Palestine Journal of Mathematics

Vol. 4(1) (2015) , 144–155

q-ANALOGUES OF SAIGO FRACTIONAL INTEGRALS AND DERIVATIVES OF GENERALIZED BASIC HYPERGEOMETRIC SERIES

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Communicated by P. K. Banerji

MSC 2010 Classifications: 33D15, 26A33, 33D05 .

Keywords and phrases: Basic hypergeometric series, q-analogue of Saigo's fractional integral operators, q-analogue of Saigo's fractional derivatives.

The support provided by the University Grants Commission, New Delhi through Rajiv Gandhi National Fellowship to one of the authors Subhash Alha is gratefully acknowledged.

Abstract In the present paper, we obtain q-analogues of Saigo fractional integrals and derivatives of generalized basic hypergeometric series. Similar results for some simpler functions and polynomials have also been derived as special cases of our main findings.

1 Introduction

We first give some definitions and notations, which have been taken from the book by Gasper and Rahman [2].

The ***q*-shifted factorial** (q-analogue of Pochhammer symbol) is defined as

$$(a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right),\ n \in N \tag{1.1}$$

with $(a;q)_0 = 1$, $q \neq 1$.
If we consider $(a;q)_\infty$ then as the infinite product diverges when $a \neq 0$ and $|q| \geq 1$, therefore whenever $(a;q)_\infty$appears in a formula, we shall assume that $|q| < 1$.
Also, for any complex number α, we have

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{1.2}$$

where the principal value of q^α is taken.

The ***q*-analogue of power function** $(z-a)^\alpha$is defined as

$$(z-a)_q^\alpha = z^\alpha \left({}^{a}\!/\!_{z};q\right)_\alpha$$

$$= z^\alpha \prod_{j=0}^{\infty} \left[\frac{1-\left({}^{a}\!/\!_{z}\right) q^j}{1-\left({}^{a}\!/\!_{z}\right) q^{j+\alpha}}\right] = z^\alpha \frac{\left({}^{a}\!/\!_{z};q\right)_\infty}{\left(q^\alpha {}^{a}\!/\!_{z};q\right)_\infty},\ 0<|q|<1,\ (z \neq 0)\,. \tag{1.3}$$

The ***q*-gamma function** is defined as

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad \text{if } 0<|q|<1, \tag{1.4}$$

where $z \in \mathbb{C}/\{0,-1,-2,...\}$ and the principal values of $q^z$and $(1-q)^{1-z}$ are taken.

The ***q*-derivative** of an analytic function $f(z)$ is defined as follows

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z},\ (z \neq 0,\ q \neq 1)\,. \tag{1.5}$$

and

$$\lim_{q \to 1} D_q f(z) = \frac{df(z)}{dz}.$$

We have

$$D_q^n z^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu-n+1)} z^{\mu-n}, Re(\mu)+1>0. \tag{1.6}$$

The ***q*-integrals** of a function $f(t)$ are defined as follows

$$\int_0^a f(t)\, d_q t = a(1-q)\sum_{k=0}^{\infty} q^k f\left(aq^k\right), \tag{1.7}$$

and

$$\int_a^{\infty} f(t)\, d_q t = a(1-q)\sum_{k=1}^{\infty} q^{-k} f\left(aq^{-k}\right). \tag{1.8}$$

The **generalized basic hypergeometric series** is defined as follows

$$ {}_r\phi_s\left[\begin{array}{c} a_1,...,a_r \\ b_1,...,b_s \end{array};q,z\right] = \sum_{n=0}^{\infty}\frac{(a_1,...,a_r;q)_n}{(b_1,...,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{z^n}{(q;q)_n}, \tag{1.9}$$

with $\binom{n}{2} = n(n-1)/2$ and $(a_1,...,a_r;q)_n = (a_1;q)_n \ldots (a_r;q)_n$.

If $0<|q|<1$, series (1.9) converges absolutely for all z if $r \le s$ and for $|z|<1$ if $r = s+1$. Also if $|q|>1$, the series converges absolutely for $|z| < |b_1...b_s q| \,/\, |a_1...a_r|$.

The ***q*-binomial theorem** is given by

$$ {}_1\phi_0\left[\begin{array}{c} \alpha \\ - \end{array};q,z\right] = \frac{(\alpha z;q)_\infty}{(z;q)_\infty}, \qquad |z|<1,\ 0<|q|<1. \tag{1.10}$$

From (1.3) and (1.10), follows that

$$(z-a)_q^{\alpha} = z^{\alpha}\, {}_1\phi_0\left[\begin{array}{c} q^{-\alpha} \\ - \end{array};q,\frac{a}{z}q^{\alpha}\right]. \tag{1.11}$$

For $0<|q|<1$, ***q*-analogues of exponential functions** are given by

$$e_q^z = {}_1\phi_0\left(\begin{array}{c} 0 \\ - \end{array};q,z\right) = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},\ \ |z|<1, \tag{1.12}$$

and

$$E_q^z = {}_0\phi_0\left(\begin{array}{c} - \\ - \end{array};q,-z\right) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}} z^n}{(q;q)_n} = (-z;q)_\infty. \tag{1.13}$$

Basic hypergeometric polynomials (Koekoek *et al.*[3]).

The little q-Jacobi polynomials

$$p_n(z;a,b;q) = {}_2\phi_1\left(\begin{array}{c} q^{-n}, abq^{n+1} \\ aq \end{array};q,qz\right). \tag{1.14}$$

The little q-Legendre polynomials

$$P_n(z;q) = {}_2\phi_1\left(\begin{array}{c} q^{-n}, q^{n+1} \\ q \end{array};q,qz\right). \tag{1.15}$$

The little q-Laguerre polynomials

$$p_n(z;a,q) = {}_2\phi_1\left(\begin{array}{c} q^{-n}, 0 \\ aq \end{array};q,qz\right). \tag{1.16}$$

The q-Laguerre polynomials

$$L_n^{(a)}(z;q) = \frac{(q^{a+1};q)_n}{(q;q)_n}\, {}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{a+1} \end{array};q,-q^{n+a+1}z\right). \tag{1.17}$$

The Stieltjes-Wigert polynomials

$$S_n(z;q) = \frac{1}{(q;q)_n}\, {}_1\phi_1\left(\begin{array}{c} q^{-n} \\ 0 \end{array};q,-q^{n+1}z\right). \tag{1.18}$$

2 *q*-analogues of Saigo fractional integrals and derivatives(

Garg & Chanchlani [1])

The *q*-analogues of Saigo fractional integrals of order $\alpha \in \mathbb{C}\ (Re\,(\alpha) > 0)$, for a real valued function $f\,(x)$ on $(0,\infty)$, are defined as follows

$$I_q^{\alpha,\beta,\eta} f\,(x) = \frac{x^{-\beta-1}}{\Gamma_q\,(\alpha)} \int_0^x \left(tq/x;q\right)_{\alpha-1}$$

$$\times \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m \left(q^{-\eta};q\right)_m}{\left(q^{\alpha};q\right)_m \left(q;q\right)_m} \left(-1\right)^m q^{(\eta-\beta)m} q^{-\binom{m}{2}} \left(\frac{t}{x}-1\right)_q^m f\,(t)\,d_q t, \tag{2.1}$$

and

$$J_q^{\alpha,\beta,\eta} f\,(x) = \frac{q^{-\alpha(\alpha+1)/2-\beta}}{\Gamma_q\,(\alpha)} \int_x^{\infty} \left(x/t;q\right)_{\alpha-1} t^{-\beta-1}$$

$$\times \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m \left(q^{-\eta};q\right)_m}{\left(q^{\alpha};q\right)_m \left(q;q\right)_m} \left(-1\right)^m q^{(\eta-\beta)m} q^{-\binom{m}{2}} \left(\frac{x}{qt}-1\right)_q^m f\left(tq^{1-\alpha}\right) d_q t. \tag{2.2}$$

where $0 < |q| < 1$, β and η being real or complex.

Definitions given by (2.1) and (2.2), in view of (1.7) and (1.8) can be written as

$$I_q^{\alpha,\beta,\eta} f\,(x) = x^{-\beta}\left(1-q\right)^{\alpha}$$

$$\times \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m \left(q^{-\eta};q\right)_m}{\left(q;q\right)_m} q^{(\eta-\beta+1)m} \sum_{k=0}^{\infty} q^k \frac{\left(q^{\alpha+m};q\right)_k}{\left(q;q\right)_k} f\left(xq^{k+m}\right). \tag{2.3}$$

and

$$J_q^{\alpha,\beta,\eta} f\,(x) = x^{-\beta} q^{-\alpha(\alpha+1)/2}\left(1-q\right)^{\alpha}$$

$$\times \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m \left(q^{-\eta};q\right)_m}{\left(q;q\right)_m} q^{\eta m} \sum_{k=0}^{\infty} q^{\beta k} \frac{\left(q^{\alpha+m};q\right)_k}{\left(q;q\right)_k} f\left(xq^{-\alpha-k-m}\right). \tag{2.4}$$

The ***q*-analogues of Saigo fractional derivatives** of order $\alpha \in \mathbb{C}\ (m-1 < Re\,(\alpha) \le m,\ m \in \mathbb{N})$, for a real valued function $f\,(x)$ on $(0,\infty)$, are defined as follows

$$D_q^{\alpha,\,\beta,\,\eta} f\,(x) = D_q^m I_q^{-\alpha+m,\,-\beta-m,\,\alpha+\eta-m} f\,(x) \tag{2.5}$$

and

$$P_q^{\alpha,\beta,\eta} f\,(x) = q^{\alpha(\alpha+\beta)}\left(-q^{-(\alpha+\beta)} D_q\right)^m J_q^{-\alpha+m,-\beta-m,\alpha+\eta} f\,(x), \tag{2.6}$$

where $0 < |q| < 1$, β and η being real or complex, $I_q^{\alpha,\beta,\eta}$ and $J_q^{\alpha,\beta,\eta}$ are given by (2.1) and (2.2) respectively.

For $0 < |q| < 1$, $Re\,(\alpha) > 0$, β, η and μ being real or complex, **images of power function under fractional *q*-integrals** $I_q^{\alpha,\beta,\eta}$ **and** $J_q^{\alpha,\beta,\eta}$ are given by

$$I_q^{\alpha,\beta,\eta}\left(x^{\mu}\right) = \frac{\Gamma_q\,(\mu+1)\,\Gamma_q\,(\mu-\beta+\eta+1)}{\Gamma_q\,(\mu-\beta+1)\,\Gamma_q\,(\mu+\alpha+\eta+1)} x^{\mu-\beta}, \tag{2.7}$$

provided $Re(\mu+1) > 0$ and $Re(\mu-\beta+\eta+1) > 0$.
and

$$J_q^{\alpha,\beta,\eta}\left(x^{\mu}\right) = \frac{\Gamma_q\,(\beta-\mu)\,\Gamma_q\,(\eta-\mu)}{\Gamma_q\,(-\mu)\,\Gamma_q\,(\beta+\alpha-\mu+\eta)} x^{\mu-\beta} q^{-\alpha\mu-\alpha(\alpha+1)/2}, \tag{2.8}$$

provided $Re(\beta-\mu) > 0$ and $Re(\eta-\mu) > 0$.

For $0 < |q| < 1$, $n-1 < Re\,(\alpha) \le n,\ n \in \mathbb{N}$, β, η and μ being real or complex, **images of power function under fractional *q*-derivatives** $D_q^{\alpha,\beta,\eta}$ **and** $P_q^{\alpha,\beta,\eta}$ are given by

$$D_q^{\alpha,\beta,\eta}\left(x^{\mu}\right) = \frac{\Gamma_q\,(\mu+1)\,\Gamma_q\,(\mu+\alpha+\beta+\eta+1)}{\Gamma_q\,(\mu+\beta+1)\,\Gamma_q\,(\mu+\eta+1)} x^{\mu+\beta}, \tag{2.9}$$

provided $Re(\mu+1) > 0$ and $Re(\mu+\alpha+\beta+\eta+1) > 0$.
and

$$P_q^{\alpha,\beta,\eta}\left(x^{\mu}\right) = \frac{\Gamma_q\,(-\beta-\mu)\,\Gamma_q\,(\alpha+\eta-\mu)}{\Gamma_q\,(-\mu)\,\Gamma_q\,(-\beta+\eta-\mu)} q^{\alpha(\beta+\mu)+\alpha(\alpha-1)/2} x^{\mu+\beta}, \tag{2.10}$$

provided $Re(-\beta-\mu) > 0$ and $Re(\alpha+\eta-\mu) > 0$.

3 q-analogues of Saigo fractional integrals and derivatives of generalized basic hypergeometric series

Theorem 3.1. *Let* $\alpha,\beta,\eta,\rho,\lambda\in\mathbb{C},\ 0<|q|<1$ *and* $Re\,(\alpha)>0$*, then for*

(i) $Re\,(\lambda+1)>0$ and $Re\,(\lambda-\beta+\eta+1)>0$*, we have*

$$I_q^{\alpha,\beta,\eta}\left(x^{\lambda}\,{}_r\phi_s\left[\begin{array}{c}a_1,...,a_r\\ b_1,...,b_s\end{array};q,\rho x\right]\right)$$

$$=\frac{\Gamma_q(\lambda+1)\,\Gamma_q(\lambda-\beta+\eta+1)}{\Gamma_q(\lambda-\beta+1)\,\Gamma_q(\lambda+\alpha+\eta+1)}x^{\lambda-\beta}\,{}_{r+2}\phi_{s+2}\left[\begin{array}{c}a_1,...,a_r,q^{\lambda+1},q^{\lambda-\beta+\eta+1}\\ b_1,...,b_s,q^{\lambda-\beta+1},q^{\lambda+\alpha+\eta+1}\end{array};q,\rho x\right]. \tag{3.1}$$

(ii) $Re\,(\beta+\lambda)>0$ and $Re\,(\eta+\lambda)>0,\ \lambda\neq 0$*, we have*

$$J_q^{\alpha,\beta,\eta}\left(x^{-\lambda}\,{}_r\phi_s\left[\begin{array}{c}a_1,...,a_r\\ b_1,...,b_s\end{array};q,{}^{\rho}\!/_{x}\right]\right)$$

$$=\frac{\Gamma_q(\beta+\lambda)\,\Gamma_q(\eta+\lambda)}{\Gamma_q(\lambda)\,\Gamma_q(\beta+\alpha+\eta+\lambda)}x^{-\lambda-\beta}q^{\alpha\lambda-\alpha(\alpha+1)/2}\,{}_{r+2}\phi_{s+2}\left[\begin{array}{c}a_1,...,a_r,q^{\beta+\lambda},q^{\eta+\lambda}\\ b_1,...,b_s,q^{\lambda},q^{\beta+\alpha+\eta+\lambda}\end{array};q,{}^{\rho q^{\alpha}}\!/_{x}\right]. \tag{3.2}$$

We can not directly put $\lambda=0$ *in* (3.2) *as in the case* $\Gamma_q(\lambda)$ *appears in the denominator of r.h.s. of the equation, but if we take limit of* (3.2) *as* $\lambda\to 0$*, we arrive at the following result which we obtain here independently for convenience of derivation.*

(iii) $Re\,(\beta)>0$ and $Re\,(\eta)>0$*, we have*

$$J_q^{\alpha,\beta,\eta}\left({}_r\phi_s\left[\begin{array}{c}a_1,...,a_r\\ b_1,...,b_s\end{array};q,{}^{\rho}\!/_{x}\right]\right)$$

$$=q^{-\alpha(\alpha-1)/2}(-1)^{1+s-r}\rho x^{-\beta-1}\frac{\Gamma_q(\beta+1)\,\Gamma_q(\eta+1)}{\Gamma_q(\alpha+\beta+\eta+1)}\frac{(1-a_1)(1-a_2)...(1-a_r)}{(1-b_1)(1-b_2)...(1-b_s)}\frac{1}{(1-q)}$$

$$\times\,{}_{r+2}\phi_{s+2}\left[\begin{array}{c}a_1q,...,a_rq,q^{\beta+1},q^{\eta+1}\\ b_1q,...,b_sq,q^2,q^{\beta+\alpha+\eta+1}\end{array};q,{}^{\rho q^{1+\alpha+s-r}}\!/_{x}\right]. \tag{3.3}$$

Here $I_q^{\alpha,\beta,\eta}$ *and* $J_q^{\alpha,\beta,\eta}$ *are defined by* (2.1) *and* (2.2) *respectively and* ${}_r\phi_s$ *is basic hypergeometric series given by* (1.9)*. We also assume that the conditions of convergence of* ${}_r\phi_s$ *as given with its definitions are satisfied.*

Proof. (i) On using the definition (1.9), the left side of (3.1) becomes

$$I_q^{\alpha,\beta,\eta}\left(\sum_{n=0}^{\infty}\frac{(a_1;q)_n...(a_r;q)_n}{(b_1;q)_n...(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{\rho^n x^{\lambda+n}}{(q;q)_n}\right). \tag{3.4}$$

On interchanging the order of fractional q-integral and summation, we have

$$\sum_{n=0}^{\infty}\frac{(a_1;q)_n...(a_r;q)_n\rho^n}{(b_1;q)_n...(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}I_q^{\alpha,\beta,\eta}\left(\frac{x^{\lambda+n}}{(q;q)_n}\right). \tag{3.5}$$

On using (2.7), it becomes

$$\sum_{n=0}^{\infty}\frac{(a_1;q)_n...(a_r;q)_n\rho^n}{(b_1;q)_n...(b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{\Gamma_q(\lambda+n+1)\,\Gamma_q(\lambda+n-\beta+\eta+1)}{\Gamma_q(\lambda+n-\beta+1)\,\Gamma_q(\lambda+n+\alpha+\eta+1)}x^{\lambda+n-\beta}. \tag{3.6}$$

Using (1.4) and simplifying, we arrive at

$$\frac{\Gamma_q(\lambda+1)\Gamma_q(\lambda-\beta+\eta+1)}{\Gamma_q(\lambda-\beta+1)\Gamma_q(\lambda+\alpha+\eta+1)}x^{\lambda-\beta}$$

$$\times\sum_{n=0}^{\infty}\frac{(a_1;q)_n\dots(a_r;q)_n}{(b_1;q)_n\dots(b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{(q^{\lambda+1};q)_n(q^{\lambda-\beta+\eta+1};q)_n}{(q^{\lambda-\beta+1};q)_n(q^{\lambda+\alpha+\eta+1};q)_n}(\rho x)^n. \tag{3.7}$$

Interpreting (3.7) with (1.9) it gives the right hand side of (3.1).

(ii) The result (3.2) can be proved on similar lines.

(iii) On using the definition (1.9) the left side of (3.3) becomes

$$J_q^{\alpha,\beta,\eta}\left(\sum_{n=0}^{\infty}\frac{(a_1;q)_n\dots(a_r;q)_n}{(b_1;q)_n\dots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{\rho^n x^{-n}}{(q;q)_n}\right) \tag{3.8}$$

On interchanging the order of fractional q-integral and summation, we get its value as

$$\sum_{n=0}^{\infty}\frac{(a_1;q)_n\dots(a_r;q)_n\rho^n}{(b_1;q)_n\dots(b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} J_q^{\alpha,\beta,\eta}\left(x^{-n}\right). \tag{3.9}$$

On using (2.8), it becomes

$$q^{-\alpha(\alpha+1)/2}\sum_{n=1}^{\infty}\frac{(a_1;q)_n\dots(a_r;q)_n\rho^n}{(b_1;q)_n\dots(b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{\Gamma_q(\beta+n)\Gamma_q(\eta+n)}{\Gamma_q(n)\Gamma_q(\beta+\alpha+\eta+n)}x^{-n-\beta}q^{\alpha n}. \tag{3.10}$$

Changing the summation index to run from 0 to ∞, using the result

$$(a;\,q)_{n+1}=(1-a)(aq;\,q)_n \tag{3.11}$$

and doing some simplifications, we arrive at

$$q^{-\alpha(\alpha-1)/2}(-1)^{1+s-r}\rho x^{-\beta-1}\frac{\Gamma_q(\beta+1)\Gamma_q(\eta+1)}{\Gamma_q(\beta+\alpha+\eta+1)}\frac{(1-a_1)(1-a_2)\dots(1-a_r)}{(1-b_1)(1-b_2)\dots(1-b_s)}\frac{1}{(1-q)}$$

$$\times\sum_{n=0}^{\infty}\frac{(a_1q;q)_n\dots(a_rq;q)_n}{(b_1q;q)_n\dots(b_sq;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{(q^{\beta+1};q)_n(q^{\eta+1};q)_n}{(q^2;q)_n(q^{\beta+\alpha+\eta+1};q)_n}\rho^n x^{-n}q^{n(1+\alpha+s-r)}. \tag{3.12}$$

Interpreting (3.12) with (1.9) it gives the right side of (3.3).

□

If we take $\beta=0$ in (3.1) and (3.2), the *q*-analogues of Saigo fractional integrals reduce to *q*-analogues of Kober fractional integrals and we arrive at the results obtained by Yadav and Purohit [5] and Yadav et al. [7].

If we take $\beta=-\alpha$ in (3.1) and (3.2), the *q*-analogues of Saigo fractional integrals reduce to *q*-analogues of Riemann-Liouville and Weyl fractional integrals and we arrive at the results obtained by Yadav and Purohit [4] and Yadav and Purohit [6].

Theorem 3.2. *Let* $\alpha,\beta,\eta,\rho,\lambda\in\mathbb{C},\ 0<|q|<1$ *and* $Re(\alpha)>0$*, then for* *(i)* $Re\,(\lambda+1)>0$ and $Re\,(\lambda+\alpha+\beta+\eta+1)>0$, *we have*

$$D_q^{\alpha,\beta,\eta}\left(x^{\lambda}{}_r\phi_s\left[\begin{array}{c}a_1,...,a_r\\b_1,...,b_s\end{array};q,\rho x\right]\right)$$

$$=\frac{\Gamma_q(\lambda+1)\,\Gamma_q(\lambda+\alpha+\beta+\eta+1)}{\Gamma_q(\lambda+\beta+1)\,\Gamma_q(\lambda+\eta+1)}x^{\lambda+\beta}{}_{r+2}\phi_{s+2}\left[\begin{array}{c}a_1,...,a_r,q^{\lambda+1},q^{\lambda+\alpha+\beta+\eta+1}\\b_1,...,b_s,q^{\lambda+\beta+1},,q^{\lambda+\eta+1}\end{array};q,\rho x\right].\tag{3.13}$$

(ii) $Re\,(\lambda-\beta)>0$ and $Re\,(\alpha+\eta+\lambda)>0,\ \lambda\neq 0$, *we have*

$$P_q^{\alpha,\beta,\eta}\left(x^{-\lambda}{}_r\phi_s\left[\begin{array}{c}a_1,...,a_r\\b_1,...,b_s\end{array};q,\rho/x\right]\right)$$

$$=q^{\alpha(\beta-\lambda)+\alpha(\alpha-1)/2}x^{\beta-\lambda}\frac{\Gamma_q(\lambda-\beta)\,\Gamma_q(\alpha+\eta+\lambda)}{\Gamma_q(\lambda)\,\Gamma_q(\lambda+\eta-\beta)}{}_{r+2}\phi_{s+2}\left[\begin{array}{c}a_1,...,a_r,q^{\lambda-\beta},q^{\alpha+\eta+\lambda}\\b_1,...,b_s,q^{\lambda},q^{\lambda+\eta-\beta}\end{array};q,\rho/xq^{\alpha}\right].\tag{3.14}$$

The result (3.14) *for* $\lambda=0$ *can be obtained independently as follows.*

(iii) $Re\,(-\beta)>0$ and $Re\,(\alpha+\eta)>0$, *we have*

$$P_q^{\alpha,\beta,\eta}\left({}_r\phi_s\left[\begin{array}{c}a_1,...,a_r\\b_1,...,b_s\end{array};q,\rho/x\right]\right)$$

$$=q^{\alpha(\alpha+2\beta-3)/2}(-1)^{1+s-r}\rho x^{\beta-1}\frac{\Gamma_q(-\beta+1)\,\Gamma_q(\eta+\alpha+1)}{\Gamma_q(-\beta+\eta+1)}\frac{(1-a_1)(1-a_2)...(1-a_r)}{(1-b_1)(1-b_2)...(1-b_s)}\frac{1}{(1-q)}$$

$$\times{}_{r+2}\phi_{s+2}\left[\begin{array}{c}a_1q,...,a_rq,q^{-\beta+1},q^{\eta+\alpha+1}\\b_1q,...,b_sq,q^2,q^{-\beta+\eta+1}\end{array};q,\rho q^{1-\alpha+s-r}/x\right].\tag{3.15}$$

where $D_q^{\alpha,\beta,\eta}$ *and* $P_q^{\alpha,\beta,\eta}$*are defined by* (2.5) *and* (2.6) *respectively and* ${}_r\phi_s$ *is basic hypergeometric series given by* (1.9). *We also assume that the conditions of convergence of* ${}_r\phi_s$ *as given with its definitions are satisfied.*

Proof. (i) On using the definition (1.9), the left side of (3.13) becomes

$$D_q^{\alpha,\beta,\eta}\sum_{n=0}^{\infty}\frac{(a_1;q)_n...(a_r;q)_n}{(b_1;q)_n...(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{\rho^n x^{\lambda+n}}{(q;q)_n}.\tag{3.16}$$

On interchanging the order of fractional q-derivative and summation, we have

$$\sum_{n=0}^{\infty}\frac{(a_1;q)_n...(a_r;q)_n\rho^n}{(b_1;q)_n...(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}D_q^{\alpha,\beta,\eta}\left(\frac{x^{\lambda+n}}{(q;q)_n}\right).\tag{3.17}$$

On using (2.9), it becomes

$$\sum_{n=0}^{\infty}\frac{(a_1;q)_n...(a_r;q)_n\rho^n}{(b_1;q)_n...(b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{\Gamma_q(\lambda+n+1)\,\Gamma_q(\lambda+n+\alpha+\beta+\eta+1)}{\Gamma_q(\lambda+n+\beta+1)\,\Gamma_q(\lambda+n+\eta+1)}x^{\lambda+n+\beta}.\tag{3.18}$$

Using (1.4) and simplifying, we get

$$\frac{\Gamma_q(\lambda+1)\,\Gamma_q(\lambda+\alpha+\beta+\eta+1)}{\Gamma_q(\lambda+\beta+1)\,\Gamma_q(\lambda+\eta+1)}x^{\lambda+\beta}$$

$$\times\sum_{n=0}^{\infty}\frac{(a_1;q)_n\,...\,(a_r;q)_n}{(b_1;q)_n\,...\,(b_s;q)_n\,(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{(q^{\lambda+1};q)_n\,(q^{\lambda+\alpha+\beta+\eta+1};q)_n}{(q^{\lambda+\beta+1};q)_n\,(q^{\lambda+\eta+1};q)_n}(\rho x)^n. \tag{3.19}$$

Interpreting (3.7) with (1.9) it gives the right hand side of (3.13).

(ii) The result (3.14) can be proved on similar lines.

(iii) On using definition (1.9), the left side of (3.15) becomes

$$P_q^{\alpha,\beta,\eta}\left(\sum_{n=0}^{\infty}\frac{(a_1;q)_n\,...\,(a_r;q)_n}{(b_1;q)_n\,...\,(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{\rho^n x^{-n}}{(q;q)_n}\right) \tag{3.20}$$

On interchanging the order of fractional q-derivative and summation, we have

$$\sum_{n=0}^{\infty}\frac{(a_1;q)_n\,...\,(a_r;q)_n\,\rho^n}{(b_1;q)_n\,...\,(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}P_q^{\alpha,\beta,\eta}\left(\frac{x^{-n}}{(q;q)_n}\right). \tag{3.21}$$

On using (2.9), it becomes

$$q^{\alpha(\alpha-1)/2}\sum_{n=1}^{\infty}\frac{(a_1;q)_n\,...\,(a_r;q)_n\,\rho^n}{(b_1;q)_n\,...\,(b_s;q)_n\,(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}$$

$$\times\frac{\Gamma_q(-\beta+n)\,\Gamma_q(n+\eta+\alpha)}{\Gamma_q(n)\,\Gamma_q(-\beta+\eta+n)}x^{-n+\beta}q^{\alpha(\beta-n)}. \tag{3.22}$$

Changing the summation index to run from 0 to ∞, using the result

$$(a;\,q)_{n+1}=(1-a)\,(aq;\,q)_n\,, \tag{3.23}$$

and doing some simplifications, we get its value as

$$q^{\alpha(\alpha+2\beta-3)/2}\,(-1)^{1+s-r}\,\rho x^{\beta-1}\frac{\Gamma_q(-\beta+1)\,\Gamma_q(\eta+\alpha+1)}{\Gamma_q(-\beta+\eta+1)}\frac{(1-a_1)(1-a_2)\,...\,(1-a_r)}{(1-b_1)(1-b_2)\,...\,(1-b_s)}\frac{1}{(1-q)}$$

$$\times\sum_{n=0}^{\infty}\frac{(a_1q;q)_n\,...\,(a_rq;q)_n}{(b_1q;q)_n\,...\,(b_sq;q)_n\,(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{(q^{-\beta+1};q)_n\,(q^{\eta+\alpha+1};q)_n}{(q^2;q)_n\,(q^{-\beta+\eta+1};q)_n}\rho^n x^{-n}q^{n(1-\alpha+s-r)}. \tag{3.24}$$

Interpreting (3.24) with (1.9) it gives the right hand side of (3.15).

□

Special Cases

Corollary 3.3.

(i) If we take $r=1,\ s=0,\ \lambda=0,\ \rho=1,\ a_1=0$ in Theorems 1(i) and 2(i), the function ${}_r\phi_s$ reduces to e_q^x defined by (1.12) *and we obtain following results*

$$I_q^{\alpha,\beta,\eta}\left(e_q^x\right)=\frac{\Gamma_q\left(-\beta+\eta+1\right)}{\Gamma_q\left(-\beta+1\right)\Gamma_q\left(\alpha+\eta+1\right)}x^{-\beta}{}_3\phi_2\left[\begin{matrix}0,q,q^{-\beta+\eta+1}\\q^{-\beta+1},q^{\alpha+\eta+1}\end{matrix};q,x\right],\qquad(3.25)$$

provided $Re\left(-\beta+\eta+1\right)>0$ and $|x|<1$.

$$D_q^{\alpha,\beta,\eta}\left(e_q^x\right)=\frac{\Gamma_q\left(\alpha+\beta+\eta+1\right)}{\Gamma_q\left(\beta+1\right)\Gamma_q\left(\eta+1\right)}x^{\beta}{}_3\phi_2\left[\begin{matrix}0,q,q^{\alpha+\beta+\eta+1}\\q^{\beta+1},q^{\eta+1}\end{matrix};q,x\right],\qquad(3.26)$$

provided $Re\left(\alpha+\beta+\eta+1\right)>0$ and $|x|<1$.

(ii) If we take $r=1,\ s=0,\ \rho=1,\ a_1=0$ *, in Theorems 1(iii) and 2(iii), we obtain the following results*

$$J_q^{\alpha,\beta,\eta}\left(e_q^{1/x}\right)=q^{-\alpha(\alpha-1)/2}x^{-\beta-1}\frac{\Gamma_q\left(\beta+1\right)\Gamma_q\left(\eta+1\right)}{\Gamma_q\left(\beta+\alpha+\eta+1\right)}\frac{1}{(1-q)}$$
$$\times{}_3\phi_2\left[\begin{matrix}0,q^{\beta+1},q^{\eta+1}\\q^2,q^{\beta+\alpha+\eta+1}\end{matrix};q,q^{\alpha}/x\right],\qquad(3.27)$$

provided $Re\left(\beta+1\right)>0$, $Re\left(\eta+1\right)>0$ *and* $|x|>|q^{\alpha}|$.

$$P_q^{\alpha,\beta,\eta}\left(e_q^{1/x}\right)=q^{\alpha(\alpha+2\beta-3)/2}x^{\beta-1}\frac{\Gamma_q\left(-\beta+1\right)\Gamma_q\left(\eta+\alpha+1\right)}{\Gamma_q\left(-\beta+\eta+1\right)}\frac{1}{(1-q)}$$
$$\times{}_3\phi_2\left[\begin{matrix}0,q^{-\beta+1},q^{\eta+\alpha+1}\\q^2,q^{-\beta+\eta+1}\end{matrix};q,q^{-\alpha}/x\right],\qquad(3.28)$$

provided $Re\left(-\beta+1\right)>0$, $Re\left(\eta+\alpha+1\right)>0$ *and* $|x|>|q^{-\alpha}|$.

Corollary 3.4.

(i) If we take $r=s=0,\ \lambda=0,\ \rho=-1$, *in Theorems 1(i) and 2(i), the function* ${}_r\phi_s$ *reduces to* E_q^x *defined by* (1.13), *we obtain following results*

$$I_q^{\alpha,\beta,\eta}\left(E_q^x\right)=\frac{\Gamma_q\left(-\beta+\eta+1\right)}{\Gamma_q\left(-\beta+1\right)\Gamma_q\left(\alpha+\eta+1\right)}x^{-\beta}{}_2\phi_2\left[\begin{matrix}q,q^{-\beta+\eta+1}\\q^{-\beta+1},q^{\alpha+\eta+1}\end{matrix};q,-x\right],\qquad(3.29)$$

provided $Re\left(-\beta+\eta+1\right)>0$.

$$D_q^{\alpha,\beta,\eta}\left(E_q^x\right)=\frac{\Gamma_q\left(\alpha+\beta+\eta+1\right)}{\Gamma_q\left(\beta+1\right)\Gamma_q\left(\eta+1\right)}x^{\beta}{}_2\phi_2\left[\begin{matrix}q,q^{\alpha+\beta+\eta+1}\\q^{\beta+1},q^{\eta+1}\end{matrix};q,-x\right],\qquad(3.30)$$

provided $Re\left(\alpha+\beta+\eta+1\right)>0$.

(ii) If we take $r=s=0,\ \ \rho=-1$, *in Theorems 1(iii) and 2(iii), we obtain the following results*

$$J_q^{\alpha,\beta,\eta}\left(E_q^{-1/x}\right)=q^{-\alpha(\alpha-1)/2}x^{-\beta-1}\frac{\Gamma_q\left(\beta+1\right)\Gamma_q\left(\eta+1\right)}{\Gamma_q\left(\beta+\alpha+\eta+1\right)}\frac{1}{(1-q)}$$
$$\times{}_2\phi_2\left[\begin{matrix}q^{\beta+1},q^{\eta+1}\\q^2,q^{\beta+\alpha+\eta+1}\end{matrix};q,-q^{1+\alpha}/x\right],\qquad(3.31)$$

provided $Re\left(\beta+1\right)>0$, $Re\left(\eta+1\right)>0$.

$$P_q^{\alpha,\beta,\eta}\left(E_q^{-1/x}\right)=q^{\alpha(\alpha+2\beta-3)/2}x^{\beta-1}\frac{\Gamma_q\left(-\beta+1\right)\Gamma_q\left(\eta+\alpha+1\right)}{\Gamma_q\left(-\beta+\eta+1\right)}\frac{1}{(1-q)}$$
$$\times{}_2\phi_2\left[\begin{matrix}q^{-\beta+1},q^{\eta+\alpha+1}\\q^2,q^{-\beta+\eta+1}\end{matrix};q,-q^{1-\alpha}/x\right],\qquad(3.32)$$

provided $Re\left(-\beta+1\right)>0$, $Re\left(\eta+\alpha+1\right)>0$.

Corollary 3.5. *If we take $r = 1,\ s = 0,\ a_1 = q^{-\nu}, \nu \in C$, in Theorems 1(i), 1(ii), 2(i) and 2(ii) and use in* (1.11), *we get*

$$I_q^{\alpha,\beta,\eta}\left(x^{\lambda}\left(1-\rho x q^{-\nu}\right)_q^{\nu}\right) = \frac{\Gamma_q(\lambda+1)\,\Gamma_q(\lambda-\beta+\eta+1)}{\Gamma_q(\lambda-\beta+1)\,\Gamma_q(\lambda+\alpha+\eta+1)} x^{\lambda-\beta} \times {}_3\phi_2\left[\begin{matrix} q^{-\nu}, q^{\lambda+1}, q^{\lambda-\beta+\eta+1} \\ q^{\lambda-\beta+1}, q^{\lambda+\alpha+\eta+1} \end{matrix} ; q, \rho x\right], \tag{3.33}$$

provided $Re(\lambda+1) > 0$, $Re(\lambda-\beta+\eta+1) > 0$*and* $|\rho x| < 1$.

$$J_q^{\alpha,\beta,\eta}\left(x^{-\lambda}\left(1-{}^{\rho q^{-\nu}}\!/_{x}\right)_q^{\nu}\right) = \frac{\Gamma_q(\beta+\lambda)\,\Gamma_q(\eta+\lambda)}{\Gamma_q(\lambda)\,\Gamma_q(\beta+\alpha+\eta+\lambda)} x^{-\lambda-\beta} q^{\alpha\lambda-\alpha(\alpha+1)/2} \times {}_3\phi_2\left[\begin{matrix} q^{-\nu}, q^{\beta+\lambda}, q^{\eta+\lambda} \\ q^{\lambda}, q^{\beta+\alpha+\eta+\lambda} \end{matrix} ; q, {}^{\rho q^{\alpha}}\!/_{x}\right], \tag{3.34}$$

provided $Re(\beta+\lambda) > 0$, $Re(\eta+\lambda) > 0$ *and* $|x| > |\rho q^{\alpha}|$.

$$D_q^{\alpha,\beta,\eta}\left(x^{\lambda}\left(1-\rho x q^{-\nu}\right)_q^{\nu}\right) = \frac{\Gamma_q(\lambda+1)\,\Gamma_q(\lambda+\alpha+\beta+\eta+1)}{\Gamma_q(\lambda+\beta+1)\,\Gamma_q(\lambda+\eta+1)} x^{\lambda+\beta} \times {}_3\phi_2\left[\begin{matrix} q^{-\nu}, q^{\lambda+1}, q^{\lambda+\alpha+\beta+\eta+1} \\ q^{\lambda+\beta+1}, q^{\lambda+\eta+1} \end{matrix} ; q, \rho x\right], \tag{3.35}$$

provided $Re(\lambda+1) > 0$, $Re(\lambda+\alpha+\beta+\eta+1) > 0$ *and* $|\rho x| < 1$.

$$P_q^{\alpha,\beta,\eta}\left(x^{-\lambda}\left(1-{}^{\rho q^{-\nu}}\!/_{x}\right)_q^{\nu}\right) = q^{\alpha(\beta-\lambda)+\alpha(\alpha-1)/2} x^{\beta-\lambda} \frac{\Gamma_q(\lambda-\beta)\,\Gamma_q(\alpha+\eta+\lambda)}{\Gamma_q(\lambda)\,\Gamma_q(\lambda+\eta-\beta)} \times {}_3\phi_2\left[\begin{matrix} q^{-\nu}, q^{\lambda-\beta}, q^{\alpha+\eta+\lambda} \\ q^{\lambda}, q^{\lambda+\eta-\beta} \end{matrix} ; q, {}^{\rho}\!/_{xq^{\alpha}}\right], \tag{3.36}$$

provided $Re(\lambda-\beta) > 0$, $Re(\alpha+\eta+\lambda) > 0$ *and* $|x| > |\rho q^{-\alpha}|$.

Corollary 3.6.

(i) If we take $r = 2,\ s = 1,\ \lambda = 0,\ \rho = q,\ a_1 = q^{-n},\ a_2 = abq^{n+1}$ and $b_1 = aq$, *in Theorems 1(i), 2(i), the function* ${}_r\phi_s$ *reduces to Little q-Jacobi polynomials defined by* (1.14) *and we obtain following results*

$$I_q^{\alpha,\beta,\eta}\left(p_n(x;a,b;q)\right) = \frac{\Gamma_q(-\beta+\eta+1)}{\Gamma_q(-\beta+1)\,\Gamma_q(\alpha+\eta+1)} x^{-\beta} {}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+1}, q, q^{-\beta+\eta+1} \\ aq, q^{-\beta+1}, q^{\alpha+\eta+1} \end{matrix} ; q, qx\right], \tag{3.37}$$

provided $Re(-\beta+\eta+1) > 0$ *and* $|qx| < 1$.

$$D_q^{\alpha,\beta,\eta}\left(p_n(x;a,b;q)\right) = \frac{\Gamma_q(\alpha+\beta+\eta+1)}{\Gamma_q(\beta+1)\,\Gamma_q(\eta+1)} x^{\beta} {}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+1}, q, q^{\alpha+\beta+\eta+1} \\ aq, q^{\beta+1}, q^{\eta+1} \end{matrix} ; q, qx\right], \tag{3.38}$$

/ provided $Re(\alpha+\beta+\eta+1) > 0$ *and* $|qx| < 1$.

(ii) If we take $r = 2,\ s = 1,\ \rho = q,\ a_1 = q^{-n}, a_2 = abq^{n+1}$ and $b_1 = aq$, *in Theorems 1(iii) and 2(iii), we obtain following results*

$$J_q^{\alpha,\beta,\eta}\left(p_n\left({}^{1}\!/_{x};a,b;q\right)\right) = q^{-\alpha(\alpha-1)/2+1} x^{-\beta-1} \frac{\Gamma_q(\beta+1)\,\Gamma_q(\eta+1)}{\Gamma_q(\beta+\alpha+\eta+1)} \frac{(1-q^{-n})\left(1-abq^{n+1}\right)}{(1-aq)} \times \frac{1}{(1-q)} {}_4\phi_3\left[\begin{matrix} q^{-n+1}, abq^{n+2}, q^{\beta+1}, q^{\eta+1} \\ aq^2, q^2, q^{\beta+\alpha+\eta+1} \end{matrix} ; q, {}^{q^{1+\alpha}}\!/_{x}\right]. \tag{3.39}$$

Provided $Re(\beta+1) > 0$, $Re(\eta+1) > 0$*and* $|x| > \left|q^{1+\alpha}\right|$.

$$P_q^{\alpha,\beta,\eta}\left(p_n\left({}^1\!/_x;a,b;q\right)\right)=q^{\alpha(\alpha+2\beta-3)/2+1}x^{\beta-1}\frac{\Gamma_q(-\beta+1)\,\Gamma_q(\eta+\alpha+1)}{\Gamma_q(-\beta+\eta+1)}\frac{(1-q^{-n})\left(1-abq^{n+1}\right)}{(1-aq)}$$
$$\times\frac{1}{(1-q)}{}_4\phi_3\left[\begin{array}{c}q^{-n+1},abq^{n+2},q^{-\beta+1},q^{\eta+\alpha+1}\\aq^2,q^2,q^{-\beta+\eta+1}\end{array};q,{}^{q^{1-\alpha}}\!/_x\right]. \tag{3.40}$$

Provided $Re\,(-\beta+1)>0$, $Re\,(\eta+\alpha+1)>0$ *and* $|x|>\left|q^{1-\alpha}\right|$.

Corollary 3.7.

(i) We set $r=2,\ s=1,\ \lambda=0,\ a_1=q^{-n},\ a_2=q^{n+1},\ b_1=q,\ \rho=q$ *in Theorems 1(i), 2(i) and use* (1.15) *to get the following results,*

$$I_q^{\alpha,\beta,\eta}\left(p_n(x;q)\right)=\frac{\Gamma_q(-\beta+\eta+1)}{\Gamma_q(-\beta+1)\,\Gamma_q(\alpha+\eta+1)}x^{-\beta}{}_4\phi_3\left[\begin{array}{c}q^{-n},q^{n+1},q,q^{-\beta+\eta+1}\\q,q^{-\beta+1},q^{\alpha+\eta+1}\end{array};q,qx\right], \tag{3.41}$$

provided $Re\,(-\beta+\eta+1)>0$ *and* $|qx|<1$.

$$D_q^{\alpha,\beta,\eta}\left(p_n(x;q)\right)=\frac{\Gamma_q(\alpha+\beta+\eta+1)}{\Gamma_q(\beta+1)\,\Gamma_q(\eta+1)}x^{\beta}{}_4\phi_3\left[\begin{array}{c}q^{-n},q^{n+1},q,q^{\alpha+\beta+\eta+1}\\q,q^{\beta+1},q^{\eta+1}\end{array};q,qx\right], \tag{3.42}$$

provided $Re\,(\alpha+\beta+\eta+1)$ *and* $|qx|<1$.

(ii) We set $r=2,\ s=1,\ a_1=q^{-n},\ a_2=q^{n+1},\ b_1=q\ \ \rho=q$, *in Theorems 1(iii) and 2(iii), to get the following results*

$$J_q^{\alpha,\beta,\eta}\left(p_n(1/x;q)\right)=q^{-\alpha(\alpha-1)/2+1}x^{-\beta-1}\frac{\Gamma_q(\beta+1)\,\Gamma_q(\eta+1)}{\Gamma_q(\beta+\alpha+\eta+1)}\frac{(1-q^{-n})\left(1-q^{n+1}\right)}{(1-q)^2}$$
$$\times{}_4\phi_3\left[\begin{array}{c}q^{-n+1},q^{n+2},q^{\beta+1},q^{\eta+1}\\q^2,q^2,q^{\beta+\alpha+\eta+1}\end{array};q,{}^{q^{1+\alpha}}\!/_x\right], \tag{3.43}$$

provided $Re\,(\beta+1)>0$, $Re\,(\eta+1)>0$*and* $|x|>\left|q^{1+\alpha}\right|$.

$$P_q^{\alpha,\beta,\eta}\left(p_n(1/x;q)\right)=q^{\alpha(\alpha+2\beta-3)/2+1}x^{\beta-1}\frac{\Gamma_q(-\beta+1)\,\Gamma_q(\eta+\alpha+1)}{\Gamma_q(-\beta+\eta+1)}\frac{(1-q^{-n})\left(1-q^{n+1}\right)}{(1-q)^2}$$
$$\times{}_4\phi_3\left[\begin{array}{c}q^{-n+1},q^{n+2},q^{-\beta+1},q^{\eta+\alpha+1}\\q^2,q^2,q^{-\beta+\eta+1}\end{array};q,{}^{q^{1-\alpha}}\!/_x\right], \tag{3.44}$$

provided $Re\,(-\beta+1)>0$, $Re\,(\eta+\alpha+1)>0$ *and* $|x|>\left|q^{1-\alpha}\right|$.

Corollary 3.8.

(i) We set $r=2,\ s=1,\ \lambda=0,\ a_1=q^{-n},\ a_2=0,\ b_1=aq,\ \rho=q$ *in Theorems 1(i) and 2(i) and use* (1.16) *to get the following results*

$$I_q^{\alpha,\beta,\eta}\left(p_n(x;a,q)\right)=\frac{\Gamma_q(-\beta+\eta+1)}{\Gamma_q(-\beta+1)\,\Gamma_q(\alpha+\eta+1)}x^{-\beta}{}_4\phi_3\left[\begin{array}{c}q^{-n},0,q,q^{-\beta+\eta+1}\\aq,q^{-\beta+1},q^{\alpha+\eta+1}\end{array};q,qx\right], \tag{3.45}$$

provided $Re\,(-\beta+\eta+1)>0$ *and* $|qx|<1$.

$$D_q^{\alpha,\beta,\eta}\left(p_n(x;a,q)\right)=\frac{\Gamma_q(\alpha+\beta+\eta+1)}{\Gamma_q(\beta+1)\,\Gamma_q(\eta+1)}x^{\beta}{}_4\phi_3\left[\begin{array}{c}q^{-n},0,q,q^{\alpha+\beta+\eta+1}\\aq,q^{\beta+1},q^{\eta+1}\end{array};q,qx\right], \tag{3.46}$$

provided $Re\,(\alpha+\beta+\eta+1)$ *and* $|qx|<1$.

(ii) We set $r = 2,\ s = 1,\ a_1 = q^{-n},\ a_2 = 0,\ b_1 = aq,\ \rho = q$, in Theorems 1(iii) and 2(iii), to get the following results

$$J_q^{\alpha,\beta,\eta}\left(p_n\left(1/x;a,q\right)\right) = q^{-\alpha(\alpha-1)/2+1}x^{-\beta-1}\frac{\Gamma_q\left(\beta+1\right)\Gamma_q\left(\eta+1\right)}{\Gamma_q\left(\beta+\alpha+\eta+1\right)}\frac{\left(1-q^{-n}\right)}{\left(1-aq\right)}$$

$$\times\frac{1}{\left(1-q\right)}{}_4\phi_3\left[\begin{matrix} q^{-n+1}, 0, q^{\beta+1}, q^{\eta+1} \\ aq^2, q^2, q^{\beta+\alpha+\eta+1}\end{matrix};q,{q^{1+\alpha}}/{x}\right], \tag{3.47}$$

provided $Re\left(\beta+1\right) > 0$, $Re\left(\eta+1\right) > 0$and $|x| > \left|q^{1+\alpha}\right|$.

$$P_q^{\alpha,\beta,\eta}\left(p_n\left(1/x;a,q\right)\right) = q^{\alpha(\alpha+2\beta-3)/2+1}x^{\beta-1}\frac{\Gamma_q\left(-\beta+1\right)\Gamma_q\left(\eta+\alpha+1\right)}{\Gamma_q\left(-\beta+\eta+1\right)}\frac{\left(1-q^{-n}\right)}{\left(1-aq\right)}$$

$$\times\frac{1}{\left(1-q\right)}{}_4\phi_3\left[\begin{matrix} q^{-n+1}, 0, q^{-\beta+1}, q^{\eta+\alpha+1} \\ aq^2, q^2, q^{-\beta+\eta+1}\end{matrix};q,{q^{1-\alpha}}/{x}\right], \tag{3.48}$$

provided $Re\left(-\beta+1\right) > 0$, $Re\left(\eta+\alpha+1\right) > 0$ and $|x| > \left|q^{1-\alpha}\right|$.

Corollary 3.9. *(i) We set $r = 1,\ s = 1,\ \lambda = 0, a_1 = q^{-n},\ b_1 = q^{a+1},\ \rho = -q^{n+a+1}$, in Theorems 1(i) and 2(i) and use (1.17) to get the following results*

$$I_q^{\alpha,\beta,\eta}\left(L_n^{(a)}\left(x;q\right)\right) = \frac{(q;q)_n}{\left(q^{a+1};q\right)_n}\frac{\Gamma_q\left(-\beta+\eta+1\right)}{\Gamma_q\left(-\beta+1\right)\Gamma_q\left(\alpha+\eta+1\right)}x^{-\beta}$$

$$\times{}_3\phi_3\left[\begin{matrix} q^{-n}, q, q^{-\beta+\eta+1} \\ q^{a+1}, q^{-\beta+1}, q^{\alpha+\eta+1}\end{matrix};q,-q^{n+a+1}x\right], \tag{3.49}$$

provided $Re\left(-\beta+\eta+1\right) > 0$.

$$D_q^{\alpha,\beta,\eta}\left(L_n^{(a)}\left(x;q\right)\right) = \frac{\Gamma_q\left(\alpha+\beta+\eta+1\right)}{\Gamma_q\left(\beta+1\right)\Gamma_q\left(\eta+1\right)}x^{\beta}{}_3\phi_3\left[\begin{matrix} q^{-n}, q, q^{\alpha+\beta+\eta+1} \\ q^{a+1}, q^{\beta+1}, q^{\eta+1}\end{matrix};q,-q^{n+a+1}x\right], \tag{3.50}$$

provided $Re\left(\alpha+\beta+\eta+1\right)$.

(ii) We set $r = 1,\ s = 1,\ a_1 = q^{-n},\ b_1 = q^{a+1},\ \rho = -q^{n+a+1}$, in Theorems 1(c) and 2(c), to get the following results

$$J_q^{\alpha,\beta,\eta}\left(L_n^{(a)}\left(1/x;q\right)\right) = q^{-\alpha(\alpha-1)/2+n+a+1}x^{-\beta-1}\frac{(q;q)_n}{\left(q^{a+1};q\right)_n}\frac{\Gamma_q\left(\beta+1\right)\Gamma_q\left(\eta+1\right)}{\Gamma_q\left(\beta+\alpha+\eta+1\right)}\frac{\left(1-q^{-n}\right)}{\left(1-q^{a+1}\right)}$$

$$\times\frac{1}{\left(1-q\right)}{}_3\phi_3\left[\begin{matrix} q^{-n+1}, q^{\beta+1}, q^{\eta+1} \\ q^{a+2}, q^2, q^{\beta+\alpha+\eta+1}\end{matrix};q,{-q^{2+n+a+\alpha}}/{x}\right], \tag{3.51}$$

provided $Re\left(\beta+1\right) > 0$, $Re\left(\eta+1\right) > 0$.

$$P_q^{\alpha,\beta,\eta}\left(L_n^{(a)}\left(1/x;q\right)\right) = q^{\alpha(\alpha+2\beta-3)/2+n+a+1}x^{\beta-1}\frac{\Gamma_q\left(-\beta+1\right)\Gamma_q\left(\eta+\alpha+1\right)}{\Gamma_q\left(-\beta+\eta+1\right)}\frac{\left(1-q^{-n}\right)}{\left(1-q^{a+1}\right)}\frac{1}{\left(1-q\right)}$$

$$\times{}_3\phi_3\left[\begin{matrix} q^{-n+1}, q^{-\beta+1}, q^{\eta+\alpha+1} \\ q^{a+2}, q^2, q^{-\beta+\eta+1}\end{matrix};q,{-q^{2+n+a-\alpha}}/{x}\right], \tag{3.52}$$

provided $Re\left(-\beta+1\right) > 0$, $Re\left(\eta+\alpha+1\right) > 0$.

Corollary 3.10.

(i) *We set* $r = 1,\ s = 1,\ \lambda = 0, a_1 = q^{-n},\ b_1 = 0$ *and* $\rho = -q^{n+1}$ *in Theorems 1(i) and 2(i), and use* (1.18) *to get the following results*

$$I_q^{\alpha,\beta,\eta}\left(S_n\left(x;q\right)\right) = \frac{\Gamma_q\left(-\beta+\eta+1\right)}{\Gamma_q\left(-\beta+1\right)\Gamma_q\left(\alpha+\eta+1\right)} x^{-\beta} {}_3\phi_3\left[\begin{matrix} q^{-n}, q, q^{-\beta+\eta+1} \\ 0, q^{-\beta+1}, q^{\alpha+\eta+1}\end{matrix}; q, -q^{n+1}x\right], \tag{3.53}$$

provided $Re\left(-\beta+\eta+1\right) > 0$.

$$D_q^{\alpha,\beta,\eta}\left(S_n\left(x;q\right)\right) = \frac{\Gamma_q\left(\alpha+\beta+\eta+1\right)}{\Gamma_q\left(\beta+1\right)\Gamma_q\left(\eta+1\right)} x^{\beta} {}_3\phi_3\left[\begin{matrix} q^{-n}, q, q^{\alpha+\beta+\eta+1} \\ 0, q^{\beta+1}, q^{\eta+1}\end{matrix}; q, -q^{n+1}x\right], \tag{3.54}$$

provided $Re\left(\alpha+\beta+\eta+1\right) > 0$.

(ii) *We set* $r = 1,\ s = 1,\ a_1 = q^{-n},\ b_1 = 0$ *and* $\rho = -q^{n+1}$ *, in Theorems 1(iii) and 2(iii), to get the following results*

$$\begin{aligned} J_q^{\alpha,\beta,\eta}\left(S_n\left({}^1\!/_x;q\right)\right) &= q^{-\alpha(\alpha-1)/2+n+1} x^{-\beta-1} \frac{\Gamma_q\left(\beta+1\right)\Gamma_q\left(\eta+1\right)}{\Gamma_q\left(\beta+\alpha+\eta+1\right)} \frac{\left(1-q^{-n}\right)}{\left(1-q\right)} \\ &\times {}_3\phi_3\left[\begin{matrix} q^{-n+1}, q^{\beta+1}, q^{\eta+1} \\ 0, q^2, q^{\beta+\alpha+\eta+1}\end{matrix}; q, -q^{2+n+\alpha}\!/_x\right], \end{aligned} \tag{3.55}$$

provided $Re\left(\beta+1\right) > 0$*,* $Re\left(\eta+1\right) > 0$.

$$\begin{aligned} P_q^{\alpha,\beta,\eta}\left(S_n\left({}^1\!/_x;q\right)\right) &= q^{\alpha(\alpha+2\beta-1)/2+n+1} x^{\beta-1} \frac{\Gamma_q\left(-\beta+1\right)\Gamma_q\left(\eta+\alpha+1\right)}{\Gamma_q\left(-\beta+\eta+1\right)} \frac{\left(1-q^{-n}\right)}{\left(1-q\right)} \\ &\times {}_3\phi_3\left[\begin{matrix} q^{-n+1}, q^{-\beta+1}, q^{\eta+\alpha+1} \\ 0, q^2, q^{-\beta+\eta+1}\end{matrix}; q, -q^{2+n-\alpha}\!/_x\right], \end{aligned} \tag{3.56}$$

provided $Re\left(-\beta+1\right) > 0$*,* $Re\left(\eta+\alpha+1\right) > 0$.

References

[1] M. Garg and L. Chanchlani, *q*-analogues of Saigo's fractional calculus operators, *Bull. Math. Anal. Appl.* **3(4)**, 169–179 (2011).

[2] G. Gasper and M. Rahman, *Basic Hypergeometric Series 2nd ed.*, Cambridge University Press, Cambridge (2004).

[3] R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their* q-*analogues*, Springer Monographs in Mathematics, Springer-Verlag, Berlin (2010).

[4] R. K. Yadav and S. D.Purohit, Applications of Riemann-Liouville fractional *q*-integral operator to basic hypergeometric functions, *Acta Cienc. Indica* **30(3)**, 593–600 (2004).

[5] R. K. Yadav, and S D. Purohit, On applications of Kober fractional *q*-integral operator to certain basic hypergeometric functions, *J. Rajasthan Acad. Phy. Sci.* **5(4)**, 437–448 (2006).

[6] R. K. Yadav and S. D. Purohit, On applications of Weyl fractional *q*-integral operator to generalized basic hypergeometric functions, *Kyungpook Math. J.* **46(2)**, 235–245 (2006).

[7] R. K. Yadav, S. D. Purohit and S. L. Kalla, On generalized Weyl fractional *q*-integral operator involving generalized basic hypergeometric functions *Fract. Calc. Appl. Anal.* **11(2)**, 129–142 (2008).

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Received: May 11, 2014.

Accepted: May 26, 2014.