^M$$
$$-h\sum_{i=0}^{N-1} u_i^0\eta_i^0 + h\tau\sum_{j=1}^{M}(u_0^j)_{\bar{t}}\eta_0^j \tag{3.2}$$

$$-h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M}\lambda_i^j(u_i^j)_x(\eta_i^j)_x = h\tau\sum_{i=1}^{N-1}\sum_{j=1}^{M}(\lambda_i^j(u_i^j)_x)_{\bar{x}} -$$
$$\tau\sum_{j=1}^{M}\lambda_{N-1}^j(u_{N-1}^j)_x\eta_N^j + \tau\sum_{j=1}^{M}\lambda_0^j(u_0^j)_x\eta_0^j. \tag{3.3}$$

Using (3.2),(3.3), from (3.1) we obtain

$$h\tau\sum_{i=1}^{N-1}\sum_{j=1}^{M}[-(u_i^j)_{\bar{t}} + (\lambda_i^j(u_i^j)_x)_{\bar{x}} - f_i^j]\eta_i^j = h\sum_{i=0}^{N-1} u_i^0\eta_i^0$$
$$-h\tau\sum_{j=1}^{M}(u_0^j)_{\bar{t}}\eta_0^j + \tau\sum_{j=1}^{M}[\lambda_0^j(u_0^j)_x\eta_0^j + \lambda_{N-1}^j(u_{N-1}^j)_x\eta_N^j]$$
$$-h\sum_{i=0}^{N-1}\phi_i\eta_i^0 + \tau\sum_{j=1}^{M}[\eta_0^j(Y_0)^j - \eta_N^j(Y_1)^j]. \tag{3.4}$$

Setting η_i^j equal to zero at every points in the network in the above equation, we obtain the difference approximations problem for (2.1)-(2.4):

$$(u_i^j)_{\bar{t}} - (\lambda_i^j(u_i^j)_x)_{\bar{x}} - f_i^j = 0, i=\overline{1,N-1}, j=\overline{1,M}, \tag{3.5}$$

$$u_i^0 = \phi_i, i=\overline{0,N-1} \tag{3.6}$$

$$-\lambda_0^j(u_0^j)_x - Y_0{}^j + h(u_0^j)_{\bar{t}} + hf_0^j = 0, j=\overline{1,M} \tag{3.7}$$

$$\lambda_{N-1}^j(u_{N-1}^j)_x - Y_1{}^j = 0, j=\overline{1,M} \tag{3.8}$$

Approximate the function $y_0, y_1, F(u(x,t),v), Q(u), B(u)$,then the functional (2.7) is can be written as follows:

$$I_n(v) = \beta_0\tau\sum_{j=1}^{M}[u_0^j - (y_0)^j]^2 + \beta_1\tau\sum_{j=1}^{M}[u_N^j - (y_1)^j]^2$$
$$+\alpha\|v-\omega\|_{E_N}^2 + h\tau A_n\sum_{i=0}^{N-1}\sum_{j=1}^{M}[F(u_i^j,v) + Q(u_i^j) + B(u_i^j)] \tag{3.9}$$

3.2 The Stability Estimstes of DAP

We are going to give the estimates of stability for the difference approximations problem (DAP) (3.5)-(3.8) and an estimate on v (see Farag [14]). We recall that:

Theorem 3.1. *Suppose that the all functions in the system (2.1)-(2.4) satisfy the above enumerated conditions. Moreover, we assume that the function $\lambda_i(u,v)$ satisfies the Lipschitz condition with respect to v, i.e $|\lambda_i(u(x,t),v+\delta v) - \lambda_i(u(x,t),v)| \le L\|\delta v\|_{E_N}$ for every $(x,t)\in\Omega$ and for every $v,\delta v\in V$, where $L>0$ is a constant. Then the estimates of stability for DAP (3.5)-(3.8) are*

$$\|u\|_{L_2(\overline{\omega}_{h\tau})}^2 \le C_2[\|\phi\|_{L_2(\overline{\omega}_h)}^2 + \|f\|_{L_2(\overline{\omega}_{h\tau})}^2 + \|Y_0\|_{L_2(\overline{\omega}_\tau)}^2 + \|Y_1\|_{L_2(\overline{\omega}_\tau)}^2]$$
$$\|u_x\|_{L_2(\overline{\omega}_{h\tau})}^2 \le C_2[\|\phi\|_{L_2(\overline{\omega}_h)}^2 + \|f\|_{L_2(\overline{\omega}_{h\tau})}^2 + \|Y_0\|_{L_2(\overline{\omega}_\tau)}^2 + \|Y_1\|_{L_2(\overline{\omega}_\tau)}^2]$$
$$\max_j\|u^j\|_{L_2(\overline{\omega}_h)}^2 \le C_2[\|\phi\|_{L_2(\overline{\omega}_h)}^2 + \|f\|_{L_2(\overline{\omega}_{h\tau})}^2 + \|Y_0\|_{L_2(\overline{\omega}_\tau)}^2 + \|Y_1\|_{L_2(\overline{\omega}_\tau)}^2]$$

where the positive constant C_2 is independing of ϕ,u , Y_0,$Y_1$and f.

Theorem 3.2. *Suppose that the all functions in the system (2.1)-(2.4) satsify the above enumerated conditions. Moreover, we assume that the function $\lambda_i(u,v)$ satisfies the Lipschitz condition with respect to v, i.e $|\lambda_i(u(x,t), v+\delta v) - \lambda_i(u(x,t), v)| \le L\|\delta v\|_{E_N}$ for every $(x,t) \in \Omega$ and for every $v, \delta v \in V$, where $L > 0$ is a constant. Then the stability estimation of the solution of DAP (3.5)-(3.8) on v is*

$$h\sum_{i=0}^{N-1}(\delta u_i^j)^2 + h\tau\sum_{i=0}^{N-1}\sum_{j=0}^{M}(\delta u_i^j)^2 + h\tau\sum_{i=0}^{N-1}\sum_{j=0}^{M}(\delta u_i^j)_x^2 \le C_9\|\delta v\|_{E_N}^2$$

where the positive constant C_9 is independing of δu and δv.

4 Examples and Applications

An interesting and well investigated problem is the identification of coefficients in partial differential equations [15-18]. In constract to this, the identification of nonlinear phenomina is less developed. This refers also to the nonlinear boundary conditions for the heat equation.
The outlined of the algorithm for solving OCP problem are as follows:

1- Given $It = 0, \epsilon' > 0, A_{It} > 0, \epsilon > 0$ and $v^{(It)} \in V$.
2- At each iteration It, do
  Solve (3.5)-(3.8), then find $u(., v^{(It)})$.
  Minimze $\Phi(v^{It})$ to find optimal control $v_*^{(It+1)}$ using PQI method [12].
End do.
3- If $\|\Phi(v^{It+1}) - \Phi(v^{It})\| < \epsilon$, then Stop, else, go to Step 4.
4- Set $v^{(It+1)} = v^{(It)}, A_{It+1} = \epsilon' A_{It}, It = It + 1$ and go to Step 2.

The numerical results were carried out for the following examples:

EXAMPLE 1: Let us accepte that the data of the optimal control problem (2.1)-(2.5) are given as $l = 0.8, T = 0.001, \alpha = 1, \beta_0 = \beta_1 = 1, \phi(x) = x, y_0 = t, y_1 = 0.8 + t, Y_0(t) = \frac{1}{1+t^2}, Y_1(t) = \frac{1}{1+(0.8+t)^2}$.

The iteration number , It, for the function to be minimized $\Phi(v)$, the exact, approximate values of $\lambda(u,v)$ with the approximate control values v^* and the absolute error: $\Pi = |\frac{\lambda_{exact} - \lambda_{approx}}{\lambda_{exact}}|$ are tabulated in table 1. It is clear that the absolute error decreases as the number of terms (nc) in $\lambda(u,v) = \sum_{k=1}^{nc} v_k u^k$ increase.

In Table 2, we report the number NEF of function evaluations neeed to attain the solution with an accuracy on the modified function $\Phi(v)$ of the order 10^{-6}. The above algorithm takes 6 iterations for decreasing $\Phi(v)$ to the value $0.8393609E - 04$.

Table 1

It	λ_{exact}	λ_{approx}	$\Pi = \lvert\frac{\lambda_{exact} - \lambda_{approx}}{\lambda_{exact}}\rvert$
1	.7352941E+00	.1224296E+00	.8334957E+00
2	.7352941E+00	.4115356E+00	.4403116E+00
3	.7352941E+00	.4743363E+00	.3549026E+00
4	.7352941E+00	.6568843E+00	.1066374E+00
5	.7352941E+00	.7134509E+00	.2970675E-01
6	.7352941E+00	.7150049E+00	.2759337E-01

Table 2

It	$\Phi(v)$	$J_\alpha(v)$	$P_n(v)$	A_n	NEF
0	15.2245300	15.2245300	0.0000000	0.0000000	1
1	12.5949100	12.5939900	9.240004E-04	1.0000000	169
2	5.0931420	5.0926800	4.620002E-04	5.000000E-01	506
3	1.4406350	1.4404040	2.310001E-04	2.500000E-01	674
4	6.352364E-02	6.340814E-02	1.155001E-04	1.250000E-01	842
5	7.121307E-04	6.543807E-04	5.775003E-05	6.250000E-02	1010
6	8.731989E-05	8.371051E-05	3.609377E-06	3.125000E-02	1176

EXAMPLE 2: $u = x + t, \lambda = ln(\frac{1}{1-u}), x \in [0, 0.9], t \in [0, 0.001]$

Knowing the computed optimal control values $v^* \in V$ obtained by using the previous numerical algorithm, we can calculate the approximate values of the unknown coefficient $\lambda(u, v)$ can be represented in a series as $\lambda(u, v) = \sum_{k=1}^{nc} v_k u^k$. In the below figure, the curves denoted by $\lambda_1^*, \lambda_2^*, \cdots$ are the approximate values of $\lambda(u, v)$ with v^*, and $\lambda Exact$ is the exact value of $\lambda(u, v)$. Obviously by increasing nc, the coefficients $\lambda(u, v)$ will agree with precise ones.

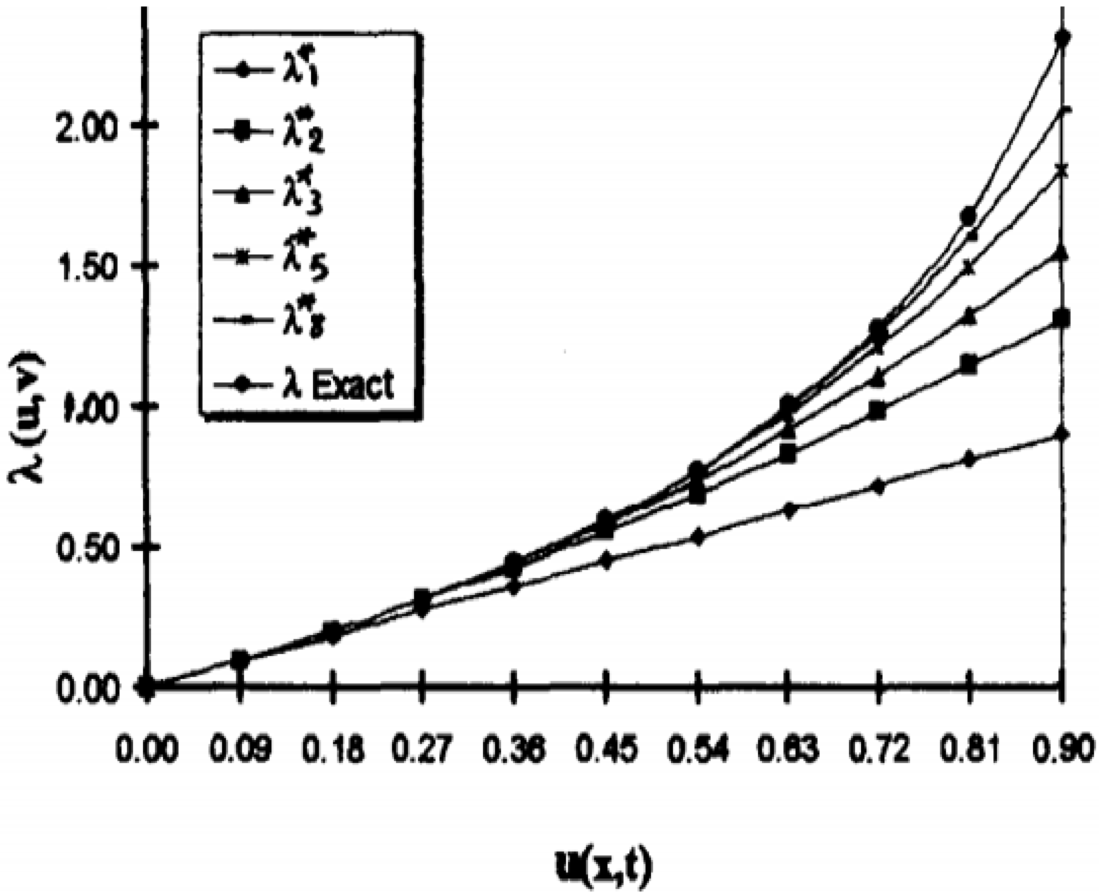

Figure 1. The identification of coefficient $\lambda(u, v)$

5 The Convergence Theorem

To come to the convergence theorem, we have to do the following assumptions:

1) For the problems (2.2)-(2.4),(2.7) and (3.5)-(3.8),(3.9), we define:

$$\Phi^* = \inf_{v \in V} \Phi(v), I_n^* = \inf_{v \in V} I_n(v), n = 1, 2, \cdots .$$

2) Let for any $n \gg 1$, there exists an approximate lower bounded value of the functional $I_n(v)$ and also exists a discrete control $v \in V$ such that

$$I_n^* \leq I_n(v) \leq I_n^* + \varepsilon_n, \tag{5.1}$$

where $\varepsilon_n \geq 0$ and $\varepsilon_n \to 0$ at $n \to \infty$.

3) In domain $\Omega = [0, l] \times [0, T]$, we construct the net such that $h_n = \tau_n$ and $\lim_{n\to\infty} N_n = \lim_{n\to\infty} M_n = \infty$, $h_n = \frac{l}{N_n} = \tau_n = \frac{T}{M_n}$.

Now, we prove that the convergence of the difference approximations of the optimal control problem (3.5)-(3.8),(3.9). The proof of the theorem will be prepared by the following two lemmas.

Lemma 5.1. *If the above assumptions are fullfilled, then for any control $v \in V$, there exists a number $0 < n_0 < n$ such that*

$$|I_n(v) - \Phi(v)| \leq \delta, \delta > 0. \tag{5.2}$$

Proof. Suppose that $u(x,t)$ and $U = u_i^j$ are the solutions of problems (2.2)-(2.4) and (3.5)-(3.8) respectively for a discrete control $v \in V$. From the work by Ladyzenskaya [9,p. 293] the interpolations $\{\hat{U}_\Delta(x,t)\}$ are uniformly bounded in $V_2^{1,0}(\Omega)$. It is possible to choose subsequence from $\{\hat{U}_\Delta(x,t)\}$ is weakly convergence to $Z(x,t) \in V_2^{1,0}(\Omega)$ and thier derivatives $\{\frac{\partial \hat{U}_\Delta(x,t)}{\partial x}\}, \{\frac{\partial \hat{U}_\Delta(x,t)}{\partial t}\}$ also are weakly convergence to the functions $\frac{\partial Z(x,t)}{\partial x}, \frac{\partial Z(x,t)}{\partial t} \in W_2^{1,1}(\Omega)$ correspondingly.

However, proceeding as in the results of Ladyzenskaya [9,p. 345], $Z(x,t)$ is the solution of the problem (2.2)-(2.4),i.e $Z(x,t) = u(x,t)$. Results in [9,p. 289] imply that the functions $\hat{U}_\Delta(x,t)$ converge in $L_2(\Omega)$ to $u(x,t)$, functions $\hat{U}_\Delta(0,t), \hat{U}_\Delta(l,t)$ converge in $L_2(0,T)$ to $u(0,t), u(l,t)$ and the function $\hat{U}_\Delta(x,0)$ converges in $L_2(0,l)$ to $u(x,0)$.

Let $\hat{y}_0, \hat{y}_1$ denote the piecewise constant fulfillement of net functions $(y_0)^j, (y_1)^j$ correspondingly, then in vitrue of results of [9,p. 301] we have

$$\|\hat{y}_0(t) - y_0(t)\|_{L_2(0,T)} \to 0, \|\hat{y}_1(t) - y_1(t)\|_{L_2(0,T)} \to 0, \quad at \quad n \to \infty. \tag{5.3}$$

Besides, we have

$$\tau \sum_{j=1}^{M} [U_0^j - (y_0)^j]^2 = \|\hat{U}(0,t) - \hat{y}_0(t)\|^2_{L_2(0,T)}, \tag{5.4}$$

$$\tau \sum_{j=1}^{M} [U_N^j - (y_1)^j]^2 = \|\hat{U}(l,t) - \hat{y}_1(t)\|^2_{L_2(0,T)}. \tag{5.5}$$

Using the forms of the functions $\Phi(v), I_n(v)$ and the last two equalities, we have

$$\begin{aligned}|I_n(v) - \Phi(v)| \leq C_1[&\|\hat{U}(0,t) - u(0,t)\|_{L_2(0,T)} + \|\hat{y}(t) - y_0(t)\|_{L_2(0,T)} \\ &+\|\hat{U}(l,t) - u(l,t)\|_{L_2(0,T)} + \|\hat{y}_1(t) - y_1(t)\|_{L_2(0,T)}] \\ &+C_2 A_n[\|F(\hat{U},v) - F(u,v)\|_{L_2(\Omega)} \\ &+\|Q(\hat{U}) - Q(u)\|_{L_2(\Omega)} + \|B(\hat{U}) - B(u)\|_{L_2(\Omega)}].\end{aligned} \tag{5.6}$$

Employing the equality (5.3) in (5.6), we can choose a number $0 < n_0 < n$ for any discrete control $v \in V$ such that the relation (5.2) is valid. Then the Lemma 5.1 is proved. □

Lemma 5.2. *Assume that the above assumptions satisfied and $\delta > 0$. Then for any sequence of control $\{v_n\} \in V$, there exists a number $0 < n_0 < n$ such that*

$$|\Phi(v_n) - I_n(v_n)| \leq \delta. \tag{5.7}$$

Proof. Let $U_n = U(\overline{v})$ be the solution of the problem (3.5)-(3.8) at $\overline{v} = v_n$ and $u_n = u(x,t,\hat{v})$ be the solution of problem (2.2)-(2.4) at $\hat{v} = v_n$ and denote the $\hat{U}_n(x,t)$ piecewise constant net functions $U(\overline{v})$.

Applying the techniqe described in Lemma 5.1, in the proof,we obtain

$$\|u_n(0,t) - \hat{U}_n(0,t)\|_{L_2(0,T)} + \|u_n(l,t) - \hat{U}_n(l,t)\|_{L_2(0,T)} \to 0, \quad at \quad n \to \infty. \tag{5.8}$$

$$\|u_n(x,t) - \hat{U}_n(x,t)\|_{L_2(\Omega)} \to 0, \quad at \quad n \to \infty. \tag{5.9}$$

Thanks to (5.8),(5.9) and the form of $\Phi(v_n), I_n(v_n)$, we get

$$\begin{aligned}|\Phi(v_n) - I_n(v_n)| \leq C_3[&\|u_n(0,t) - \hat{U}_n(0,t)\|_{L_2(0,T)} + \|y_0(t) - \hat{y}_0(t)\|_{L_2(0,T)} \\ &+\|u_n(l,t) - \hat{U}_n(l,t)\|_{L_2(0,T)} + \|y_1(t) - \hat{y}_1(t)\|_{L_2(0,T)}] \\ &+C_4 A_n[\|F_n(u_n,v) - F(\hat{U}_n,v)\|_{L_2(\Omega)} \\ &+\|Q_n(u_n) - Q(\hat{U}_n)\|_{L_2(\Omega)} + \|B_n(u_n) - B(\hat{U}_n)\|_{L_2(\Omega)}].\end{aligned} \tag{5.10}$$

Employing the equalities (5.8),(5.9) in (5.10), the estimat (5.7) is valid. Then the Lemma 5.2 is proved.

□

Theorem 5.3. *Under the above assumptions, if n is (big enough), then*

$$\lim_{n\to\infty} I_n^* = \Phi^*. \tag{5.11}$$

Besides, if the discrete control $v \in V$ satisfies the relation (5.1), then the sequence of control $\{v_n\} \in V$ is a minimizing sequence for the problem (2.2)-(2.4),(2.7),i.e

$$\lim_{n\to\infty} \Phi(v_n) = \Phi^*. \tag{5.12}$$

Proof. The function $\Phi(v)$ is bounded below, then we find $v_\delta \in V, \delta > 0$ such that $\Phi^* \leq \Phi(v_\delta) < \Phi^* + \frac{\delta}{2}$.

It thus follow from Lemma 5.1 that $|I_n(v_\delta) - \Phi(v)| \leq \frac{\delta}{2}$. But $\Phi_n^* \leq \Phi(v_\delta) \leq I(v_\delta) + \frac{\delta}{2} < \Phi^* + \delta$ then we obtain

$$\lim_{n\to\infty} I_n^* \leq \Phi^*. \tag{5.13}$$

The reasoning used in the proof of (5.13), applied here, proves that In virtue of arbitrariness of $\delta > 0$, we obtain

$$\Phi^* \leq \lim_{n\to\infty} I_n^*. \tag{5.14}$$

It follows from (5.13) and (5.14) that $\lim_{n\to\infty} I_n^*$ exists and (5.4) is fullfilled.

Finally, if the controls $v_n \in V$ satisfy the conditions of theorem, then

$$|\Phi(v_n) - \Phi^*| \leq |\Phi(v_n) - I_n(v_n)| + |I_n(v) - I_n^*| + |I_n^* - \Phi^*| \to 0, \qquad n \to \infty. \tag{5.15}$$

This gives The relation (5.11) and the proof is completed. □

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