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On Exact Frames in Topological Algebras

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Abstract. We present necessary and sufficient conditions for a frame in topological algebras to be exact.

1 Introduction and Preliminaries

Let $\mathcal{A}$ be a linear space over the complex field $\mathbb{C}$ (or the real field $\mathbb{R}$). $\mathcal{A}$ is said to be a complex (or real) *algebra* if for all $x, y \in \mathcal{A}$, the product xy is defined and $xy \in \mathcal{A}$ satisfies the following conditions

(i) $x(yz) = (xy)z = xyz$,

(ii) $x(y+z) = xy + xz$,

(iii) $(y+z)x = yx + zx \ \ (z \in \mathcal{A})$,

(iv) $(\lambda x)(\mu y) = (\lambda\mu)xy$, for all $\lambda, \mu \in \mathbb{C}$.

An algebra $\mathcal{A}$ with a Hausdorff topology is called a *semi-topological algebra* if the maps: $(x, y) \mapsto x + y$ from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ and $(\lambda, x) \mapsto \lambda x$ from $\mathbb{C} \times \mathcal{A}$ to $\mathcal{A}$ are continuous and the map: $(x, y) \mapsto xy$ is separately continuous. A semi-topological algebra is said to be a *topological algebra* if the map: $(x, y) \mapsto xy$ is jointly continuous. For basics on topological algebra we refer [2] and for linear expansion system in topological vector spaces, see [3].

Throughout this paper, $(\mathcal{A}, \tau)$ denotes a real (or complex) locally convex separable topological algebra, assumed to be commutative. By $\mathcal{A}^*$ and $\mathcal{A}'$, we denote the topological dual and the algebraic dual of $\mathcal{A}$, respectively. For a Hausdorff locally convex topology τ in $\mathcal{A}$ and $Z \subset \mathcal{A}$, $[Z]^\tau$ shall denote the τ-closure of the span of Z in $\mathcal{A}$. A pair $(\{x_n\}, \{f_n\}) \subset \mathcal{A} \times \mathcal{A}'$ is called a biorthogonal system if $f_i(x_j) = \delta_{i,j}$ for all $i, j \in \mathbb{N}$, where $\delta_{i,j}$ denotes the Kronecker delta.

Definition 1.1. [7] A countable sequence $\mathcal{F} \equiv \{x_n\} \subset \mathcal{A}$ is a *τ-frame* for $(\mathcal{A}, \tau)$ if there exists a sequence $\{f_n\} \subset \mathcal{A}'$, such that for each $x \in \mathcal{A}$

$$x = \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i \ \left(= \tau\text{-}\sum_{i=1}^{\infty} f_i(x)x_i \right),$$

where the sequence $\{\sum_{i=1}^{n} f_i(x)x_i\}$ converges in the topology τ of $\mathcal{A}$.

Remark 1.2. The sequence $\{f_n\} \subset \mathcal{A}'$ is called an *associated sequence of functionals*, which need not be unique. The associated functionals f_n $(n \in \mathbb{N})$ need not be continuous.

Definition 1.3. [7] A τ-frame $\mathcal{F} \equiv \{x_n\}$ for $(\mathcal{A}, \tau)$ is a *τ-Schauder frame* for $\mathcal{A}$ if all associated functionals f_n $(n \in \mathbb{N})$ are τ-continuous.

In this paper, we give necessary and sufficient conditions for a τ-frame in topological algebras to be τ-exact. Since this work is continuation of a paper by authors [7], we consider the structure of a topological algebra.

First we recall some basic notations and definitions to make the paper self-contained. Let $\langle \mathcal{A}, \mathcal{B} \rangle$ be a non-degenerate dual pair of topological algebras $\mathcal{A}$ and $\mathcal{B}$ over the same complex (or real) field and τ be any polar topology on $\mathcal{A}$ given by a set of weakly bounded subsets of $\mathcal{B}$. We consider the following polar topologies of $\langle \mathcal{A}, \mathcal{B} \rangle$ and denote them as follows. The details can be found in [8].

(i) $\sigma(\mathcal{A},\mathcal{B})$ for the weak topology on $\mathcal{A}$, the topology of uniform convergence on the finite subsets of $\mathcal{B}$.

(ii) $\beta(\mathcal{A},\mathcal{B})$ for the strong topology on $\mathcal{A}$, the topology of uniform convergence on the $\sigma(\mathcal{B},\mathcal{A})$-bounded subsets of $\mathcal{B}$.

(iii) $\beta^*(\mathcal{A},\mathcal{B})$ for the strong* topology on $\mathcal{A}$, the topology of uniform convergence on the $\beta(\mathcal{B},\mathcal{A})$-bounded subsets of $\mathcal{B}$.

It is easy to see that $\sigma(\mathcal{A},\mathcal{B}) \leq \beta^*(\mathcal{A},\mathcal{B}) \leq \beta(\mathcal{A},\mathcal{B})$. Some interesting properties on β duality can be found in [1].

Definition 1.4. [8] The sequential dual $\mathcal{A}^\dagger$ of $\mathcal{A}$ is defined by

$$\mathcal{A}^\dagger = \{f \in \mathcal{A}' : f(x_n) \to 0 \text{ for all } \tau\text{-null sequences } \{x_n\} \subset \mathcal{A}\}.$$

Note that $\mathcal{A}^* \subset \mathcal{A}^\dagger \subset \mathcal{A}'$.

Definition 1.5. [8] A subset K of $\mathcal{A}'$ is said to be τ-limited if for every τ-null sequence $\{x_n\}$ in $\mathcal{A}$, we have

$$\lim_{n\to\infty} \sup_{f\in K} f(x_n) = 0.$$

Remark 1.6. Let $\tau^\dagger$ be the topology on $\mathcal{A}$ defined to be generated by the class $\mathcal{U}$ of all absolutely convex subsets of $\mathcal{A}$ such that every τ-null sequence in $\mathcal{A}$ eventually belongs to U, for each $U \in \mathcal{U}$. The topology $\tau^\dagger$ is, in fact, the topology of uniform convergence on the τ-limited subsets of $\mathcal{A}^\dagger$, and $(\mathcal{A},\tau^\dagger)^*$=$\mathcal{A}^\dagger$.

2 Main Results

We start with the definition of exact frames in topological algebras.

Definition 2.1. A τ-frame $\mathcal{F} \equiv \{x_n\}$ for $(\mathcal{A},\tau)$ is said to be *τ-exact*, if for all $j \in \mathbb{N}$ the sequence $\{x_n\}_{n\neq j}$ is not a τ-frame for $\mathcal{A}$.

Remark 2.2. Recall that a τ-frame $\{x_n\}$ for $\mathcal{A}$ is said to be *τ-ω-linearly independent* if

$$\tau\text{-}\sum_{i=1}^{\infty} \alpha_i x_i = 0 \Rightarrow \alpha_i = 0 \text{ for all } i \in \mathbb{N},\ (\alpha_i \text{ are scalars}).$$

A τ-exact frame $\mathcal{F} \equiv \{x_n\}$ for $\mathcal{A}$ is τ-ω-linearly independent. Indeed, let $\tau\text{-}\sum_{i=1}^{\infty} \alpha_i x_i = 0$ for some scalar coefficients $\{\alpha_i\}$. Assume that $\alpha_j \neq 0$ for some j. Then, $x_j = \tau\text{-}\sum\limits_{\substack{i=1\\ i\neq j}}^{\infty} \frac{-\alpha_i}{\alpha_j} x_i$, a contradiction.

The following proposition gives a sufficient condition for a τ-frame to be τ-exact.

Proposition 2.3. *A τ-frame $\mathcal{F} \equiv \{x_n\}$ for $(\mathcal{A},\tau)$ is τ-exact, provided for each $j \in \mathbb{N}$,*

$$x_j \notin [x_1,\ldots,x_{j-1},x_{j+1},\ldots]^\tau.$$

Proof. Assume that $x_j \notin [x_1,\ldots,x_{j-1},x_{j+1},\ldots]^\tau$ for each $j \in \mathbb{N}$. If possible, let $\mathcal{F}$ be not τ-exact. Then, for some j, $\{x_n\}_{n\neq j}$ forms a frame for $\mathcal{A}$. So, $[x_1,\ldots,x_{j-1},x_{j+1},\ldots]^\tau = \mathcal{A}$. Thus, by τ-completeness of frames, $x_j \in [x_1,\ldots,x_{j-1},x_{j+1},\ldots]^\tau$, which is a contradiction to the hypothesis. Hence $\mathcal{F}$ is τ-exact. □

Corollary 2.4. *Let $\mathcal{F} \equiv \{x_n\}$ be a τ-frame for $(\mathcal{A},\tau)$. If there exists a sequence $\{g_n\} \subset \mathcal{A}^*$ such that $(\{x_n\},\{g_n\})$ is a biorthogonal system, then $\mathcal{F}$ is τ-exact.*

Proof. Suppose that there exists a sequence $\{g_n\} \subset \mathcal{A}^*$ such that (x_n,g_n) is a biorthogonal system. It is enough to show that $x_j \notin [x_1,\ldots,x_{j-1},x_{j+1},\ldots]^\tau$ for each $j \in \mathbb{N}$. Then, by Proposition 2.3 the result follows. If possible, let $x_j \in [x_1,\ldots,x_{j-1},x_{j+1},\ldots]^\tau$ for some j. Let $\Phi_\mathbb{N}$ be the family of all finite subsets of $\mathbb{N}$. Then, for some $F \in \Phi_\mathbb{N}\backslash\{j\}$

$$\left| g_j(x_j - \sum_{m\in F} \alpha_m x_m) \right| \leq \frac{1}{2}.$$

However, this is absurd since (x_n,g_n) is biorthogonal. So, $x_j \notin [x_1,\ldots,x_{j-1},x_{j+1},\ldots]^\tau$ for each $j \in \mathbb{N}$. The result is proved. □

It would be interesting to know the exactness of a given τ-frame for $\mathcal{A}$ under topologies generated by pairs associated with $\mathcal{A}$. In this direction we have the following example of a $\beta(\mathcal{A},\mathcal{B})$-exact frame for $\mathcal{A}$.

Example 2.5. Let $\mathcal{A} = \{\{\xi_j\} \subset \mathbb{C} : \sum_{i=1}^{\infty} |\xi_i| < \infty\}$ and $\mathcal{B} = \{\{\xi_j\} \subset \mathbb{C} : \lim_{n\to\infty} \xi_n = 0\}$. Let τ be the topology induced by the metric d on $\mathcal{A}$ which is defined as

$$d(x,y) = \sum_{j=1}^{\infty} |\xi_j - \eta_j|,\ x = \{\xi_i\}, y = \{\eta_i\} \in \mathcal{A}.$$

Then, $(\mathcal{A},\tau)$ is a locally convex separable topological algebra under pointwise multiplication. Define $\{x_n\} \subset \mathcal{A}$ by

$$x_1 = e_1 \text{ and } x_n = (-1)^{n+1}e_1 + e_n,\ n \geq 2,$$

where e_n denote the canonical unit vector, i.e., $e_n = \{0,0,....,\underbrace{1}_{nth},0,0,0,....\}$ $(n \in \mathbb{N})$.

Choose $f_1 = e_1 + e_2 - e_3 + e_4 - e_5 + \dots$ and $f_n = e_n,\ n \geq 2$.
Then, $x = \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i$ for each $x \in \mathcal{A}$. Hence $\mathcal{F} \equiv \{x_n\}$ is τ-frame for $\mathcal{A}$.
Furthermore, each $f_n \in (\mathcal{A},\beta(\mathcal{A},\mathcal{B}))^* = \{\{\xi_j\} \subset \mathbb{C} : \sup\limits_{1\leq j<\infty} |\xi_j| < \infty\} \subset \mathcal{A}'$ and $f_i(x_j) = \delta_{i,j}$ for all $i,j \in \mathbb{N}$. Therefore, by Corollary 2.4, the frame $\{x_n\}$ is $\beta(\mathcal{A},\mathcal{B})$-exact frame for $\mathcal{A}$.

If we have two comparable topologies on a topological algebra, then the exactness of τ-Schauder frames is preserved while moving to a finer topology. This is shown in the following proposition.

Proposition 2.6. *Assume that τ_1 and τ_2 are Hausdorff locally convex topologies on $\mathcal{A}$ such that τ_2 is finer than τ_1. Then, every τ_1-exact Schauder frame in $\mathcal{A}$ is τ_2-exact Schauder frame.*

Proof. If $\mathcal{F} \equiv \{x_n\}$ is a τ_1-exact frame, then $\{x_n\}_{n\neq j}$ is not a frame for $\mathcal{A}$ for all $j \in \mathbb{N}$. Let $j_0 \in \mathbb{N}$ be fixed but arbitrary. Then, there exists some $x \in \mathcal{A}$ such that $x \neq \tau_1\text{-}\sum_{n\neq j_0} f_n(x)x_n$, where $\{f_n\} \subset (\mathcal{A},\tau_1)^*$. Since τ_2 is finer than τ_1, therefore $(\mathcal{A},\tau_2)^* \subseteq (\mathcal{A},\tau_1)^*$. So, there does not exist any $\{f_n\} \subset (\mathcal{A},\tau_2)^*$ such that $x \in \mathcal{A}$ can be expressed as $\tau_2\text{-}\sum_{n\neq j_0} f_n(x)x_n$. That is, $\mathcal{F}$ is τ_2-exact. Thus, every τ_1-exact frame is τ_2-exact. □

Remark 2.7. In Proposition 2.6 if $\tau_1^\dagger$ is finer than τ_2, then τ_2-ω-linearly independent Schauder frame for the topological algebra $\mathcal{A}$ is τ_1-ω-linearly independent. Indeed, let $\mathcal{F} \equiv \{x_n\}$ be τ_2-ω-linearly independent Schauder frame for $\mathcal{A}$ and $\tau_1 - \sum_{i=1}^{\infty} \alpha_i x_i = 0$ for some scalars $\{\alpha_n\}$. Then, $\{\sum_{i=1}^{n} \alpha_i x_i\}$ is τ_1-null sequence. Since $\tau_2 \leq \tau_1^\dagger$, every τ_1-null sequence in $\mathcal{A}$ is τ_2-null. So, the sequence $\{\sum_{i=1}^{n} \alpha_i x_i\}$ is τ_2-null. Hence $\tau_2\text{-}\sum_{i=1}^{\infty} \alpha_i x_i = 0$. This implies that $\alpha_i = 0$ for all $i \in \mathbb{N}$, as $\mathcal{F}$ is τ_2-ω-linearly independent. Therefore, $\mathcal{F}$ is τ_1-ω-linearly independent Schauder frame for $\mathcal{A}$.

Remark 2.8. The converse of the Proposition 2.6 is not true. That is, a τ-exact Schauder frame for $(\mathcal{A},\tau)$ need not be μ-exact Schauder frame for $(\mathcal{A},\mu)$, where $\mu \leq \tau$. More precisely, the τ-exactness of a τ-Schauder frame for $\mathcal{A}$ is not stable under topologies generated by certain dual pairs. In this direction, we observed that, a $\beta(\mathcal{A},\mathcal{B})$-exact Schauder frame is not a $\sigma(\mathcal{A},\mathcal{B})$-exact Schauder frame for the underlying space. This is justified in the following example.

Example 2.9. Let $(\mathcal{A},\tau)$ and $\mathcal{B}$ be the topological algebras given in Example 2.5. Define a sequence $\{x_n\} \subset \mathcal{A}$ by

$$x_1 = e_1,\ x_n = e_{n+1} - e_n, \quad n \geq 2.$$

Choose $f_1 = e_1,\ f_n = -e_n,\ n \geq 2$. Then, each $f_n \in (\mathcal{A},\beta(\mathcal{A},\mathcal{B}))^* \subset \mathcal{A}'$. One can easily see that $\{x_n\}$ is a τ- Schauder frame for $\mathcal{A}$. Furthermore, $f_i(x_j) = \delta_{i,j}$ for all $i,j \in \mathbb{N}$. Hence by Corollary 2.4, the τ- Schauder frame $\{x_n\}$ is a $\beta(\mathcal{A},\mathcal{B})$-exact Schauder frame for $\mathcal{A}$. To show $\{x_n\}$ is not a $\sigma(\mathcal{A},\mathcal{B})$-exact Schauder frame, it enough to show that $\{x_n\}$ is not $\sigma(\mathcal{A},\mathcal{B})$-$\omega$-linearly independent.
Choose $\alpha_1 = 1$ and $\alpha_i = -1$ $(i \geq 2)$.
Then

$$\begin{aligned}
\sigma(\mathcal{A},\mathcal{B})\text{-}\sum_{i=1}^{\infty} \alpha_i x_i &= \sum_{i=1}^{\infty} \alpha_i x_i(y) \\
&= y_1 - \sum_{i=2}^{\infty} (y_{i+1} - y_i) \\
&= 0 \quad \text{for all } y = \{y_i\} \in \mathcal{B}.
\end{aligned}$$

Hence $\{x_n\}$ is not a $\sigma(\mathcal{A},\mathcal{B})$-exact Schauder frame for $\mathcal{A}$.

Next we give necessary and sufficient conditions for a τ-Schauder frame in a topological algebra to be exact.

Theorem 2.10. *A finitely linearly independent τ-Schauder frame $\mathcal{F} \equiv \{x_n\}$ for $(\mathcal{A},\tau)$ is τ-exact if and only if whenever $\tau\text{-}\lim_{n\to\infty}\sum_{i=1}^n \alpha_i x_i = 0$, we have $\lim_{n\to\infty}\alpha_n = 0$.*

Proof. Since $\mathcal{F} \equiv \{x_n\}$ is a τ-Schauder frame for $\mathcal{A}$, $[x_n]^\tau = \mathcal{A}$ and each $x \in \mathcal{A}$ can be expressed as

$$x = \tau\text{-}\lim_{n\to\infty}\sum_{i=1}^n f_i(x)x_i,$$

where $\{f_i\} \subset \mathcal{A}^*$.
For a fixed $k \in \mathbb{N}$, by hypothesis, we have

$$\tau\text{-}\lim_{n\to\infty}\left(\sum_{i=1}^{n+k} f_i(x)x_i - \sum_{i=1}^{n} f_i(x)x_i\right) = 0 \Rightarrow \lim_{n\to\infty}[f_{n+k}(x) - f_n(x)] = 0.$$

Therefore, $\lim_{n\to\infty} f_n(x) \sim \lim_{n\to\infty} f_{n,k}(x)$ exists. Choose $g_j(x) = \lim_{n\to\infty} f_{n,j}(x)$, where g_j are defined on $\mathrm{span}\{x_n\}_n$ by

$$g_j\left(\sum_{i=1}^n \alpha_i x_i\right) = \alpha_j,\ \{\alpha_i\}_{i=1}^n \text{ are scalars } \ (j,n\in\mathbb{N}).$$

Notice that, by hypothesis $\{x_n\}$ is finitely linearly independent, so g_j are well-defined. Furthermore, each g_j is a τ-continuous linear functional on $\mathcal{A}$ and $g_i(x_j) = \delta_{ij}$ for all $i,j \in \mathbb{N}$. Thus, by Corollary 2.4, $\mathcal{F}$ is a τ-exact Schauder frame for $\mathcal{A}$.
For the reverse part, let $\mathcal{F}$ be a τ-exact Schauder frame for $(\mathcal{A},\tau)$. Then, by Remark 2.2, $\mathcal{F}$ is a τ-ω-linearly independent Schauder frame. Suppose $\tau\text{-}\lim_{n\to\infty}\sum_{i=1}^n \alpha_i x_i = 0$. Then, clearly we have $\lim_{n\to\infty}\alpha_n = 0$. $\square$

To conclude the paper, we show that a ω-linearly independent Schauder frame in a coarser topology becomes an exact Schauder in finer topology on the underlying space.

Proposition 2.11. *Assume that τ_1 and τ_2 are Hausdorff locally convex topologies on $\mathcal{A}$ such that τ_2 is finer than τ_1. Then, a τ_1-ω-linearly independent Schauder frame $\mathcal{F} \equiv \{x_n\}$ in $\mathcal{A}$ is τ_2-exact if $\mathcal{F}$ is a finitely linearly independent τ_2-Schauder frame for $\mathcal{A}$.*

Proof. If possible, suppose $\mathcal{F} \equiv \{x_n\}$ is not τ_2-exact, then by Theorem 2.10, there exists a sequence of scalars $\{\alpha_n\}_{n=1}^\infty$ such that $\tau_2\text{-}\lim_{n\to\infty}\sum_{i=1}^n \alpha_i x_i = 0$ and $\lim_{n\to\infty}\alpha_n \neq 0$. Since τ_2 is finer than τ_1, we have $\tau_1\text{-}\lim_{n\to\infty}\sum_{i=1}^n \alpha_i x_i = 0$. This gives $\alpha_i = 0$ for each $i \in \mathbb{N}$. That is, $\lim_{n\to\infty}\alpha_n = 0$, a contradiction. Hence $\mathcal{F}$ is a τ_2-exact Schauder frame for $\mathcal{A}$. $\square$

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