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ON GENERALIZED n-DERIVATIONS IN NEAR-RINGS

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Abstract. In this paper, we introduce the notion of generalized n-derivation in near-ring N and investigate several identities involving generalized n-derivations of a prime near-ring N which force N to be a commutative ring. Finally some more related results are also obtained.

1. INTRODUCTION

Throughout the paper, N will denote a zero symmetric left near-ring. N is called zero symmetric if $0x = 0$ holds for all $x \in N$ (Recall that in a left near-ring $x0 = 0$ for all $x \in N$). N is called a prime near-ring if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. It is called semiprime if $xNx = \{0\}$ implies $x = 0$. Given an integer $n > 1$, near-ring N is said to be n-torsion free, if for $x \in N$, $nx = 0$ implies $x = 0$. The symbol Z will denote the multiplicative center of N, that is, $Z = \{x \in N \mid xy = yx \text{ for all } y \in N\}$. For any $x, y \in N$ the symbols $[x, y] = xy - yx$ and $(x, y) = x + y - x - y$ stand for multiplicative commutator and additive commutator of x and y respectively, while the symbol xoy will denote $xy + yx$. For terminologies concerning near-rings, we refer to G.Pilz [15].

An additive map $d : N \longrightarrow N$ is called a derivation if $d(xy) = d(x)y+xd(y)$ (or equivalently $d(xy) = xd(y) + d(x)y$) holds for all $x, y \in N$. The concept of derivation has been generalized in several ways by various authors. Generalized derivation has been introduced already in rings by M. Brešar [6]. Also the notions of generalized derivation, permuting tri-generalized derivation have been introduced in near-rings by Öznur Gölbasi [8] and M.A.öztürk etc. [13] respectively. An additive mapping $f : N \longrightarrow N$ is called a right generalized derivation with associated derivation d if $f(xy) = f(x)y + xd(y)$, for all $x, y \in N$ and f is called a left generalized derivation with associated derivation d if $f(xy) = d(x)y + xf(y)$, for all $x, y \in N$. f is called a generalized derivation with associated derivation d if it is both a left as well as a right generalized derivation with associated derivation d. Motivated by the concept of tri-derivation, Park [14] introduced the notion of permuting n-derivation in rings. Further, the authors introduced and studied the notion of permuting n-derivation in near-rings (for reference see [3]). In the present paper, inspired by these concepts, we define generalized n-derivation in near-rings and study some properties involved there, which gives a generalization of n-derivation of near-rings.
A map $D : \underbrace{N \times N \times \cdots \times N}_{n-\text{times}} \longrightarrow N$ is said to be permuting if the equation $D(x_1, x_2, \cdots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ holds for all $x_1, x_2, \cdots, x_n \in N$ and for every permutation $\pi \in S_n$ where S_n is the permutation group on $\{1, 2, \cdots, n\}$. A map $d : N \to N$ defined by $d(x) = D(x, x, \cdots, x)$ for all $x \in N$ where $D : \underbrace{N \times N \times \cdots \times N}_{n-\text{times}} \to N$ is a permuting map, is called the trace of D.
Let n be a fixed positive integer. An n-additive (i.e.; additive in each argument) mapping $D : N \times N \times \cdots \times N \longrightarrow N$ is called an n-derivation if the relations

$$D(x_1x_1', x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_1' + x_1D(x_1', x_2, \cdots, x_n)$$

$$D(x_1, x_2x_2', \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_2' + x_2D(x_1, x_2', \cdots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \cdots, x_nx_n') = D(x_1, x_2, \cdots, x_n)x_n' + x_nD(x_1, x_2, \cdots, x_n')$$

hold for all $x_1, x_1^{'}, x_2, x_2^{'}, \cdots, x_n, x_n^{'} \in N$. If in addition D is a permuting map then D is called a permuting n-derivation of N (see [3] for further reference). An n-additive mapping $F : N \times N \times \cdots \times N \longrightarrow N$ is called a right generalized n-derivation of N with associated n-derivation D if the relations

$$F(x_1x_1^{'}, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n)$$

$$F(x_1, x_2x_2^{'}, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)x_2^{'} + x_2D(x_1, x_2^{'}, \cdots, x_n)$$

$$\vdots$$

$$F(x_1, x_2, \cdots, x_nx_n^{'}) = F(x_1, x_2, \cdots, x_n)x_n^{'} + x_nD(x_1, x_2, \cdots, x_n^{'})$$

hold for all $x_1, x_1^{'}, x_2, x_2^{'}, \cdots, x_n, x_n^{'} \in N$. If in addition both F and D are permuting maps then F is called a permuting right generalized n-derivation of N with associated permuting n-derivation D. An n-additive mapping $F : N \times N \times \cdots \times N \longrightarrow N$ is called a left generalized n-derivation of N with associated n-derivation D if the relations

$$F(x_1x_1^{'}, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_1^{'} + x_1F(x_1^{'}, x_2, \cdots, x_n)$$

$$F(x_1, x_2x_2^{'}, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_2^{'} + x_2F(x_1, x_2^{'}, \cdots, x_n)$$

$$\vdots$$

$$F(x_1, x_2, \cdots, x_nx_n^{'}) = D(x_1, x_2, \cdots, x_n)x_n^{'} + x_nF(x_1, x_2, \cdots, x_n^{'})$$

hold for all $x_1, x_1^{'}, x_2, x_2^{'}, \cdots, x_n, x_n^{'} \in N$. If in addition both F and D are permuting maps then F is called a permuting left generalized n-derivation of N with associated permuting n-derivation D. Lastly an n-additive mapping $F : N \times N \times \cdots \times N \longrightarrow N$ is called a generalized n-derivation of N with associated n-derivation D if it is both a right generalized n-derivation as well as a left generalized n-derivation of N with associated n-derivation D. If in addition both F and D are permuting maps then F is called a permuting generalized n-derivation of N with associated permuting n-derivation D.

For an example of a left generalized n-derivation, let n be a fixed positive integer, S a commutative left near-ring. Then $N_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b, 0 \in S \right\}$ is a non-commutative zero symmetric left near-ring with regard to matrix addition and matrix multiplication. Define $D_1 : \underbrace{N_1 \times N_1 \times \cdots \times N_1}_{n-\text{times}} \longrightarrow N_1$ such that

$$D_1\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & a_1a_2\cdots a_n \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that D_1 is an n-derivation of N_1. Define $F_1 : N_1 \times N_1 \times \cdots \times N_1 \longrightarrow N_1$ such that

$$F_1\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & b_1b_2\cdots b_n \\ 0 & 0 \end{pmatrix}.$$

It can be easily verified that F_1 is a left generalized n-derivation of N_1 with associated n-derivation D_1 but not a right generalized n-derivation of N_1 with associated n-derivation D_1. It can be also seen that F_1 is a permuting left generalized n-derivation of N_1 with associated permuting n-derivation D_1 but not a permuting right generalized n-derivation of N_1 with associated permuting n-derivation D_1.

For an example of right generalized n-derivation, consider $N_2 = \left\{ \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \mid c, d, 0 \in S \right\}$. It can be easily shown that N_2 is a non-commutative zero symmetric left near-ring with regard to matrix addition and matrix multiplication. Define $D_2 : \underbrace{N_2 \times N_2 \times \cdots \times N_2}_{n-\text{times}} \longrightarrow N_2$ such that

$$D_2\left(\begin{pmatrix} 0 & c_1 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} 0 & c_2 \\ 0 & d_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & c_n \\ 0 & d_n \end{pmatrix}\right) = \begin{pmatrix} 0 & c_1c_2\cdots c_n \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that D_2 is an n-derivation of N_2. Define $F_2 : N_2 \times N_2 \times \cdots \times N_2 \longrightarrow N_2$ such that

$$F_2\left(\begin{pmatrix} 0 & c_1 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} 0 & c_2 \\ 0 & d_2 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & c_n \\ 0 & d_n \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & d_1 d_2 \cdots d_n \end{pmatrix}.$$

It can be easily verified that F_2 is a right generalized n-derivation of N_2 with associated n-derivation D_2 but not a left generalized n-derivation of N_2 with associated n-derivation D_2. It can be also seen that F_2 is a permuting right generalized n-derivation of N_2 with associated permuting n-derivation D_2 but not a permuting left generalized n-derivation of N_2 with associated permuting n-derivation D_2.
For an example of generalized n-derivation,
consider $N_3 = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z, 0 \in S \right\}$. It can be seen that N_3 is a non-commutative zero symmetric left near-ring with regard to matrix addition and matrix multiplication. Define $D_3 : \underbrace{N_3 \times N_3 \times \cdots \times N_3}_{n-\text{times}} \longrightarrow N_3$ such that

$$D_3\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & z_n \end{pmatrix}\right) = \begin{pmatrix} 0 & x_1 x_2 \cdots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that D_3 is an n-derivation of N_3. Define $F_3 : N_3 \times N_3 \times \cdots \times N_3 \longrightarrow N_3$ such that

$$F_3\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & z_n \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily verified that F_3 is a generalized n-derivation (i.e.; both left generalized n-derivation and right generalized n-derivation) of N_3 with associated n-derivation D_3. It can be also easily seen that F_3 is permuting generalized n-derivation with associated permuting n-derivation D_3.
It is to be noted that if in the above examples we take S to be a distributive near-ring, then F_1, F_2 and F_3 become left generalized n-derivation, right generalized n-derivation and generalized n-derivation associated with n-derivations D_1, D_2 and D_3 respectively. However these are not permuting left generalized n-derivation, permuting right generalized n-derivation and permuting generalized n-derivation respectively.
Recently many authors have studied commutativity of rings satisfying certain properties and identities involving derivations, generalized derivations, permuting n-derivations etc.(see for detail reference [1,2,6,7,11,12,14,16]). Also commutativity behavior of prime near-rings satisfying certain properties and identities involving derivations, generalized derivations, permuting tri-generalized derivations, permuting n-derivations etc. have been investigated by several authors (see [3,4,5,8,9,10,13] where further references can be found). Now our purpose is to study the commutativity behavior of prime near-rings which admit suitably constrained generalized n-derivations. In fact, our results generalize, extend and compliment several results obtained earlier on generalized derivations, permuting tri-generalized derivations and permuting n-derivations. For example Theorems 3.2 – 3.4,3.6 & 3.7 of [3], Theorem 2.6 of [8], Theorems 3.1,3.2,3.5,3.6 of [9], Theorem 3.1 of [10] and Lemmas 9 & 10 of [13] etc.- to mention a few only. Some other related results have been also discussed.

2. PRELIMINARY RESULTS

We begin with the following lemmas which are essential for developing the proofs of our main results. Proofs of Lemmas 2.1 & 2.2 can be seen in [4] and [5] respectively while Lemmas 2.3-2.5 have been essentially proved in [3].

Lemma 2.1. Let N be a prime near-ring.

(i) If $z \in Z \setminus \{0\}$ then z is not a zero divisor.

(ii) If $Z \setminus \{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.

Lemma 2.2. Let N be a prime near-ring. If $z \in Z \setminus \{0\}$ and x is an element of N such that $xz \in Z$ or $zx \in Z$ then $x \in Z$.

Lemma 2.3. Let N be a near-ring. Then D is a permuting n-derivation of N if and only if $D(x_1x_1^{'}, x_2, \cdots, x_n) = x_1D(x_1^{'}, x_2, \cdots, x_n) + D(x_1, x_2, \cdots, x_n)x_1^{'}$ for all $x_1, x_1^{'}, x_2, \cdots, x_n \in N$.

Lemma 2.4. Let N be prime near-ring and D a nonzero permuting n-derivation of N. If $D(N, N, \cdots, N)x = \{0\}$ where $x \in N$, then $x = 0$.

Lemma 2.5. Let D be a nonzero permuting n-derivation of prime near-ring N such that $D(N, N, \cdots, N) \subseteq Z$. Then N is a commutative ring.

Remark 2.1. It can be easily shown that above Lemmas 2.3 $-$ 2.5 also hold if D is a nonzero n-derivation of near-ring N.

Lemma 2.6. F is a right generalized n-derivation of N with associated n-derivation D if and only if

$$\begin{aligned}
F(x_1x_1^{'}, x_2, \cdots, x_n) &= x_1D(x_1^{'}, x_2, \cdots, x_n) + F(x_1, x_2, \cdots, x_n)x_1^{'} \\
F(x_1, x_2x_2^{'}, \cdots, x_n) &= x_2D(x_1, x_2^{'}, \cdots, x_n) + F(x_1, x_2, \cdots, x_n)x_2^{'} \\
&\vdots \\
F(x_1, x_2, \cdots, x_nx_n^{'}) &= x_nD(x_1, x_2, \cdots, x_n^{'}) + F(x_1, x_2, \cdots, x_n)x_n^{'}
\end{aligned}$$

hold for all $x_1, x_1^{'}, x_2, x_2^{'}, \cdots, x_n, x_n^{'} \in N$.

Proof. Let F be a right generalized n-derivation of N with associated n-derivation D. Then $F(x_1x_1^{'}, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n)$, for all $x_1, x_1^{'}, x_2, \cdots, x_n \in N$.
Consider

$$\begin{aligned}
F(x_1(x_1^{'} + x_1^{'}), x_2, \cdots, x_n) &= F(x_1, x_2, \cdots, x_n)(x_1^{'} + x_1^{'}) + x_1D(x_1^{'} + x_1^{'}, x_2, \cdots, x_n) \\
&= F(x_1, x_2, \cdots, x_n)x_1^{'} + F(x_1, x_2, \cdots, x_n)x_1^{'} \\
&\quad + x_1D(x_1^{'}, x_2, \cdots, x_n) + x_1D(x_1^{'}, x_2, \cdots, x_n).
\end{aligned}$$

Also

$$\begin{aligned}
F(x_1(x_1^{'} + x_1^{'}), x_2, \cdots, x_n) &= F(x_1x_1^{'}, x_2, \cdots, x_n) + F(x_1x_1^{'}, x_2, \cdots, x_n) \\
&= F(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n) \\
&\quad + F(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n).
\end{aligned}$$

Combining the above two equalities we find that $F(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n) = x_1D(x_1^{'}, x_2, \cdots, x_n) + F(x_1, x_2, \cdots, x_n)x_1^{'}$, for all $x_1, x_1^{'}, x_2, \cdots, x_n \in N$. Similarly we can prove the remaining $(n-1)$ relations. Converse can be proved in a similar manner.

Lemma 2.7. Let N be a near-ring admitting a right generalized n-derivation F with associated n-derivation D of N. Then,

$$\begin{aligned}
\{F(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n)\}y &= F(x_1, x_2, \cdots, x_n)x_1^{'}y \\
&\quad + x_1D(x_1^{'}, x_2, \cdots, x_n)y, \\
\{F(x_1, x_2, \cdots, x_n)x_2^{'} + x_2D(x_1, x_2^{'}, \cdots, x_n)\}y &= F(x_1, x_2, \cdots, x_n)x_2^{'}y \\
&\quad + x_2D(x_1, x_2^{'}, \cdots, x_n)y, \\
&\vdots \\
\{F(x_1, x_2, \cdots, x_n)x_n^{'} + x_nD(x_1, x_2, \cdots, x_n^{'})\}y &= F(x_1, x_2, \cdots, x_n)x_n^{'}y \\
&\quad + x_nD(x_1, x_2, \cdots, x_n^{'})y,
\end{aligned}$$

hold for all $x_1, x_1^{'}, x_2, x_2^{'}, \cdots, x_n, x_n^{'}, y \in N$.

Proof. For all $x_1, x_1^{'}, x_1^{''}, x_2, \cdots, x_n \in N$,

$$\begin{aligned} F((x_1x_1^{'})x_1^{''}, x_2, \cdots, x_n) &= F(x_1x_1^{'}, x_2, \cdots, x_n)x_1^{''} + (x_1x_1^{'})D(x_1^{''}, x_2, \cdots, x_n) \\ &= \{F(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n)\}x_1^{''} \\ &\quad +(x_1x_1^{'})D(x_1^{''}, x_2, \cdots, x_n). \end{aligned}$$

Also

$$\begin{aligned} F(x_1(x_1^{'}x_1^{''}), x_2, \cdots, x_n) &= F(x_1, x_2, \cdots, x_n)x_1^{'}x_1^{''} + x_1D(x_1^{'}x_1^{''}, x_2, \cdots, x_n) \\ &= F(x_1, x_2, \cdots, x_n)x_1^{'}x_1^{''} + x_1\{D(x_1^{'}, x_2, \cdots, x_n)x_1^{''} \\ &\quad +x_1^{'}D(x_1^{''}, x_2, \cdots, x_n)\} \\ &= F(x_1, x_2, \cdots, x_n)x_1^{'}x_1^{''} + x_1D(x_1^{'}, x_2, \cdots, x_n)x_1^{''} \\ &\quad +x_1x_1^{'}D(x_1^{''}, x_2, \cdots, x_n). \end{aligned}$$

Combining the above two relations, we get

$$\begin{aligned} \{F(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n)\}x_1^{''} &= F(x_1, x_2, \cdots, x_n)x_1^{'}x_1^{''} \\ &\quad +x_1D(x_1^{'}, x_2, \cdots, x_n)x_1^{''}. \end{aligned}$$

Putting y in place of $x_1^{''}$, we find that

$$\begin{aligned} \{F(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n)\}y &= F(x_1, x_2, \cdots, x_n)x_1^{'}y \\ &\quad +x_1D(x_1^{'}, x_2, \cdots, x_n)y. \end{aligned}$$

Similarly other $(n-1)$ relations can be proved.

Using Lemma 2.6 and similar techniques as used to prove the above lemma, one can easily get the following:

Lemma 2.8. Let N be a near-ring admitting a right generalized n-derivation F with associated n-derivation D of N. Then,

$$\begin{aligned} \{x_1D(x_1^{'}, x_2, \cdots, x_n) + F(x_1, x_2, \cdots, x_n)x_1^{'}\}y &= x_1D(x_1^{'}, x_2, \cdots, x_n)y \\ &\quad +F(x_1, x_2, \cdots, x_n)x_1^{'}y, \\ \{x_2D(x_1, x_2^{'}, \cdots, x_n) + F(x_1, x_2, \cdots, x_n)x_2^{'}\}y &= x_2D(x_1, x_2^{'}, \cdots, x_n)y \\ &\quad +F(x_1, x_2, \cdots, x_n)x_2^{'}y, \\ &\vdots \\ \{x_nD(x_1, x_2, \cdots, x_n^{'}) + F(x_1, x_2, \cdots, x_n)x_n^{'}\}y &= x_nD(x_1, x_2, \cdots, x_n^{'})y \\ &\quad +F(x_1, x_2, \cdots, x_n)x_n^{'}y, \end{aligned}$$

hold for all $x_1, x_1^{'}, x_2, x_2^{'}, \cdots, x_n, x_n^{'}, y \in N$.

Lemma 2.9. F is a left generalized n-derivation of N with associated n-derivation D if and only if

$$\begin{gathered} F(x_1x_1^{'}, x_2, \cdots, x_n) = x_1F(x_1^{'}, x_2, \cdots, x_n) + D(x_1, x_2, \cdots, x_n)x_1^{'}, \\ F(x_1, x_2x_2^{'}, \cdots, x_n) = x_2F(x_1, x_2^{'}, \cdots, x_n) + D(x_1, x_2, \cdots, x_n)x_2^{'}, \\ \vdots \\ F(x_1, x_2, \cdots, x_nx_n^{'}) = x_nF(x_1, x_2, \cdots, x_n^{'}) + D(x_1, x_2, \cdots, x_n)x_n^{'} \end{gathered}$$

hold for all $x_1, x_1^{'}, x_2, x_2^{'}, \cdots, x_n, x_n^{'} \in N$.
Proof. Use same arguments as used in the proof of Lemma 2.6.

Lemma 2.10. Let N be a near-ring admitting a generalized n-derivation F with associated n-derivation D of N. Then,

$$\begin{aligned} \{D(x_1, x_2, \cdots, x_n)x_1^{'} + x_1F(x_1^{'}, x_2, \cdots, x_n)\}y &= D(x_1, x_2, \cdots, x_n)x_1^{'}y \\ &\quad +x_1F(x_1^{'}, x_2, \cdots, x_n)y, \end{aligned}$$

$$\begin{aligned}\{D(x_1,x_2,\cdots,x_n)x_2^{'}+x_2F(x_1,x_2{'},\cdots,x_n)\}y &= D(x_1,x_2,\cdots,x_n)x_2^{'}y\\ &\quad +x_2F(x_1,x_2^{'},\cdots,x_n)y,\\ &\vdots\\ \{D(x_1,x_2,\cdots,x_n)x_n^{'}+x_nF(x_1,x_2,\cdots,x_n^{'})\}y &= D(x_1,x_2,\cdots,x_n)x_n^{'}y\\ &\quad +x_nF(x_1,x_2,\cdots,x_n^{'})y,\end{aligned}$$

hold for all $x_1,x_1^{'},x_2,x_2^{'},\cdots,x_n,x_n^{'},y\in N$.

Proof. For all $x_1,x_1^{'},x_1^{''},x_2,\cdots,x_n\in N$,

$$\begin{aligned}F((x_1x_1^{'})x_1^{''},x_2,\cdots,x_n) &= F(x_1x_1^{'},x_2,\cdots,x_n)x_1^{''}+(x_1x_1^{'})D(x_1^{''},x_2,\cdots,x_n)\\ &= \{D(x_1,x_2,\cdots,x_n)x_1^{'}+x_1F(x_1^{'},x_2,\cdots,x_n)\}x_1^{''}\\ &\quad +(x_1x_1^{'})D(x_1^{''},x_2,\cdots,x_n).\end{aligned}$$

Also

$$\begin{aligned}F(x_1(x_1^{'}x_1^{''}),x_2,\cdots,x_n) &= D(x_1,x_2,\cdots,x_n)x_1^{'}x_1^{''}+x_1F(x_1^{'}x_1^{''},x_2,\cdots,x_n)\\ &= D(x_1,x_2,\cdots,x_n)x_1^{'}x_1^{''}+x_1\{F(x_1^{'},x_2,\cdots,x_n)x_1^{''}\\ &\quad +x_1^{'}D(x_1^{''},x_2,\cdots,x_n)\}\\ &= D(x_1,x_2,\cdots,x_n)x_1^{'}x_1^{''}+x_1F(x_1^{'},x_2,\cdots,x_n)x_1^{''}\\ &\quad +x_1x_1^{'}D(x_1^{''},x_2,\cdots,x_n).\end{aligned}$$

Combining the above two relations, we get

$$\begin{aligned}\{D(x_1,x_2,\cdots,x_n)x_1^{'}+x_1F(x_1^{'},x_2,\cdots,x_n)\}x_1^{''} &= D(x_1,x_2,\cdots,x_n)x_1^{'}x_1^{''}\\ &\quad +x_1F(x_1^{'},x_2,\cdots,x_n)x_1^{''}.\end{aligned}$$

Putting y in place of $x_1^{''}$, we find that

$$\begin{aligned}\{D(x_1,x_2,\cdots,x_n)x_1^{'}+x_1F(x_1^{'},x_2,\cdots,x_n)\}y &= D(x_1,x_2,\cdots,x_n)x_1^{'}y\\ &\quad +x_1F(x_1^{'},x_2,\cdots,x_n)y.\end{aligned}$$

Similarly other $(n-1)$ relations can be shown.

Lemma 2.11. Let N be a near-ring admitting a generalized n-derivation F with associated n-derivation D of N. Then,

$$\begin{aligned}\{x_1F(x_1^{'},x_2,\cdots,x_n)+D(x_1,x_2,\cdots,x_n)x_1^{'}\}y &= x_1F(x_1^{'},x_2,\cdots,x_n)y\\ &\quad +D(x_1,x_2,\cdots,x_n)x_1^{'}y,\\ \{x_2F(x_1,x_2^{'},\cdots,x_n)+D(x_1,x_2,\cdots,x_n)x_2^{'}\}y &= x_2F(x_1,x_2^{'},\cdots,x_n)y\\ &\quad +D(x_1,x_2,\cdots,x_n)x_2^{'}y,\\ &\vdots\\ \{x_nF(x_1,x_2,\cdots,x_n^{'})+D(x_1,x_2,\cdots,x_n)x_n^{'}\}y &= x_nF(x_1,x_2,\cdots,x_n^{'})y\\ &\quad +D(x_1,x_2,\cdots,x_n)x_n^{'}y,\end{aligned}$$

hold for all $x_1,x_1^{'},x_2,x_2^{'},\cdots,x_n,x_n^{'},y\in N$.

Proof. Using Lemmas 2.6, 2.9 and the same trick as used in the proof of above lemma, one can get its proof easily.

Lemma 2.12. Let N be prime near-ring admitting a generalized n-derivation F with associated nonzero n-derivation D of N and $x\in N$.

(i) If $xF(N,N,\cdots,N)=\{0\}$, then $x=0$.

(ii) If $F(N,N,\cdots,N)x=\{0\}$, then $x=0$.

Proof. (i) Given that $xF(x_1x_1^{'}, x_2, \cdots, x_n) = 0$ for all $x_1, x_1^{'}, \cdots, x_n \in N$. This yields that $x\{F(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n)\} = 0$. By hypothesis we have $xND(x_1^{'}, x_2, \cdots, x_n) = \{0\}$. But since N is a prime near-ring and $D \neq 0$, we have $x = 0$.

(ii) It can be proved in a similar way by using Lemma 2.10.

Lemma 2.13. Let N be near-ring admitting a generalized n-derivation F with associated n-derivation D of N. Then $F(Z, N, N, \cdots, N) \subseteq Z$.

Proof. Let $z \in Z$, then $F(zr_1, r_2, \cdots, r_n) = F(r_1z, r_2, \cdots, r_n)$ for all $r_1, r_2, \cdots, r_n \in N$. Using Lemma 2.9 we have $F(z, r_2, \cdots, r_n)r_1 + zD(r_1, r_2, \cdots, r_n) = r_1F(z, r_2, \cdots, r_n) + D(r_1, r_2, \cdots, r_n)z$. Which in turn gives us $F(z, r_2, \cdots, r_n)r_1 = r_1F(z, r_2, \cdots, r_n)$, that is, $F(Z, N, N, \cdots, N) \subseteq Z$.

3. MAIN RESULTS

Recently Öznur Gölbasi [8, Theorem 2.6] proved that if N is a prime near-ring with a nonzero generalized derivation f such that $f(N) \subseteq Z$ then $(N, +)$ is an abelian group. Moreover if N is 2-torsion free, then N is a commutative ring. The following result shows that "2-torsion free restriction" in the above result used by Öznur Gölbasi is superfluous. In fact, for generalized n-derivation in a prime near-ring N we have obtained the following.

Theorem 3.1. Let N be a prime near-ring admitting a nonzero generalized n-derivation F with associated n-derivation D of N. If $F(N, N, \cdots, N) \subseteq Z$, then N is a commutative ring.

Proof. For all $x_1, x_1^{'}, \cdots, x_n \in N$

$$F(x_1x_1^{'}, x_2, \cdots, x_n) = D(x_1, x_2, \cdots, x_n)x_1^{'} + x_1F(x_1^{'}, x_2, \cdots, x_n) \in Z. \tag{3.1}$$

Hence $\{D(x_1, x_2, \cdots, x_n)x_1^{'} + x_1F(x_1^{'}, x_2, \cdots, x_n)\}x_1 = x_1\{D(x_1, x_2, \cdots, x_n)x_1^{'} + x_1F(x_1^{'}, x_2, \cdots, x_n)\}$. By hypothesis and Lemma 2.10 we obtain $D(x_1, x_2, \cdots, x_n)x_1^{'}x_1 = x_1D(x_1, x_2, \cdots, x_n)x_1^{'}$, putting $x_1^{'}y$ where $y \in N$ for $x_1^{'}$ in the preceding relation and using it again we get $D(x_1, x_2, \cdots, x_n)x_1^{'}(yx_1 - x_1y) = 0$ i.e,; $D(x_1, x_2, \cdots, x_n)N(yx_1 - x_1y) = \{0\}$. But primeness of N yields that for each fixed x_1 either $x_1 \in Z$ or $D(x_1, x_2, \cdots, x_n) = 0$ for all $x_2, x_3, \cdots, x_n \in N$. If first case holds then $D(x_1t, x_2, \cdots, x_n) = D(tx_1, x_2, \cdots, x_n)$ for all $t, x_2, \cdots, x_n \in N$. Using Lemma 2.3 and Remark 2.1 we obtain that $D(x_1, x_2, \cdots, x_n)t + x_1D(t, x_2, \cdots, x_n) = tD(x_1, x_2, \cdots, x_n) + D(t, x_2, \cdots, x_n)x_1$ for all $t, x_2, \cdots, x_n \in N$,that is, $D(x_1, x_2, \cdots, x_n) \in Z$ and second case implies $D(x_1, x_2, \cdots, x_n) = 0$ that is,

$0 = D(x_1, x_2, \cdots, x_n) \in Z$. Including both the cases we get $D(x_1, x_2, \cdots, x_n) \in Z$ for all $x_1, x_2, \cdots, x_n \in N$ i.e.; $D(N, N, \cdots, N) \subseteq Z$. If $D \neq 0$, then by Lemma 2.5 and Remark 2.1, N is a commutative ring. On the other hand if $D = 0$, then equation (3.1) takes the form $F(x_1x_1^{'}, x_2, \cdots, x_n) = x_1F(x_1^{'}, x_2, \cdots, x_n)$ for all $x_1, x_1^{'}, \cdots, x_n \in N$. By hypothesis and Lemma 2.2, $x_1 \in Z$ i.e.; $N = Z$. Thus we conclude that N is a commutative near-ring. Since $N \neq \{0\}$, there exists $0 \neq p \in N = Z$ such that $p + p \in N = Z$. By Lemma 2.1(ii) we find that N is a commutative ring.

Corollary 3.1 ([3], Theorem 3.2). Let N be a prime near-ring admitting a nonzero permuting n-derivation D such that $D(N, N, ..., N) \subseteq Z$ then N is a commutative ring.

Recently Öznur Gölbasi [9, Theorem 3.1. and 3.2.] showed that if f is a generalized derivation of a prime near-ring N with associated nonzero derivation d such that $f([x, y]) = 0$ for all $x, y \in N$ or $f([x, y]) = \pm[x, y]$ for all $x, y \in N$, then N is a commutative ring. While proving the theorem it has been assumed that f is a left generalized derivation with associated nonzero derivation d. We have extended these results in the setting of left generalized n-derivations in prime near-rings by establishing the following theorems.

Theorem 3.2**.** Let N be a prime near-ring admitting a left generalized n-derivation F with associated nonzero n-derivation D of N. If $F([x, y], r_2, r_3, \cdots, r_n) = 0$ for all $x, y, r_2, r_3, \cdots, r_n \in N$, then N is commutative ring.

Proof. Since $F([x,y],r_2,\cdots,r_n)=0$, substituting xy for y we obtain $F(x[x,y],r_2,\cdots,r_n)=0$,that is, $D(x,r_2,\cdots,r_n)[x,y]+xF([x,y],r_2,\cdots,r_n)=0$. By hypothesis we get $D(x,r_2,\cdots,r_n)[x,y]=0$ that is,

$$D(x,r_2,\cdots,r_n)xy=D(x,r_2,\cdots,r_n)yx. \tag{3.2}$$

Putting yz for y in (3.2) and using it again we have $D(x,r_2,\cdots,r_n)y(xz-zx)=0$ i.e.; $D(x,r_2,\cdots,r_n)N[x,z]=\{0\}$. For each fixed $x\in N$ primeness of N yields either $x\in Z$ or $D(x,r_2,\cdots,r_n)=0$ for all $r_2,\cdots,r_n\in N$. If first case holds then $D(xt,r_2,\cdots,r_n)=D(tx,r_2,\cdots,r_n)$ for all $t,r_2,\cdots,r_n\in N$. Using Lemma 2.3 and Remark 2.1 we obtain that $D(x,r_2,\cdots,r_n)t+xD(t,r_2,\cdots,r_n)=tD(x,r_2,\cdots,r_n)+D(t,r_2,\cdots,r_n)x$ for all $t,r_2,\cdots,r_n\in N$ i.e.; $D(x,r_2,\cdots,r_n)\in Z$ and second case implies $D(x,r_2,\cdots,r_n)=0$, that is, $0=D(x,r_2,\cdots,r_n)\in Z$. Including both the cases we get $D(x,r_2,\cdots,r_n)\in Z$ for all $x,r_2,\cdots,r_n\in N$,that is,
$D(N,N,\cdots,N)\subseteq Z$, hence by Lemma 2.5 and Remark 2.1, N is a commutative ring.

Theorem 3.3. Let N be a prime near-ring admitting a left generalized n-derivation F with associated nonzero n-derivation D of N. If $F([x,y],r_2,r_3,\cdots,r_n)=\pm[x,y]$ for all $x,y,r_2,r_3,\cdots,r_n\in N$, then N is commutative ring.

Proof. Since $F([x,y],r_2,\cdots,r_n)=\pm[x,y]$. Substituting xy for y we obtain
$F(x[x,y],r_2,\cdots,r_n)=\pm x[x,y]$ i.e.; $D(x,r_2,\cdots,r_n)[x,y]+xF([x,y],r_2,\cdots,r_n)=\pm x[x,y]$. By hypothesis we get $D(x,r_2,\cdots,r_n)[x,y]=0$ that is, $D(x,r_2,\cdots,r_n)xy=D(x,r_2,\cdots,r_n)yx$, which is identical with (3.2) of Theorem 3.2. Now arguing in the same way as in the Theorem 3.2 we conclude that N is a commutative ring.

The conclusion of Theorems 3.2 and 3.3 remain valid if we replace the product $[x,y]$ by xoy. In fact, we obtain the following results.

Theorem 3.4. Let N be a prime near-ring admitting a left generalized n-derivation F with associated nonzero n-derivation D of N. If $F(xoy,r_2,r_3,\cdots,r_n)=0$ for all $x,y,r_2,r_3,\cdots,r_n\in N$, then N is commutative ring.

Proof. Given that $F(xoy,r_2,\cdots,r_n)=0$. Substituting xy for y we get $F(x(xoy),r_2,\cdots,r_n)=0$ i.e.; $D(x,r_2,\cdots,r_n)(xoy)+xF(xoy,r_2,\cdots,r_n)=0$. By hypothesis we get $D(x,r_2,\cdots,r_n)(xoy)=0$, that is,

$$D(x,r_2,\cdots,r_n)xy=-D(x,r_2,\cdots,r_n)yx. \tag{3.3}$$

Putting yz for y in (3.3) we have $D(x,r_2,\cdots,r_n)xyz=-D(x,r_2,\cdots,r_n)yzx$, that is $D(x,r_2,\cdots,r_n)xyz+D(x,r_2,\cdots,r_n)yzx=0$. Now substituting the values from (3.3) in the preceding relation we get $\{-D(x,r_2,\cdots,r_n)yx\}z+D(x,r_2,\cdots,r_n)yzx=0$ that is $D(x,r_2,\cdots,r_n)y(-x)z+D(x,r_2,\cdots,r_n)yzx=0$. Replacing x by $-x$ in the preceding relation we have $D(-x,r_2,\cdots,r_n)yxz+D(-x,r_2,\cdots,r_n)yz(-x)=0$, in turn we get $D(-x,r_2,\cdots,r_n)y(xz-zx)=0$ or $D(-x,r_2,\cdots,r_n)N[x,z]=\{0\}$. For each fixed $x\in N$ primeness of N yields either $x\in Z$ or $D(-x,r_2,\cdots,r_n)=0$. If first case holds then $D(xt,r_2,\cdots,r_n)$ = $D(tx,r_2,\cdots,r_n)$ for all $t,r_2,\cdots,r_n\in N$. Using Lemma 2.3 and Remark 2.1 we obtain that $D(x,r_2,\cdots,r_n)t+xD(t,r_2,\cdots,r_n)=tD(x,r_2,\cdots,r_n)+D(t,r_2,\cdots,r_n)x$ for all $t,r_2,\cdots,r_n\in N$ i.e.; $D(x,r_2,\cdots,r_n)\in Z$ and second case implies $-D(x,r_2,\cdots,r_n)=0$ that is, $0=D(x,r_2,\cdots,r_n)\in Z$. Combining both the cases we get $D(x,r_2,\cdots,r_n)\in Z$ for all $x,r_2,\cdots,r_n\in N$ i.e.; $D(N,N,\cdots,N)\subseteq Z$ hence by Lemma 2.5 and Remark 2.1, N is a commutative ring.

Theorem 3.5. Let N be a prime near-ring admitting a left generalized n-derivation F with associated nonzero n-derivation D of N. If $F(xoy,r_2,r_3,\cdots,r_n)=\pm(xoy)$ for all $x,y,r_2,r_3,\cdots,r_n\in N$, then N is a commutative ring.

Proof. We have $F(xoy,r_2,\cdots,r_n)=\pm(xoy)$. Substituting xy for y we obtain
$F(x(xoy),r_2,\cdots,r_n)=\pm x(xoy)$ i.e.; $D(x,r_2,\cdots,r_n)(xoy)+xF(xoy,r_2,\cdots,r_n)=\pm x(xoy)$. By hypothesis we get $D(x,r_2,\cdots,r_n)(xoy)=0$, i.e; $D(x,r_2,\cdots,r_n)xy=-D(x,r_2,\cdots,r_n)yx$, which is identical with (3.3) of Theorem 3.4. Now arguing in the same way as in the Theorem 3.4 we conclude that N is a commutative ring.

Theorem 3.6. Let N be a prime near-ring admitting a left generalized n-derivation F with associated nonzero n-derivation D of N. If $F([x,y],r_2,r_3,\cdots,r_n)=\pm(xoy)$ for all $x,y,r_2,r_3,\cdots,r_n\in N$, then N is a commutative ring.

Proof. We have $F([x,y],r_2,\cdots,r_n)=\pm(xoy)$. Substituting xy for y we obtain $F(x[x,y],r_2,\cdots,r_n)=\pm x(xoy)$ i.e.; $D(x,r_2,\cdots,r_n)[x,y]+xF([x,y],r_2,\cdots,r_n)=\pm x(xoy)$. By hypothesis we get $D(x,r_2,\cdots,r_n)[x,y]=0$ that is, $D(x,r_2,\cdots,r_n)xy=D(x,r_2,\cdots,r_n)yx$, which is identical with (3.2) of Theorem 3.2. Now arguing in the same way as in the Theorem 3.2 we conclude that N is a commutative ring.

Theorem 3.7. Let N be a prime near-ring admitting a left generalized n-derivation F with associated nonzero n-derivation D of N. If $F(xoy,r_2,r_3,\cdots,r_n)=\pm[x,y]$ for all $x,y,r_2,r_3,\cdots,r_n\in N$, then N is a commutative ring.

Proof. Since $F(xoy,r_2,\cdots,r_n)=\pm[x,y]$. Substituting xy for y we obtain $F(x(xoy),r_2,\cdots,r_n)=\pm x[x,y]$ i.e.; $D(x,r_2,\cdots,r_n)(xoy)+xF(xoy,r_2,\cdots,r_n)=\pm x[x,y]$. By hypothesis we get $D(x,r_2,\cdots,r_n)(xoy)=0$ that is,

$$D(x,r_2,\cdots,r_n)xy=-D(x,r_2,\cdots,r_n)yx,$$

which is identical with (3.3) of Theorem 3.4. Now arguing in the same way as in the Theorem 3.4 we conclude that N is a commutative ring.

Theorem 3.8. Let N be a prime near-ring admitting a generalized n-derivation F with associated nonzero n-derivation D of N. If $F([x,y],r_2,r_3,\cdots,r_n)\in Z$ for all $x,y,r_2,r_3,\cdots,r_n\in N$, then N is commutative ring or $D(Z,N,N,\cdots,N)=\{0\}$.

Proof. For all $x,y,r_2,r_3,\cdots,r_n\in N$,

$$F([x,y],r_2,\cdots,r_n)\in Z. \tag{3.4}$$

Now we have two cases,
CaseI: If $Z=\{0\}$, it follows $F([x,y],r_2,\cdots,r_n)=0$ for all $x,y,r_2,r_3,\cdots,r_n\in N$. Now by Theorem 3.2 we conclude that N is a commutative ring.
CaseII: If $Z\neq\{0\}$, replacing y by yz in (3.4), where $z\in Z$, we get $D(z,r_2,\cdots,r_n)[x,y]+zF([x,y],r_2,\cdots,r_n)\in Z$ for all $x,y,r_2,r_3,\cdots,r_n\in N, z\in Z$. Using (3.4) together with Lemma 2.10, preceding relation forces $D(z,r_2,\cdots,r_n)[x,y]\in Z$. Since $z\in Z$, $D(zt,r_2,\cdots,r_n)$ = $D(tz,r_2,\cdots,r_n)$ for all $t,r_2,\cdots,r_n\in N$. Using Lemma 2.3 and Remark 2.1 we obtain that $D(z,r_2,\cdots,r_n)t+zD(t,r_2,\cdots,r_n)=tD(z,r_2,\cdots,r_n)+D(t,r_2,\cdots,r_n)z$ for all $t,r_2,\cdots,r_n\in N$ i.e.; $D(z,r_2,\cdots,r_n)\in Z$. Now we infer that $D(z,r_2,\cdots,r_n)[[x,y],t]=0$ for all $t\in N$. But if $D(Z,N,N,\cdots,N)\neq\{0\}$ then by Lemma 2.1(i) we have $[[x,y],t]=0$ i.e.; $[x,y]\in Z$. Now replacing y by xy in the preceding relation $[[x,y],t]=0$, we have $[x,y][x,t]=0$ which in turn gives us $[x,y]N[x,t]=\{0\}$. In particular we have $[x,y]N[x,y]=\{0\}$. In light of primeness of N we obtain that $[x,y]=0$ and hence N is a commutative near-ring i.e; $N=Z$. Since $N\neq\{0\}$, there exists $p\in N\setminus\{0\}$. Hence $p+p\in N=Z$ and by Lemma 2.1(ii), we conclude that N is a commutative ring.

Theorem 3.9. Let N be a 2-torsion free prime near-ring admitting a generalized n-derivation F with associated nonzero n-derivation D of N. If $F(xoy,r_2,r_3,\cdots,r_n)\in Z$ for all $x,y,r_2,r_3,\cdots,r_n\in N$, then N is a commutative ring or $D(Z,N,N,\cdots,N)=\{0\}$.

Proof. For all $x,y,r_2,r_3,\cdots,r_n\in N$,

$$F(xoy,r_2,\cdots,r_n)\in Z. \tag{3.5}$$

Now we separate the proof in two cases,
CaseI: If $Z=\{0\}$, it follows $F(xoy,r_2,\cdots,r_n)=0$ for all $x,y,r_2,r_3,\cdots,r_n\in N$. Hence by Theorem 3.4 we conclude that N is a commutative ring.
CaseII: If $Z\neq\{0\}$, replacing y by yz in (3.5), where $z\in Z$, we get $D(z,r_2,\cdots,r_n)(xoy)+zF(xoy,r_2,\cdots,r_n)\in Z$ for all $x,y,r_2,r_3,\cdots,r_n\in N, z\in Z$. Using (3.5) together with Lemma 2.10, preceding relation forces $D(z,r_2,\cdots,r_n)(xoy)\in Z$. Since $z\in Z$, $D(zt,r_2,\cdots,r_n)$ = $D(tz,r_2,\cdots,r_n)$ for all $t,r_2,\cdots,r_n\in N$. Using Lemma 2.3 and Remark 2.1 we obtain that $D(z,r_2,\cdots,r_n)t+zD(t,r_2,\cdots,r_n)=tD(z,r_2,\cdots,r_n)+D(t,r_2,\cdots,r_n)z$ for all

$t, r_2, \cdots, r_n \in N$ i.e.; $D(z, r_2, \cdots, r_n) \in Z$ and hence we infer that $D(z, r_2, \cdots, r_n)[xoy, t] = 0$ for all $t \in N$. But if $D(Z, N, N, \cdots, N) \neq \{0\}$ then by Lemma 2.1(i) we have $[xoy, t] = 0$ i.e., $(xoy) \in Z$. Let $0 \neq y \in Z$. Hence $xoy = y(x + x), x^2oy = y(x^2 + x^2)$, it follows by Lemma 2.2 that $x + x \in Z, x^2 + x^2 \in Z$ for all $x \in N$. Thus $(x + x)xt = x(x + x)t = (x^2 + x^2)t = t(x^2 + x^2) = tx(x + x) = (x + x)tx$ for all $x, t \in N$ and therefore $(x + x)N[x, t] = \{0\}$ for all $x, t \in N$. Once again using primeness, we get $x \in Z$ or $2x = 0$ in latter case 2-torsion freeness forces $x = 0$. Consequently, in both the cases we arrive at $x \in Z$ i.e.; $N = Z$ and therefore N is a commutative near-ring. Since $N \neq \{0\}$, there exists $p \in N \setminus \{0\}$. Hence $p + p \in N = Z$ and by Lemma 2.1(ii), we conclude that N is a commutative ring.

Very recently Öznur Gölbasi [10, Theorem 3.1.] proved that if N is a semi prime near-ring and f is a nonzero generalized derivation on N with an associated derivation d such that $f(x)y = xf(y)$ for all $x, y \in N$, then $d = 0$. While proving the theorem it has been assumed that f is a right generalized derivation of N with associated derivation d. We have extended this result in the setting of generalized n-derivation. In fact we proved the following.

Theorem 3.10. Let N be a semi prime near-ring admitting a generalized n-derivation F with associated n-derivation D of N. If $F(x_1, x_2, \cdots, x_n)y_1 = x_1F(y_1, y_2, \cdots, y_n)$ for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N$, then $D = 0$.

Proof. We have
$$F(x_1, x_2, \cdots, x_n)y_1 = x_1F(y_1, y_2, \cdots, y_n) \tag{3.6}$$
for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N$. Putting x_1z_1 in place of x_1 in the above identity (3.6), where $z_1 \in N$ and using Lemma 2.10, we get

$$\begin{aligned} x_1z_1F(y_1, y_2, \cdots, y_n) &= F(x_1z_1, x_2, \cdots, x_n)y_1 \\ &= D(x_1, x_2, \cdots, x_n)z_1y_1 + x_1F(z_1, x_2, \cdots, x_n)y_1. \end{aligned}$$

By (3.6) we find that

$$x_1z_1F(y_1, y_2, \cdots, y_n) = D(x_1, x_2, \cdots, x_n)z_1y_1 + x_1z_1F(y_1, y_2, \cdots, y_n).$$

This yields that $D(x_1, x_2, \cdots, x_n)z_1y_1 = 0$. Now replacing y_1 by $D(x_1, x_2, \cdots, x_n)$ we get $D(x_1, x_2, \cdots, x_n)ND(x_1, x_2, \cdots, x_n) = \{0\}$. But since N is a semi prime near-ring, we conclude that $D = 0$.

Corollary 3.2 ([3], Theorem 3.6). Let N be a semiprime near-ring and D a permuting n-derivation of N. If $D(x_1, x_2, \cdots, x_n)y_1 = x_1D(y_1, y_2, \cdots, y_n)$, for all $x_1, x_2, \cdots, x_n$, $y_1, y_2, \cdots, y_n \in N$, then $D = 0$.

Theorem 3.11. Let N be a prime near-ring admitting a generalized n-derivation F with associated n-derivation D of N. If $K = \{a \in N \mid [F(N, N, \cdots, N), a] = \{0\}\}$ and d stands for the trace of D, then

(i) $a \in K$ implies either $a \in Z$ or $d(a) = 0$.

(ii) $d(K) \subseteq Z$.

Proof. (i) We have
$$F(x_1, x_2, \cdots, x_n)a = aF(x_1, x_2, \cdots, x_n) \tag{3.7}$$
for all $x_1, x_2, \cdots, x_n \in N$. Putting ax_1 in place of x_1 in the above equation and using Lemma 2.10 we get

$$D(a, x_2, \cdots, x_n)x_1a + aF(x_1, x_2, \cdots, x_n)a = aD(a, x_2, \cdots, x_n)x_1 + aaF(x_1, x_2, \cdots, x_n).$$

Using the identity (3.7), we get $D(a, x_2, \cdots, x_n)x_1a = aD(a, x_2, \cdots, x_n)x_1$. Now putting x_1y_1 for x_1 in the latter relation and using it again, we have $D(a, x_2, \cdots, x_n)x_1[y_1, a] = 0$ where $y_1 \in N$. This gives us $D(a, x_2, \cdots, x_n)N[a, y_1] = \{0\}$. Since N is a prime near-ring, either $[a, y_1] = 0$ for all $y_1 \in N$ or $D(a, x_2, \cdots, x_n) = 0$ for all $x_2, \cdots, x_n \in N$. If first holds then $a \in Z$, if not then $D(a, x_2, \cdots, x_n) = 0$, and hence in particular, $D(a, a, \cdots, a) = 0$ or $d(a) = 0$.

(ii) From the above proof we observe that if $a \in K$ then either $a \in Z$ or $d(a) = 0$. But $d(a) = 0$ implies $d(a) \in Z$. If $d(a) \neq 0$ then we have $a \in Z$. In this case we have $D(xa, a \cdots, a) =$

$D(ax, a, \cdots, a)$ for all $x \in N$. Using Lemma 2.3 and Remark 2.1, we obtain that $xD(a, a, \cdots, a)+ D(x, a, \cdots, a)a = D(a, a, \cdots, a)x + aD(x, a, \cdots, a)$. This reduces to $xD(a, a, \cdots, a) = D(a, a, \cdots, a)x$, which shows that $d(a) \in Z$ and thus $d(K) \subseteq Z$.

Corollary 3.3 ([3], Theorem 3.7). Let N be any prime near-ring and D be any nonzero permuting n-derivation of N. If $K = \{a \in N \mid [D(N, N, \cdots, N), a] = \{0\}\}$ and d stands for the trace of D, then

(i) $a \in K$ implies either $a \in Z$ or $d(a) = 0$.

(ii) $d(K) \subseteq Z$.

Corollary 3.4 ([9], Theorem 3.6). If f is a generalized derivation of prime near-ring N with associated nonzero derivation d, $a \in N$ and $[f(x), a] = 0$ for all $x \in N$, then $d(a) \in Z$.

Theorem 3.12. Let N be a prime near-ring admitting a generalized n-derivation F with associated n-derivation D of N such that $D(Z, N, \cdots, N) \neq \{0\}$ and $a \in N$. If $[F(N, N, \cdots, N), a] = \{0\}$, then $a \in Z$.

Proof. Since $D(Z, N, \cdots, N) \neq \{0\}$, there exist $c \in Z, r_2, \cdots, r_n \in N$ all being non zero such that $D(c, r_2, \cdots, r_n) \neq 0$. Furthermore, as D is an n-derivation of N and $c \in Z$, $D(ct, r_2, \cdots, r_n)$ = $D(tc, r_2, \cdots, r_n)$ for all $t \in N$. By Lemma 2.3 and Remark 2.1, we infer that $D(c, r_2, \cdots, r_n)t+ cD(t, r_2, \cdots, r_n) = tD(c, r_2, \cdots, r_n) + D(t, r_2, \cdots, r_n)c$ for all $t \in N$ i.e.; $D(c, r_2, \cdots, r_n) \in Z$. By hypothesis $F(cx, r_2, \cdots, r_n)a = aF(cx, r_2, \cdots, r_n)$ for all $x \in N$ using Lemma 2.10 we have $D(c, r_2, \cdots, r_n)xa+cF(x, r_2, \cdots, r_n)a = aD(c, r_2, \cdots, r_n)x+acF(x, r_2, \cdots, r_n)$. Since both $D(c, r_2, \cdots, r_n)$ and c are elements of Z, using the hypothesis again previous equation takes the form $D(c, r_2, \cdots, r_n)[x, a] = 0$ i.e.; $D(c, r_2, \cdots, r_n)N[x, a] = \{0\}$. By primeness of N and $0 \neq D(c, r_2, \cdots, r_n)$ we obtain that $a \in Z$.

Corollary 3.5 ([9], Theorem 3.5). If f is a generalized derivation of prime near-ring N with associated nonzero derivation d such that $d(Z) \neq \{0\}$, and $a \in N$, $[f(x), a] = 0$ for all $x \in N$, then $a \in Z$.

Theorem 3.13. Let N be a prime near-ring admitting a generalized n-derivation F with associated n-derivation D of N such that $D(Z, N, \cdots, N) \neq \{0\}$. If $G : N \times N \times \cdots N \longrightarrow N$ is a map such that $[F(N, N, \cdots, N), G(N, N, \cdots, N)] = \{0\}$, then $G(N, N, \cdots, N) \subseteq Z$.

Proof. Taking $G(N, N, \cdots, N)$ instead of a in Theorem 3.12., we get the required result.

Theorem 3.14. Let N be a prime near-ring admitting a generalized n-derivation F with associated n-derivation D of N such that $D(Z, N, \cdots, N) \neq \{0\}$. If G is a nonzero generalized n-derivation of N such that $[F(N, N, \cdots, N), G(N, N, \cdots, N)] = \{0\}$, then N is a commutative ring.

Proof. Since G, a nonzero generalized n-derivation is a map from $N \times N \times \cdots N$ to N. Therefore by Theorem 3.13. we get $G(N, N, \cdots, N) \subseteq Z$. Thus N is a commutative ring by Theorem 3.1.

Theorem 3.15. Let F and G be generalized n-derivations of prime near-ring N with associated nonzero n-derivations D and H of N respectively such that $F(x_1, x_2, \cdots, x_n)H(y_1, y_2, \cdots, y_n) = -G(x_1, x_2, \cdots, x_n)D(y_1, y_2, \cdots, y_n)$ for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N$. Then $(N, +)$ is an abelian group.

Proof. For all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N$ we have,
$F(x_1, x_2, \cdots, x_n)H(y_1, y_2, \cdots, y_n) = -G(x_1, x_2, \cdots, x_n)D(y_1, y_2, \cdots, y_n)$. We substitute $y_1 + y_1^{'}$ for y_1 in preceding relation thereby obtaining,

$$F(x_1, x_2, \cdots, x_n)H(y_1 + y_1^{'}, y_2, \cdots, y_n) + G(x_1, x_2, \cdots, x_n)D(y_1 + y_1^{'}, y_2, \cdots, y_n) = 0$$

that is,

$$F(x_1, x_2, \cdots, x_n)H(y_1, y_2, \cdots, y_n) + F(x_1, x_2, \cdots, x_n)H(y_1^{'}, y_2, \cdots, y_n)$$
$$+G(x_1, x_2, \cdots, x_n)D(y_1, y_2, \cdots, y_n) + G(x_1, x_2, \cdots, x_n)D(y_1^{'}, y_2, \cdots, y_n) = 0.$$

Using the hypothesis we get,

$$F(x_1, x_2, \cdots, x_n)H(y_1, y_2, \cdots, y_n) + F(x_1, x_2, \cdots, x_n)H(y_1^{'}, y_2, \cdots, y_n)$$
$$-F(x_1, x_2, \cdots, x_n)H(y_1, y_2, \cdots, y_n) - F(x_1, x_2, \cdots, x_n)H(y_1^{'}, y_2, \cdots, y_n) = 0$$

that is, $F(x_1, x_2, \cdots, x_n)H((y_1, y_1^{'}), y_2, \cdots, y_n) = 0$. Now using Lemma 2.12$(ii)$ we get $H((y_1, y_1^{'}), y_2, \cdots, y_n) = 0$. Replacing $(y_1, y_1^{'})$ by $w(y_1, y_1^{'})$ where $w \in N$ in the previous relation and using it again we have $H(w, y_2, \cdots, y_n)(y_1, y_1^{'}) = 0$ for all $w, y_1, y_1^{'}, y_2, \cdots, y_n \in N$. Since $H \neq 0$, by Lemma 2.4 and Remark 2.1, we conclude that $(y_1, y_1^{'}) = 0$, i.e.; $(N, +)$ is an abelian group.

Corollary 3.6 ([3],Theorem 3.4). Let N be a prime near-ring with nonzero permuting n-derivations D_1 and D_2 such that

$$D_1(x_1, x_2, \cdots, x_n)D_2(y_1, y_2, \cdots, y_n) = -D_2(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n)$$

for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in N$. Then $(N, +)$ is an abelian group.

Theorem 3.16. Let F_1 and F_2 be generalized n-derivations of prime near-ring N with associated nonzero n-derivations D_1 and D_2 of N respectively such that

$$[F_1(N, N, \cdots, N), F_2(N, N, \cdots, N)] = \{0\}.$$

Then $(N, +)$ is an abelian group.

Proof. If both z and $z + z$ commute element wise with $F_2(N, N, \cdots, N)$, then

$$zF_2(x_1, x_2, \cdots, x_n) = F_2(x_1, x_2, \cdots, x_n)z$$

and

$$(z + z)F_2(x_1, x_2, \cdots, x_n) = F_2(x_1, x_2, \cdots, x_n)(z + z)$$

for all $x_1, x_2, \cdots, x_n \in N$. In particular,

$$(z + z)F_2(x_1 + x_1^{'}, x_2, \cdots, x_n) = F_2(x_1 + x_1^{'}, x_2, \cdots, x_n)(z + z) \text{ for all } x_1, x_1^{'}, \cdots, x_n \in N.$$

From the previous equalities we get $zF_2(x_1 + x_1^{'} - x_1 - x_1^{'}, x_2, \cdots, x_n) = 0$, that is, $zF_2((x_1, x_1^{'}), x_2, \cdots, x_n) = 0$. Putting $z = F_1(y_1, y_2, \cdots, y_n)$ we get

$$F_1(y_1, y_2, \cdots, y_n)F_2((x_1, x_1^{'}), x_2, \cdots, x_n) = 0.$$

By Lemma 2.12(ii) we conclude that $F_2((x_1, x_1^{'}), x_2, \cdots, x_n) = 0$. Putting $w(x_1, x_1^{'})$ in place of additive commutator $(x_1, x_1^{'})$ where $w \in N$ we have $F_2(w(x_1, x_1^{'}), x_2, \cdots, x_n) = 0$ that is,

$$D_2(w, x_2, \cdots, x_n)(x_1, x_1^{'}) + wF_2((x_1, x_1^{'}), x_2, \cdots, x_n) = 0.$$

Previous equality yields $D_2(w, x_2, \cdots, x_n)(x_1, x_1^{'}) = 0$. By Lemma 2.4 and Remark 2.1, we conclude that $(x_1, x_1^{'}) = 0$. Hence $(N, +)$ is an abelian group.

Corollary 3.7([3],Theorem 3.3). Let N be a prime near-ring and D_1 and D_2 be any two nonzero permuting n-derivations of N. If $[D_1(N, N, \cdots, N), D_2(N, N, \cdots, N)] = \{0\}$, then $(N, +)$ is an abelian group.

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