**Palestine Journal of Mathematics**

**Vol. 5(Special Issue: 1) (2016)** , 198–217 

# A DYNAMIC PROBLEM WITH ADHESION AND DAMAGE IN ELECTRO-ELASTO-VISCOPLASTICITY

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Communicated by Salim Messaoudi

MSC 2010 Classifications: Primary 74M15; Secondary 49J40, 74H20, 74H25.

Keywords and phrases: Electro-elasto-viscoplastic materials, normal compliance, adhesion, damage, differential equations, evolution equations, fixed point.

**Abstract**. In this paper, we study a mathematical problem for dynamic contact between two electro-elasto-viscoplastic bodies with damage. The contact is frictionless, modelled with a normal compliance condition involving adhesion effect of contact surfaces. Evolution of the bonding field is described by a first order differential equation. We derive variational formulation for the model and prove an existence and uniqueness result of the weak solution. The proof is based on arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities, differential equations and Banach fixed point theorem.

## 1 Introduction

The piezoelectric phenomenon represents the coupling between the mechanical and electrical behavior of a class of materials, called piezoelectric materials. In simplest terms, when a piezoelectric material is squeezed, an electric charge collects on its surface, conversely, when a piezoelectric material is subjected to a voltage drop, it mechanically deforms. Many crystalline materials exhibit piezoelectric behavior. A few materials exhibit the phenomenon strongly enough to be used in applications that take advantage of their properties. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate and polyvinylidene fluoride (a polymer film). Piezoelectric materials are used extensively as switches and actually in many engineering systems in radioelectronics, electroacoustics and measuring equipment. General models for electro-elasto-viscoplastic materials can be found in [1, 9] and, more recently, in [25]. A contact problem with normal compliance for piezoelectric materials was investigated in [2, 10, 26, 27]. The variational formulations of the corresponding problems were derived and existence and uniqueness of weak solutions were obtained. In this paper we deal study a dynamic frictionless contact problem with adhesion between two electro-elasto-viscoplastic bodies. For this, we consider a rate-type constitutive equation with two internal variables of the form

$$\begin{aligned}\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^*\nabla\varphi^\ell + \\ \int_0^t \mathcal{F}^\ell\Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^*\nabla\varphi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)), \zeta^\ell(s)\Big)\, ds,\end{aligned} \tag{1.1}$$

where $\boldsymbol{u}^\ell$ the displacement field, $\boldsymbol{\sigma}^\ell$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell)$ represent the stress and the linearized strain tensor, respectively. Here $\mathcal{A}^\ell$ is a given nonlinear function, $\mathcal{F}^\ell$ is a nonlinear constitutive function describing the viscoplastic behaviour of the material. We also consider that the viscoplastic function $\mathcal{F}^\ell$ depends on the internal state variable $\zeta^\ell$ describing the damage of the material caused by plastic deformations. $\mathcal{G}^\ell$ represents the elasticity operator. $E(\varphi^\ell) = -\nabla\varphi^\ell$ is the electric field, $\mathcal{E}^\ell = (e_{ijk})$ represents the third order piezoelectric tensor, $(\mathcal{E}^\ell)^*$ is its transposition. In (1.1) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable $t$. It follows from (1.1) that at each time moment, the stress tensor $\boldsymbol{\sigma}^\ell(t)$ is split into three parts: $\boldsymbol{\sigma}^\ell(t) = \boldsymbol{\sigma}^\ell_V(t) + \boldsymbol{\sigma}^\ell_E(t) + \boldsymbol{\sigma}^\ell_R(t)$, where $\boldsymbol{\sigma}^\ell_V(t) = \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t))$ represents the purely viscous part of the stress, $\boldsymbol{\sigma}^\ell_E(t) = (\mathcal{E}^\ell)^*\nabla\varphi^\ell(t)$ represents the electric part of the stress and $\boldsymbol{\sigma}_R^\ell(t)$ satisfies a rate-type elastic-viscoplastic with damage relation

$$\boldsymbol{\sigma}_R^\ell(t) = \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)) + \int_0^t \mathcal{F}^\ell\big(\boldsymbol{\sigma}_R^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)), \zeta^\ell(s)\big)\, ds. \tag{1.2}$$

Note also that when $\mathcal{F}^\ell = 0$ the constitutive law (1.1) becomes the Kelvin-Voigt electro-viscoelastic constitutive relation,

$$\boldsymbol{\sigma}^\ell(t) = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)) + (\mathcal{E}^\ell)^* \nabla\varphi^\ell(t). \tag{1.3}$$

Dynamic contact problems with Kelvin-Voigt materials of the form (1.3) can be found in [2, 27]. The normal compliance contact condition was first considered in [16] in the study of dynamic problems with linearly elastic and viscoelastic materials and then it was used in various references, see e.g. [14, 21]. This condition allows the interpenetration of the body's surface into the obstacle and it was justified by considering the interpenetration and deformation of surface asperities. The importance of this paper is to make the coupling of the piezoelectric problem and a frictionless contact problem with adhesion. The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [4, 10, 17, 18] and recently in the monographs [19, 20]. The novelty in all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by $\beta$, it describes the point wise fractional density of adhesion of active bonds on the contact surface, and some times referred to as the intensity of adhesion. Following [11], the bonding field satisfies the restriction $0 \le \beta \le 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction $\beta$ of the bonds is active. The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General models for damage were derived in [6, 7] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [8]. The three-dimensional case has been investigated in [15]. In all these papers the damage of the material is described with a damage function $\zeta^\ell$, restricted to have values between zero and one. When $\zeta^\ell = 1$, there is no damage in the material, when $\zeta^\ell = 0$, the material is completely damaged, when $0 < \zeta^\ell < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [8, 22, 23, 25]. In this paper the inclusion used for the evolution of the damage field is

$$\dot{\zeta}^\ell - \kappa^\ell \Delta\zeta^\ell + \partial\psi_{K^\ell}(\zeta^\ell) \ni \phi^\ell\big(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) - (\mathcal{E}^\ell)^*\nabla\varphi^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \zeta^\ell\big), \tag{1.4}$$

where $K^\ell$ denotes the set of admissible damage functions defined by

$$K^\ell = \{\xi \in H^1(\Omega^\ell);\ 0 \le \xi \le 1,\ \text{a.e. in } \Omega^\ell\}, \tag{1.5}$$

$\kappa^\ell$ is a positive coefficient, $\partial\psi_{K^\ell}$ represents the subdifferential of the indicator function of the set $K^\ell$ and $\phi^\ell$ is a given constitutive function which describes the sources of the damage in the system. Examples and mechanical interpretation of elasto-viscoplastic materials of the form (1.2) in which the function $\mathcal{F}^\ell$ does not depend on the damage parameter $\zeta^\ell$ were considered by many authors, see for instance [5, 12] and the references therein. Contact problems for materials of the form (1.1), (1.2) without damage parameter. Contact problems for elasto-viscoplastic materials of the form (1.2) are studied in[1, 25]. In this paper we consider a mathematical frictionless contact problem between two electro-elasto-viscoplastic bodies for rate-type materials of the form (1.1). The contact is modelled with normal compliance where the adhesion of the contact surfaces is taken into account and is modelled with a surface variable, the bonding field. We model the material's behavior with an electro-elasto-viscoplastic constitutive law with damage. We derive a variational formulation of the problem and prove the existence of a unique weak solution.

The paper is organized as follows. In Sect.2 we describe the mathematical models for the frictionless adhesive contact problem between two materials behavior with an electro-elasto-viscoplastic constitutive law with damage and the contact with normal compliance. In Sect.3 we list the assumption on the data and derive the variational formulation of the problem. In Sect.4 we state our main existence and uniqueness result, Theorem 4.1. The proof of the theorem is based on arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments.

## 2 Problem Statement

We consider the following physical setting. Let us consider two electro-elastic- viscoplastics bodies, occupying two bounded domains $\Omega^1$, $\Omega^2$ of the space $\mathbb{R}^d (d = 2, 3)$. For each domain $\Omega^\ell$, the boundary $\Gamma^\ell$ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts $\Gamma_1^\ell$, $\Gamma_2^\ell$ and $\Gamma_3^\ell$, on one hand, and on two measurable parts $\Gamma_a^\ell$ and $\Gamma_b^\ell$, on the other hand, such that $meas\Gamma_1^\ell > 0$, $meas\Gamma_a^\ell > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The $\Omega^\ell$ body is submitted to $\boldsymbol{f}_0^\ell$ forces and volume electric charges of density $q_0^\ell$. The bodies are assumed to be clamped on $\Gamma_1^\ell \times (0, T)$. The surface tractions $\boldsymbol{f}_2^\ell$ act on $\Gamma_2^\ell \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\ell \times (0, T)$ and a surface electric charge of density $q_2^\ell$ is prescribed on $\Gamma_b^\ell \times (0, T)$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The bodies is in adhesive contact with an obstacle, over the contact surface $\Gamma_3$. With these assumptions, the classical formulation of the mechanical frictionless contact problem with adhesion and damage between two electro-elastic-viscoplastics bodies is the following.

**Problem P.** For $\ell = 1, 2$, find a displacement field $\boldsymbol{u}^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{S}^d$, an electric potential field $\varphi^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}$, a damage field $\zeta^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}$, a bonding field $\beta : \Gamma_3 \times (0, T) \longrightarrow \mathbb{R}$ and a electric displacement field $\boldsymbol{D}^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}^d$ such that

$$\begin{aligned}&\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell + \\ &\int_0^t \mathcal{F}^\ell \Big( \boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)), \zeta^\ell(s) \Big)\, ds \end{aligned} \quad \text{in } \Omega^\ell \times (0, T), \tag{2.1}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \mathcal{B}^\ell \nabla \varphi^\ell \quad \text{in } \Omega^\ell \times (0, T), \tag{2.2}$$

$$\dot{\zeta}^\ell - \kappa^\ell \Delta \zeta^\ell + \partial \psi_{K^\ell}(\zeta^\ell) \ni \phi^\ell \big( \boldsymbol{\sigma}^\ell - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \zeta^\ell \big) \quad \text{in } \Omega^\ell \times (0, T), \tag{2.3}$$

$$\rho^\ell \ddot{\boldsymbol{u}}^\ell = \operatorname{Div} \boldsymbol{\sigma}^\ell + \boldsymbol{f}_0^\ell \quad \text{in } \Omega^\ell \times (0, T), \tag{2.4}$$

$$\operatorname{div} \boldsymbol{D}^\ell - q_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0, T), \tag{2.5}$$

$$\boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0, T), \tag{2.6}$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell = \boldsymbol{f}_2^\ell \quad \text{on } \Gamma_2^\ell \times (0, T), \tag{2.7}$$

$$\begin{cases} \sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \\ \sigma_\nu = -p_\nu([u_\nu]) + \gamma_\nu \beta^2 R_\nu([u_\nu]) \end{cases} \quad \text{on } \Gamma_3 \times (0, T), \tag{2.8}$$

$$\begin{cases} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \boldsymbol{\sigma}_\tau = p_\tau(\beta) \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]) \end{cases} \quad \text{on } \Gamma_3 \times (0, T), \tag{2.9}$$

$$\dot{\beta} = -\Big(\beta\big(\gamma_\nu(R_\nu([u_\nu]))^2 + \gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_\tau])|^2\big) - \varepsilon_a\Big)_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{2.10}$$

$$\frac{\partial \zeta^\ell}{\partial \nu^\ell} = 0 \quad \text{on } \Gamma^\ell \times (0,T), \tag{2.11}$$

$$\varphi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0,T), \tag{2.12}$$

$$\boldsymbol{D}^\ell.\boldsymbol{\nu}^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0,T), \tag{2.13}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell,\;\; \dot{\boldsymbol{u}}^\ell(0) = \boldsymbol{v}_0^\ell,\;\; \zeta^\ell(0) = \zeta_0^\ell \quad \text{in } \Omega^\ell, \tag{2.14}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.15}$$

First, equations (2.1) and (2.2) represent the electro-elastic-viscoplastic constitutive law with damage of the material in which $\varepsilon(\boldsymbol{u}^\ell)$ denotes the linearized strain tensor, $E(\varphi^\ell) = -\nabla\varphi^\ell$ is the electric field, where $\varphi^\ell$ is the electric potential, $\mathcal{A}^\ell$ and $\mathcal{G}^\ell$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively. $\mathcal{F}^\ell$ is a nonlinear constitutive function describing the viscoplastic behaviour of the material. $\mathcal{E}^\ell$ represents the piezoelectric operator, $(\mathcal{E}^\ell)^*$ is its transpose, $\mathcal{B}^\ell$ denotes the electric permittivity operator, and $\boldsymbol{D}^\ell = (D_1^\ell, ..., D_d^\ell)$ is the electric displacement vector. The evolution of the damage field is governed by the inclusion of parabolic type given by the relation (2.3). Equations (2.4) and (2.5) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, respectively. Next, the equations (2.6) and (2.7) represent the displacement and traction boundary condition, respectively. Condition (2.8) represents the normal compliance conditions with adhesion where $\gamma_\nu$ is a given adhesion coefficient and $[u_\nu] = u_\nu^1 + u_\nu^2$ stands for the displacements in normal direction. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu\beta^2 R_\nu([u_\nu])$, the adhesive traction is tensile and is proportional, with proportionality coefficient $\gamma_\nu$, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length $L$. The maximal tensile traction is $\gamma_\nu\beta^2 L$. $R_\nu$ is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator $R_\nu$, together with the operator $\boldsymbol{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter $L$ is made in what follows. Condition (2.9) represents the adhesive contact condition on the tangential plane, where $[\boldsymbol{u}_\tau] = \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2$ stands for the jump of the displacements in tangential direction. $\boldsymbol{R}_\tau$ is the truncation operator given by

$$\boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \le L, \\ L\frac{\boldsymbol{v}}{|\boldsymbol{v}|} & \text{if } |\boldsymbol{v}| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length $L$. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted.

Next, the equation (2.10) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [3], see also [24, 25] for more details. Here, besides $\gamma_\nu$, two new adhesion coefficients are involved, $\gamma_\tau$ and $\varepsilon_a$. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from (2.10), $\dot{\beta} \le 0$. Boundary condition (2.11) describes a homogeneous Neumann boundary condition where $\frac{\partial\zeta^\ell}{\partial\nu^\ell}$ is the normal derivative of $\zeta^\ell$. (2.12) and (2.13) represent the electric boundary conditions. (2.14) represents the initial displacement field, the initial velocity and the initial damage. Finally (2.15) represents the initial condition in which $\beta_0$ is the given initial bonding field.

## 3 Variational Formulation and Preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end, we need to introduce some notation and preliminary material. Here and below, $\mathbb{S}^d$ represent the space of second-order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\boldsymbol{u}^\ell.\boldsymbol{v}^\ell = u_i^\ell.v_i^\ell, \quad |\boldsymbol{v}^\ell| = (\boldsymbol{v}^\ell.\boldsymbol{v}^\ell)^{\frac{1}{2}}, \quad \forall \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell = \sigma_{ij}^\ell.\tau_{ij}^\ell, \quad |\boldsymbol{\tau}^\ell| = (\boldsymbol{\tau}^\ell.\boldsymbol{\tau}^\ell)^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell \in \mathbb{S}^d.$$

Here and below, the indices $i$ and $j$ run between 1 and $d$ and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H^\ell = \{\boldsymbol{v}^\ell = (v_i^\ell);\ v_i^\ell \in L^2(\Omega^\ell)\},\ H_1^\ell = \{\boldsymbol{v}^\ell = (v_i^\ell);\ v_i^\ell \in H^1(\Omega^\ell)\},$$
$$\mathcal{H}^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell);\ \tau_{ij}^\ell = \tau_{ji}^\ell \in L^2(\Omega^\ell)\},\ \mathcal{H}_1^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell) \in \mathcal{H}^\ell;\ \mathrm{div}\boldsymbol{\tau}^\ell \in H^\ell\}.$$

The spaces $H^\ell$, $H_1^\ell$, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell dx, \quad (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H_1^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell dx + \int_{\Omega^\ell} \nabla \boldsymbol{u}^\ell.\nabla \boldsymbol{v}^\ell dx,$$
$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell dx, \quad (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell dx + \int_{\Omega^\ell} \mathrm{div}\,\boldsymbol{\sigma}^\ell.\,\mathrm{Div}\,\boldsymbol{\tau}^\ell dx$$

and the associated norms $\|.\|_{H^\ell}$, $\|.\|_{H_1^\ell}$, $\|.\|_{\mathcal{H}^\ell}$, and $\|.\|_{\mathcal{H}_1^\ell}$ respectively. Here and below we use the notation

$$\nabla \boldsymbol{u}^\ell = (u_{i,j}^\ell),\ \varepsilon(\boldsymbol{u}^\ell) = (\varepsilon_{ij}(\boldsymbol{u}^\ell)), \quad \varepsilon_{ij}(\boldsymbol{u}^\ell) = \frac{1}{2}(u_{i,j}^\ell + u_{j,i}^\ell), \quad \forall u^\ell \in H_1^\ell,$$
$$\mathrm{Div}\,\boldsymbol{\sigma}^\ell = (\sigma_{ij,j}^\ell), \quad \forall \boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell.$$

For every element $\boldsymbol{v}^\ell \in H_1^\ell$, we also use the notation $\boldsymbol{v}^\ell$ for the trace of $\boldsymbol{v}^\ell$ on $\Gamma^\ell$ and we denote by $v_\nu^\ell$ and $\boldsymbol{v}_\tau^\ell$ the *normal* and the *tangential* components of $\boldsymbol{v}^\ell$ on the boundary $\Gamma^\ell$ given by

$$v_\nu^\ell = \boldsymbol{v}^\ell.\nu^\ell, \quad \boldsymbol{v}_\tau^\ell = \boldsymbol{v}^\ell - v_\nu^\ell \boldsymbol{\nu}^\ell.$$

Let $H'_{\Gamma^\ell}$ be the dual of $H_{\Gamma^\ell} = H^{\frac{1}{2}}(\Gamma^\ell)^d$ and let $(.,.)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell}$ denote the duality pairing between $H'_{\Gamma^\ell}$ and $H_{\Gamma^\ell}$. For every element $\boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell$ let $\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell$ be the element of $H'_{\Gamma^\ell}$ given by

$$(\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell, \boldsymbol{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell} = (\boldsymbol{\sigma}^\ell, \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + (\mathrm{Div}\,\boldsymbol{\sigma}^\ell, \boldsymbol{v}^\ell)_{H^\ell} \quad \forall \boldsymbol{v}^\ell \in H_1^\ell.$$

Denote by $\sigma_\nu^\ell$ and $\boldsymbol{\sigma}_\tau^\ell$ the *normal* and the *tangential* traces of $\boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell$, respectively. If $\boldsymbol{\sigma}^\ell$ is continuously differentiable on $\Omega^\ell \cup \Gamma^\ell$, then

$$\sigma_\nu^\ell = (\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell).\boldsymbol{\nu}^\ell, \quad \boldsymbol{\sigma}_\tau^\ell = \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell - \sigma_\nu^\ell \boldsymbol{\nu}^\ell,$$
$$(\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell, \boldsymbol{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell} = \int_{\Gamma^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell.\boldsymbol{v}^\ell da$$

fore all $\boldsymbol{v}^\ell \in H_1^\ell$, where $da$ is the surface measure element.

To obtain the variational formulation of the problem (2.1)–(2.15), we introduce for the bonding field the set

$$\mathcal{Z} = \left\{\theta \in L^\infty\big(0, T; L^2(\Gamma_3)\big);\ 0 \le \theta(t) \le 1\ \ \forall t \in [0, T],\ \text{a.e. on } \Gamma_3\right\},$$

and for the displacement field we need the closed subspace of $H_1^\ell$ defined by

$$V^\ell = \left\{\boldsymbol{v}^\ell \in H_1^\ell;\ \boldsymbol{v}^\ell = 0 \text{ on } \Gamma_1^\ell\right\}.$$

Since $meas\Gamma_1^\ell > 0$, the following Korn's inequality holds :

$$\|\varepsilon(\boldsymbol{v}^\ell)\|_{\mathcal{H}^\ell} \geq c_K \|\boldsymbol{v}^\ell\|_{H_1^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell, \tag{3.1}$$

where the constant $c_K$ denotes a positive constant which may depends only on $\Omega^\ell$, $\Gamma_1^\ell$ (see [19]). Over the space $V^\ell$ we consider the inner product given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{V^\ell} = (\varepsilon(\boldsymbol{u}^\ell), \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}, \quad \forall \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in V^\ell, \tag{3.2}$$

and let $\|.\|_{V^\ell}$ be the associated norm. It follows from Korn's inequality (3.1) that the norms $\|.\|_{H_1^\ell}$ and $\|.\|_{V^\ell}$ are equivalent on $V^\ell$. Then $(V^\ell, \|.\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.2), there exists a constant $c_0 > 0$, depending only on $\Omega^\ell$, $\Gamma_1^\ell$ and $\Gamma_3$ such that

$$\|\boldsymbol{v}^\ell\|_{L^2(\Gamma_3)^d} \leq c_0 \|\boldsymbol{v}^\ell\|_{V^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell. \tag{3.3}$$

We also introduce the spaces

$$E_0^\ell = L^2(\Omega^\ell),\ E_1^\ell = H^1(\Omega^\ell),\ W^\ell = \left\{\psi^\ell \in E_1^\ell;\ \psi^\ell = 0 \text{ on } \Gamma_a^\ell\right\},$$
$$\mathcal{W}^\ell = \left\{\boldsymbol{D}^\ell = (D_i^\ell);\ D_i^\ell \in L^2(\Omega^\ell),\ \operatorname{div} \boldsymbol{D}^\ell \in L^2(\Omega^\ell)\right\}.$$

Since $meas\Gamma_a^\ell > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi^\ell\|_{L^2(\Omega^\ell)^d} \geq c_F \|\psi^\ell\|_{H^1(\Omega^\ell)} \quad \forall \psi^\ell \in W^\ell, \tag{3.4}$$

where $c_F > 0$ is a constant which depends only on $\Omega^\ell$, $\Gamma_a^\ell$.
Over the space $W^\ell$, we consider the inner product given by

$$(\varphi^\ell, \psi^\ell)_{W^\ell} = \textstyle\int_{\Omega^\ell} \nabla\varphi^\ell.\nabla\psi^\ell dx$$

and let $\|.\|_{W^\ell}$ be the associated norm. It follows from (3.4) that $\|.\|_{H^1(\Omega^\ell)}$ and $\|.\|_{W^\ell}$ are equivalent norms on $W^\ell$ and therefore $(W^\ell, \|.\|_{W^\ell})$ is areal Hilbert space. On the space $\mathcal{W}^\ell$, we use the inner product

$$(\boldsymbol{D}^\ell, \boldsymbol{\Psi}^\ell)_{\mathcal{W}^\ell} = \int_{\Omega^\ell} \boldsymbol{D}^\ell.\boldsymbol{\Psi}^\ell dx + \int_{\Omega^\ell} \operatorname{div} \boldsymbol{D}^\ell. \operatorname{div} \boldsymbol{\Psi}^\ell dx,$$

where $\operatorname{div} \boldsymbol{D}^\ell = (\boldsymbol{D}_{i,i}^\ell)$, and the associated norm $\|.\|_{\mathcal{W}^\ell}$.

In order to simplify the notations, we define the product spaces

$$\boldsymbol{V} = V^1 \times V^2,\ \ H = H^1 \times H^2,\ \ H_1 = H_1^1 \times H_1^2,\ \ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2,$$
$$\mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, E_0 = E_0^1 \times E_0^2, E_1 = E_1^1 \times E_1^2, W = W^1 \times W^2, \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2.$$

The spaces $\boldsymbol{V}$, $E_1$, $W$ and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(.,.)_{\boldsymbol{V}}$, $(.,.)_{E_1}$, $(.,.)_W$, and $(.,.)_{\mathcal{W}}$. The associate norms will be denoted by $\|.\|_{\boldsymbol{V}}$, $\|.\|_{E_1}$, $\|.\|_W$ and $\|.\|_{\mathcal{W}}$, respectively.

Finally, for any real Hilbert space $X$, we use the classical notation for the spaces $L^p(0,T;X)$, $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$, $k \geq 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to $X$, respectively, with the norms

$$\|f\|_{C(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X,$$

$$\|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \|\dot{f}(t)\|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for areal number $r$, we use $r_+$ to represent its positive part, that is $r_+ = \max\{0, r\}$.

In the study of the Problem **P**, we consider the following assumptions:
we assume that the *viscosity operator* $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{l}\text{(a) There exists } C^1_{\mathcal{A}^\ell}, C^2_{\mathcal{A}^\ell} > 0 \text{ such that,}\\ \qquad |\mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi})| \le C^1_{\mathcal{A}^\ell}|\boldsymbol{\xi}| + C^2_{\mathcal{A}^\ell} \quad \forall \boldsymbol{\xi} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell.\\ \text{(b) There exists } m_{\mathcal{A}^\ell} > 0 \text{ such that}\\ \qquad (\mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{A}^\ell}|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2\\ \qquad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell.\\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\ell,\\ \qquad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d.\\ \text{(d) The mapping } \boldsymbol{\xi} \mapsto \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}) \text{ is continuous on } \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell.\end{array}\right. \tag{3.5}$$

The *elasticity operator* $\mathcal{G}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{l}\text{(a) There exists } L_{\mathcal{G}^\ell} > 0 \text{ such that}\\ \qquad |\mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\xi}_2)| \le L_{\mathcal{G}^\ell}|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|\\ \qquad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell.\\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\ell,\\ \qquad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d.\\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x},0) \text{ belongs to } \mathcal{H}^\ell.\end{array}\right. \tag{3.6}$$

The *viscoplasticity operator* $\mathcal{F}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{l}\text{(a) There exists } L_{\mathcal{F}^\ell} > 0 \text{ such that}\\ \qquad |\mathcal{F}^\ell(\boldsymbol{x},\boldsymbol{\eta}_1,\boldsymbol{\xi}_1,d_1) - \mathcal{F}^\ell(\boldsymbol{x},\boldsymbol{\eta}_2,\boldsymbol{\xi}_2,d_2)| \le L_{\mathcal{F}^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|\\ \qquad + |d_1 - d_2|\big), \quad \forall \boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d, \ \forall d_1, d_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell.\\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{F}^\ell(\boldsymbol{x},\boldsymbol{\eta},\boldsymbol{\xi},d) \text{ is Lebesgue measurable in } \Omega^\ell,\\ \qquad \text{for any } \boldsymbol{\eta},\boldsymbol{\xi} \in \mathbb{S}^d, \ d \in \mathbb{R}.\\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{F}^\ell(\boldsymbol{x},0,0,0) \text{ belongs to } \mathcal{H}^\ell.\end{array}\right. \tag{3.7}$$

The *damage source function* $\phi^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\left\{\begin{array}{l}\text{(a) There exists } L_{\phi^\ell} > 0 \text{ such that}\\ \qquad |\phi^\ell(\boldsymbol{x},\boldsymbol{\eta}_1,\boldsymbol{\xi}_1,\alpha_1) - \phi^\ell(\boldsymbol{x},\boldsymbol{\eta}_2,\boldsymbol{\xi}_2,\alpha_2)| \le L_{\phi^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|\\ \qquad + |\alpha_1 - \alpha_2|\big), \quad \forall \boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d \text{ and } \alpha_1,\alpha_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega^\ell.\\ \text{(b) The mapping } \boldsymbol{x} \mapsto \phi^\ell(\boldsymbol{x},\boldsymbol{\eta},\boldsymbol{\xi},\alpha) \text{ is Lebesgue measurable on } \Omega^\ell,\\ \qquad \text{for any } \boldsymbol{\eta},\boldsymbol{\xi} \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}.\\ \text{(c) The mapping } \boldsymbol{x} \mapsto \phi^\ell(\boldsymbol{x},0,0,0) \text{ belongs to } L^2(\Omega^\ell).\end{array}\right. \tag{3.8}$$

The *piezoelectric tensor* $\mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

$$\left\{\begin{array}{l}\text{(a) } \mathcal{E}^\ell(\boldsymbol{x},\tau) = (e^\ell_{ijk}(\boldsymbol{x})\tau_{jk}), \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x} \in \Omega^\ell.\\ \text{(b) } e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell), \ 1 \le i,j,k \le d.\end{array}\right. \tag{3.9}$$

Recall also that the transposed operator $(\mathcal{E}^\ell)^*$ is given by $(\mathcal{E}^\ell)^* = (e^{\ell,*}_{ijk})$ where $e^{\ell,*}_{ijk} = e^\ell_{kij}$ and the following equality hold

$$\mathcal{E}^\ell\sigma.\boldsymbol{v} = \sigma.(\mathcal{E}^\ell)^*\boldsymbol{v} \quad \forall \sigma \in \mathbb{S}^d, \ \forall \boldsymbol{v} \in \mathbb{R}^d.$$

The *electric permittivity operator* $\mathcal{B}^\ell = (b^\ell_{ij}) : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$ verifies:

$$\left\{\begin{array}{l}\text{(a) } \mathcal{B}^\ell(\boldsymbol{x},\mathbf{E}) = (b^\ell_{ij}(\boldsymbol{x})E_j) \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell.\\ \text{(b) } b^\ell_{ij} = b^\ell_{ji}, \ b^\ell_{ij} \in L^\infty(\Omega^\ell), \quad 1 \le i,j \le d.\\ \text{(c) There exists } m_{\mathcal{B}^\ell} > 0 \text{ such that } \mathcal{B}^\ell\mathbf{E}.\mathbf{E} \ge m_{\mathcal{B}^\ell}|\mathbf{E}|^2\\ \qquad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell.\end{array}\right. \tag{3.10}$$

The *normal compliance functions* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) } \exists\, L_\nu > 0 \text{ such that } |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \leq L_\nu |r_1 - r_2| \\ \qquad \forall\, r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3,\ \forall r \in \mathbb{R}. \\ \text{(c) } p_\nu(\boldsymbol{x}, r) = 0, \text{ for all } r \leq 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{3.11}$$

The *tangential compliance functions* $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) } \exists\, L_\tau > 0 \text{ such that } |p_\tau(\boldsymbol{x}, d_1) - p_\tau(\boldsymbol{x}, d_2)| \leq L_\tau |d_1 - d_2| \\ \qquad \forall\, d_1, d_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } \exists\, M_\tau > 0 \text{ such that } |p_\tau(\boldsymbol{x}, d)| \leq M_\tau \ \forall\, d \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, d) \text{ is measurable on } \Gamma_3,\ \forall d \in \mathbb{R}. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, 0) \in L^2(\Gamma_3). \end{cases} \tag{3.12}$$

We suppose that the mass density satisfies

$$\rho^\ell \in L^\infty(\Omega^\ell) \text{ and } \exists \rho_0 > 0 \text{ such that } \rho^\ell(x) \geq \rho_0 \text{ a.e. } x \in \Omega^\ell,\ \ell = 1, 2. \tag{3.13}$$

The following regularity is assumed on the density of volume forces, traction, volume electric charges and surface electric charges:

$$\begin{aligned} \mathbf{f}_0^\ell &\in L^2(0, T; L^2(\Omega^\ell)^d), \quad \mathbf{f}_2^\ell \in L^2(0, T; L^2(\Gamma_2^\ell)^d), \\ q_0^\ell &\in C(0, T; L^2(\Omega^\ell)), \quad q_2^\ell \in C(0, T; L^2(\Gamma_b^\ell)), \end{aligned} \tag{3.14}$$

$$q_2^\ell(t) = 0 \quad \text{on } \Gamma_3 \ \ \forall t \in [0, T]. \tag{3.15}$$

The adhesion coefficients $\gamma_\nu, \gamma_\tau$ and $\varepsilon_a$ satisfy the conditions

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3),\ \ \varepsilon_a \in L^2(\Gamma_3),\ \ \gamma_\nu, \gamma_\tau, \varepsilon_a \geq 0, \text{ a.e. on } \Gamma_3. \tag{3.16}$$

The microcrack diffusion coefficient verifies

$$\kappa^\ell > 0, \tag{3.17}$$

and, finally, the initial data satisfy

$$\begin{aligned} \boldsymbol{u}_0^\ell \in \boldsymbol{V}^\ell, \quad \boldsymbol{v}_0^\ell \in H^\ell, \quad \zeta_0^\ell \in K^\ell,\ \ \ell = 1, 2, \\ \beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1, \text{ a.e. on } \Gamma_3. \end{aligned} \tag{3.18}$$

where $K^\ell$ is the set of admissible damage functions defined in (1.5).

Let $a : E_1 \times E_1 \to \mathbb{R}$, be the bilinear form

$$a(\zeta, \xi) = \sum_{\ell=1}^{2} \kappa^\ell \int_{\Omega^\ell} \nabla \zeta^\ell . \nabla \xi^\ell dx. \tag{3.19}$$

We will use a modified inner product on $H$, given by

$$((\boldsymbol{u}, \boldsymbol{v}))_H = \sum_{\ell=1}^{2} (\rho^\ell \boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H,$$

that is, it is weighted with $\rho^\ell$, and we let $\|\!|.\|\!|_H$ be the associated norm, i.e.,

$$\|\!|\boldsymbol{v}\|\!|_H = ((\boldsymbol{v}, \boldsymbol{v}))_H^{\frac{1}{2}}, \quad \forall v \in H.$$

It follows from assumption (3.13) that $|||.|||_H$ and $\|.\|_H$ are equivalent norms on $H$, and the inclusion mapping of $(\boldsymbol{V}, \|.\|_{\boldsymbol{V}})$ into $(H, |||.|||_H)$ is continuous and dense. We denote by $\boldsymbol{V}'$ the dual of $\boldsymbol{V}$. Identifying $H$ with its own dual, we can write the Gelfand triple

$$\boldsymbol{V} \subset H \subset \boldsymbol{V}'.$$

Using the notation $(.,.)_{\boldsymbol{V}'\times\boldsymbol{V}}$ to represent the duality pairing between $\boldsymbol{V}'$ and $\boldsymbol{V}$ we have

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = ((\boldsymbol{u}, \boldsymbol{v}))_H, \quad \forall \boldsymbol{u} \in H, \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Finally, we denote by $\|.\|_{\boldsymbol{V}'}$ the norm on $\boldsymbol{V}'$. Using the Riesz representation theorem, we define the linear mappings $\mathbf{f} : [0,T] \to \boldsymbol{V}'$ and $q : [0,T] \to W$ as follows:

$$(\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}_0^\ell(t) \cdot \boldsymbol{v}^\ell \, dx + \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} \mathbf{f}_2^\ell(t) \cdot \boldsymbol{v}^\ell \, da \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.20}$$

$$(q(t), \zeta)_W = \sum_{\ell=1}^{2} \int_{\Omega^\ell} q_0^\ell(t)\zeta^\ell \, dx - \sum_{\ell=1}^{2} \int_{\Gamma_b^\ell} q_2^\ell(t)\zeta^\ell \, da \quad \forall \zeta \in W. \tag{3.21}$$

Next, we denote by $j_{ad} : L^\infty(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ the adhesion functional defined by

$$j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \Big( -\gamma_\nu \beta^2 R_\nu([u_\nu])[v_\nu] + p_\tau(\beta) \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])[\boldsymbol{v}_\tau] \Big) \, da. \tag{3.22}$$

In addition to the functional (3.22), we need the normal compliance functional

$$j_{\nu c}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu] \, da. \tag{3.23}$$

Keeping in mind (3.11) and (3.12), we observe that the integrals (3.22) and (3.23) are well defined and we note that conditions (3.14) imply

$$\mathbf{f} \in L^2(0,T;\boldsymbol{V}'), \quad q \in C(0,T;W). \tag{3.24}$$

By a standard procedure based on Green's formula, we derive the following variational formulation of the mechanical (2.1)–(2.15).

**Problem PV.** Find a displacement field $\boldsymbol{u} : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$, an electric potential field $\varphi : [0,T] \to W$, a damage field $\zeta : [0,T] \to E_1$, a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ and a electric displacement field $\boldsymbol{D} : [0,T] \to \mathcal{W}$ such that

$$\begin{aligned} &\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \varepsilon(\dot{\boldsymbol{u}}^\ell) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla\varphi^\ell + \\ &\int_0^t \mathcal{F}^\ell \Big( \boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla\varphi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)), \zeta^\ell(s) \Big) \, ds \end{aligned} \quad \text{in } \Omega^\ell \times (0,T) \tag{3.25}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \mathcal{B}^\ell \nabla\varphi^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{3.26}$$

$$\begin{aligned} (\ddot{u}, v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell, \ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + j_{ad}(\beta(t), \boldsymbol{u}(t), \boldsymbol{v}) + j_{\nu c}(\boldsymbol{u}(t), \boldsymbol{v}) \\ = (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \ \text{a.e. } t \in (0,T), \end{aligned} \tag{3.27}$$

$$\begin{aligned} &\zeta(t) \in K, \qquad \sum_{\ell=1}^{2} (\dot{\zeta}^\ell(t), \xi^\ell - \zeta^\ell(t))_{L^2(\Omega^\ell)} + a(\zeta(t), \xi - \zeta(t)) \geq \\ &\sum_{\ell=1}^{2} \Big( \phi^\ell \Big( \boldsymbol{\sigma}^\ell(t) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) - (\mathcal{E}^\ell)^* \nabla\varphi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \zeta^\ell(t) \Big), \xi^\ell - \zeta^\ell(t) \Big)_{L^2(\Omega^\ell)}, \\ &\quad \forall \xi \in K, \ \text{a.e. } t \in (0,T), \end{aligned} \tag{3.28}$$

$$\sum_{\ell=1}^{2}(\mathcal{B}^\ell\nabla\varphi^\ell(t),\nabla\phi^\ell)_{H^\ell}-\sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell(t)),\nabla\phi^\ell)_{H^\ell}=(q(t),\phi)_W, \tag{3.29}$$
$$\forall\phi\in W,\ \text{a.e. } t\in(0,T),$$

$$\dot\beta(t)=-\Big(\beta(t)\big(\gamma_\nu(R_\nu([u_\nu(t)]))^2+\gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_\tau(t)])|^2\big)-\varepsilon_a\Big)_+\ \text{a.e. } (0,T), \tag{3.30}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\quad \dot{\boldsymbol{u}}(0)=\boldsymbol{v}_0,\quad \zeta(0)=\zeta_0,\quad \beta(0)=\beta_0, \tag{3.31}$$

where $K=K^1\times K^2$.
We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, an electrical potential field, a bonding field and a electric displacement field. The existence of the unique solution of Problem **PV** is stated and proved in the next section.

**Remark 3.1.** We note that, in Problem **P** and in Problem **PV**, we do not need to impose explicitly the restriction $0\le\beta\le1$. Indeed, equation (3.30) guarantees that $\beta(x,t)\le\beta_0(x)$ and, therefore, assumption (3.18) shows that $\beta(x,t)\le1$ for $t\ge0$, a.e. $x\in\Gamma_3$. On the other hand, if $\beta(x,t_0)=0$ at time $t_0$, then it follows from (3.30) that $\dot\beta(x,t)=0$ for all $t\ge t_0$ and therefore, $\beta(x,t)=0$ for all $t\ge t_0$, a.e. $x\in\Gamma_3$. We conclude that $0\le\beta(x,t)\le1$ for all $t\in[0,T]$, a.e. $x\in\Gamma_3$.

## 4 Existence and Uniqueness Result

Now, we propose our existence and uniqueness result

**Theorem 4.1.** *Assume that* (3.5)–(3.18) *hold. Then there exists a unique solution* $\{\boldsymbol{u},\boldsymbol{\sigma},\varphi,\zeta,\beta,\boldsymbol{D}\}$ *to Problem* ***PV****, Moreover, the solution satisfies*

$$\boldsymbol{u}\in H^1(0,T;\boldsymbol{V})\cap C^1(0,T;H),\ \ddot{\boldsymbol{u}}\in L^2(0,T;\boldsymbol{V}'), \tag{4.1}$$
$$\boldsymbol{\sigma}\in L^2(0,T;\mathcal{H}),\ (\mathrm{Div}\,\boldsymbol{\sigma}^1,\mathrm{Div}\,\boldsymbol{\sigma}^2)\in L^2(0,T;\boldsymbol{V}'), \tag{4.2}$$
$$\varphi\in C(0,T;W), \tag{4.3}$$
$$\zeta\in H^1(0,T;E_0)\cap L^2(0,T;E_1), \tag{4.4}$$
$$\beta\in W^{1,\infty}(0,T;L^2(\Gamma_3))\cap\mathcal{Z}, \tag{4.5}$$
$$\boldsymbol{D}\in C(0,T;\mathcal{W}). \tag{4.6}$$

The functions $\boldsymbol{u}$,$\varphi$, $\zeta$, $\boldsymbol{\sigma}$, $\boldsymbol{D}$ and $\beta$ which satisfy (3.25)-(3.31) are called a weak solution of the contact Problem **P**. We conclude that, under the assumptions (3.5)– (3.18), the mechanical problem (2.1)–(2.15) has a unique weak solution satisfying (4.1)–(4.6). We turn now to the proof of Theorem 4.1 which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed point arguments. We assume in what follows that assumptions of Theorem 4.1 hold, and we consider that $C$ is a generic positive constant which depends on $\Omega^\ell$, $\Gamma_1^\ell$, $\Gamma_1^\ell$, $\Gamma_3$, $p_\nu,p_\tau$, $\mathcal{A}^\ell$, $\mathcal{B}^\ell$, $\mathcal{G}^\ell$, $\mathcal{F}^\ell$, $\mathcal{E}^\ell$, $\gamma_\nu$, $\gamma_\tau$, $\phi^\ell$, $\kappa^\ell$, and $T$. but does not depend on $t$ nor of the rest of input data, and whose value may change from place to place. Let a $\eta\in L^2(0,T;\boldsymbol{V}')$ be given. In the first step we consider the following variational problem.

**Problem $\mathbf{PV}^u_\eta$.** Find a displacement field $\boldsymbol{u}_\eta:[0,T]\to\boldsymbol{V}$ such that

$$\begin{aligned}(\ddot u_\eta(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}+(\eta(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}}\\ =(\mathbf{f}(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}}\quad\forall\boldsymbol{v}\in\boldsymbol{V},\,a.e.\ t\in(0,T),\end{aligned} \tag{4.7}$$

$$\boldsymbol{u}^\ell(0)=\boldsymbol{u}_0^\ell,\ \ \dot{\boldsymbol{u}}^\ell(0)=\boldsymbol{v}_0^\ell\quad\text{in }\Omega^\ell. \tag{4.8}$$

To solve Problem $\mathbf{PV}^u_\eta$, we apply an abstract existence and uniqueness result which we recall now, for the convenience of the reader. Let $\boldsymbol{V}$ and $H$ denote real Hilbert spaces such that $\boldsymbol{V}$ is dense in $H$ and the inclusion map is continuous, $H$ is identified with its dual and with a subspace of the dual $\boldsymbol{V}'$ of $\boldsymbol{V}$, i.e., $\boldsymbol{V} \subset H \subset \boldsymbol{V}'$, and we say that the inclusions above define a Gelfand triple. The notations $\|.\|_{\boldsymbol{V}}$, $\|.\|_{\boldsymbol{V}'}$ and $(.,.)_{\boldsymbol{V}'\times\boldsymbol{V}}$ represent the norms on $\boldsymbol{V}$ and on $\boldsymbol{V}'$ and the duality pairing between $\boldsymbol{V}'$ them, respectively. The following abstract result may be found in [25, p.48].

**Theorem 4.2.** *Let $\boldsymbol{V}$, $H$ be as above, and let $A : \boldsymbol{V} \to \boldsymbol{V}'$ be a hemicontinuous and monotone operator which satisfies*

$$(A\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \geq w\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 + \lambda \;\; \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{4.9}$$

$$\|A\boldsymbol{v}\|_{\boldsymbol{V}'} \leq C(\|\boldsymbol{v}\|_{\boldsymbol{V}} + 1) \;\; \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{4.10}$$

*for some constants $w > 0$, $C > 0$ and $\lambda \in \mathbb{R}$. Then, given $\boldsymbol{u}_0 \in H$ and $f \in L^2(0,T;\boldsymbol{V}')$, there exists a unique function $\boldsymbol{u}$ which satisfies*

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;H), \;\; \dot{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{V}'),$$
$$\dot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \mathbf{f}(t) \text{ a.e. } t \in (0,T),$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0$$

We have the following result for the problem.

**Lemma 4.3.** *There exists a unique solution to Problem $\boldsymbol{PV}^u_\eta$ and it has its regularity expressed in (4.1).*

*Proof.* We define the operator $A : \boldsymbol{V} \to \boldsymbol{V}'$ by

$$(A\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\boldsymbol{u}^\ell),\; \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \tag{4.11}$$

Using (4.11), (3.2) and (3.5) it follows that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_{\boldsymbol{V}'}^2 \leq \sum_{\ell=1}^{2} \|\mathcal{A}^\ell \varepsilon(\boldsymbol{u}^\ell) - \mathcal{A}^\ell \varepsilon(\boldsymbol{v}^\ell)\|_{\mathcal{H}^\ell}^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},$$

and keeping in mind the Krasnoselski Theorem (see [13, p.60]), we deduce that $A : \boldsymbol{V} \to \boldsymbol{V}'$ is a continuous operator. Now, by (4.11), (3.2) and (3.5), we find

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \geq m\|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{V}}^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}, \tag{4.12}$$

where the positive constant $m = \min\{m_{\mathcal{A}^1}, m_{\mathcal{A}^2}\}$. Choosing $\boldsymbol{v} = 0$ in (4.12) we obtain

$$\begin{aligned}(A\boldsymbol{u}, \boldsymbol{u})_{\boldsymbol{V}'\times\boldsymbol{V}} &\geq m\|\boldsymbol{u}\|_{\boldsymbol{V}}^2 - \|Ao\|_{\boldsymbol{V}'}^2 \|\boldsymbol{u}\|_{\boldsymbol{V}} \\ &\geq \frac{1}{2} m\|\boldsymbol{u}\|_{\boldsymbol{V}}^2 - \frac{1}{2m}\|Ao\|_{\boldsymbol{V}'}^2 \quad \forall \boldsymbol{u} \in \boldsymbol{V},\end{aligned}$$

which implies that $A$ satisfies condition (4.9) with $\omega = \frac{m}{2}$ and $\lambda = -\frac{1}{2m}\|Ao\|_{\boldsymbol{V}'}^2$. Moreover, by (4.11) and (3.5) we find

$$\|A\boldsymbol{u}\|_{\boldsymbol{V}'} \leq C^1\|u\|_{\boldsymbol{V}} + C^2 \quad \forall \boldsymbol{u} \in \boldsymbol{V}.$$

where $C^1 = \max\{C^1_{\mathcal{A}^1}, C^1_{\mathcal{A}^2}\}$ and $C^2 = \max\{C^2_{\mathcal{A}^1}, C^2_{\mathcal{A}^2}\}$. This inequality and (3.2) imply that A satisfies condition (4.10). Finally, we recall that by (3.14) and (3.20) we have $\mathbf{f} - \eta \in L^2(0,T;\boldsymbol{V}')$ and $\boldsymbol{v}_0 \in H$.
It follows now from Theorem 4.2 that there exists a unique function $\boldsymbol{v}_\eta$ which satisfies

$$\boldsymbol{v}_\eta \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;H), \;\; \dot{\boldsymbol{v}}_\eta \in L^2(0,T;\boldsymbol{V}'), \tag{4.13}$$

$$\dot{\boldsymbol{v}}_\eta(t) + A\boldsymbol{v}_\eta(t) + \eta(t) = \mathbf{f}(t), \;\; a.e.\; t \in [0,T] \tag{4.14}$$

$$\boldsymbol{v}_\eta(0) = \boldsymbol{v}_0. \tag{4.15}$$

Let $\boldsymbol{u}_\eta : [0,T] \to \boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)ds + \boldsymbol{u}_0 \quad \forall t \in [0,T]. \tag{4.16}$$

It follows from (4.11) and (4.13)–(4.16) that $\boldsymbol{u}_\eta$ is a unique solution of the variational problem $\mathbf{PV}^u_\eta$. and it satisfies the regularity expressed in (4.1). □

In the second step, let $\eta \in L^2(0,T;\boldsymbol{V}')$, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.3 and we consider the following variational problem.

**Problem $\mathbf{PV}^\varphi_\eta$.** Find the electric potential field $\varphi_\eta : [0,T] \to W$ such that

$$\sum_{\ell=1}^{2}(\mathcal{B}^\ell \nabla\varphi^\ell_\eta(t), \nabla\phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell_\eta(t)), \nabla\phi^\ell)_{H^\ell} = (q(t),\phi)_W \tag{4.17}$$
$$\forall\phi \in W,\ a.e.\ t\in(0,T).$$

We have the following result.

**Lemma 4.4.** *Problem $\boldsymbol{PV}^\varphi_\eta$ has a unique solution $\varphi_\eta$ which satisfies the regularity* (4.3).

*Proof.* We define a bilinear form: $b(.,.) : W \times W \to \mathbb{R}$ such that

$$b(\varphi,\phi) = \sum_{\ell=1}^{2}(\mathcal{B}^\ell\nabla\varphi^\ell, \nabla\phi^\ell)_{H^\ell} \quad \forall\varphi,\phi \in W. \tag{4.18}$$

We use (3.4), (3.10) and (4.18) to show that the bilinear form $b(.,.)$ is continuous, symmetric and coercive on $W$, moreover using (3.21) and the Riesz Representation Theorem we may define an element $q_\eta : [0,T] \to W$ such that

$$(q_\eta(t),\phi)_W = (q(t),\phi)_W + \sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell_\eta(t)), \nabla\phi^\ell)_{H^\ell} \quad \forall\phi\in W, t\in(0,T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$b(\varphi_\eta(t),\phi) = (q_\eta(t),\phi)_W \quad \forall\phi\in W. \tag{4.19}$$

We conclude that $\varphi_\eta(t)$ is a solution of Problem $\mathbf{PV}^\varphi_\eta$. Let $t_1, t_2 \in [0,T]$, it follows from (4.17) that

$$\|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W \le C\big(\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_{\boldsymbol{V}} + \|q(t_1) - q(t_2)\|_W\big),$$

and the previous inequality, the regularity of $\boldsymbol{u}_\eta$ and $q$ imply that $\varphi_\eta \in C(0,T;W)$. □

In the third step we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma4.3 and we consider the following initial-value problem.

**Problem $\mathbf{PV}^\beta_\eta$.** Find the adhesion field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot\beta_\eta(t) = -\Big(\beta_\eta(t)\big(\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2 + \gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{\eta\tau}(t)])|^2\big) - \varepsilon_a\Big)_+,\ a.e.\ t\in(0,T), \tag{4.20}$$
$$\beta_\eta(0) = \beta_0. \tag{4.21}$$

We have the following result.

**Lemma 4.5.** *There exists a unique solution $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}$ to Problem $\boldsymbol{PV}^\beta_\eta$.*

*Proof.* For the simplicity we suppress the dependence of various functions on $\Gamma_3$, and note that the equalities and inequalities below are valid a.e. on $\Gamma_3$. Consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta) = -\Big(\beta\big[\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2 + \gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{\eta\tau}(t)])|^2\big] - \varepsilon_a\Big)_+,$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator $R_\nu$ and $\boldsymbol{R}_\tau$ that $F_\eta$ is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, the mapping $t \to F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using the Cauchy-Lipschitz theorem (see, [25, p.48]), we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to the Problem $\mathbf{PV}_\eta^\beta$. Also, the arguments used in Remark 3.1 show that $0 \le \beta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on $\Gamma_3$. Therefore, from the definition of the set $\mathcal{Z}$, we find that $\beta_\eta(t) \in \mathcal{Z}$, which concludes the proof of the lemma. □

In the fourth step, we let $\theta \in L^2(0.T;E_0)$ be given and consider the following variational problem for the damage field.

**Problem $\mathbf{PV}_\theta^\zeta$.** *Find a damage field* $\zeta_\theta = (\zeta_\theta^1,\zeta_\theta^2) : [0,T] \to E$ such that

$$\begin{aligned}\zeta_\theta(t) \in K,\ \sum_{\ell=1}^{2}(\dot\zeta_\theta^\ell(t),\xi^\ell - \zeta_\theta^\ell(t))_{L^2(\Omega^\ell)} + a(\zeta_\theta(t),\xi-\zeta_\theta(t)) \ge \\ \sum_{\ell=1}^{2}\big(\theta^\ell(t),\xi^\ell - \zeta^\ell(t)\big)_{L^2(\Omega^\ell)}, \quad \forall\xi \in K,\ \text{a.e. } t \in (0,T).\end{aligned} \tag{4.22}$$

The following abstract result for parabolic variational inequalities (see, e.g., [25, p.47])

**Theorem 4.6.** *Let $X \subset Y = Y' \subset X'$ be a Gelfand triple. Let $F$ be a nonempty, closed, and convex set of $X$. Assume that $a(.,.) : X \times X \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\alpha > 0$ and $c_0$,*

$$a(v,v) + c_0\|v\|_Y^2 \ge \alpha\|v\|_X^2 \quad \forall v \in X.$$

*Then, for every $u_0 \in F$ and $f \in L^2(0,T;Y)$, there exists a unique function $u \in H^1(0,T;Y) \cap L^2(0,T;X)$ such that $u(0) = u_0$, $u(t) \in F\ \ \forall t \in [0,T]$, and*

$$(\dot u(t), v - u(t))_{X'\times X} + a(u(t), v-u(t)) \ge (f(t), v-u(t))_Y \ \ \forall v \in F \text{ a.e. } t \in (0,T).$$

We prove next the unique solvability of Problem $\mathbf{PV}_\theta^\zeta$.

**Lemma 4.7.** *There exists a unique solution $\zeta_\theta$ of Problem $\mathbf{PV}_\theta^\zeta$ and it satisfies*

$$\zeta_\theta \in H^1(0,T;E_0) \cap L^2(0,T;E_1).$$

*Proof.* The inclusion mapping of $(E_1,\|.\|_{E_1})$ into $(E_0,\|.\|_{E_0})$ is continuous and its range is dense. We denote by $E_1'$ the dual space of $E_1$ and, identifying the dual of $E_0$ with itself, we can write the Gelfand triple

$$E_1 \subset E_0 = E_0' \subset E_1'.$$

We use the notation $(.,.)_{E_1'\times E_1}$ to represent the duality pairing between $E'$ and $E_1$. We have

$$(\zeta,\xi)_{E_1'\times E_1} = (\zeta,\xi)_{E_0} \quad \forall\zeta \in E_0, \xi \in E_1,$$

and we note that $K$ is a closed convex set in $E_1$. Then, using (3.17), (3.19) and the fact that $\zeta_0 \in K$ in (3.18), it is easy to see that Lemma 4.7 is a straight consequence of Theorem 4.6. □

Now we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.3, $\varphi_\eta$ obtained in Lemma 4.4 and $\zeta_\theta$ obtained in Lemma 4.7 to construct the following Cauchy problem for the stress field.

**Problem $\mathbf{PV}^{\sigma}_{\eta\theta}$.** *Find a stress field* $\boldsymbol{\sigma}_{\eta\theta} = (\boldsymbol{\sigma}^1_{\eta\theta}, \boldsymbol{\sigma}^2_{\eta\theta}) : [0,T] \to \mathcal{H}$ *such that*

$$\boldsymbol{\sigma}^{\ell}_{\eta\theta}(t) = \mathcal{G}^{\ell}\boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell}_{\eta}(t)) + \int_0^t \mathcal{F}^{\ell}(\boldsymbol{\sigma}^{\ell}_{\eta\theta}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell}_{\eta}(s)), \zeta^{\ell}_{\theta}(s))\, ds, \quad \ell = 1,2, \tag{4.23}$$

for all $t \in [0,T]$.

In the study of Problem $\mathbf{PV}^{\sigma}_{\eta\theta}$ we have the following result.

**Lemma 4.8.** *There exists a unique solution of Problem* $\boldsymbol{PV}^{\sigma}_{\eta\theta}$ *and it satisfies* $\boldsymbol{\sigma}_{\eta\theta} \in L^2(0,T;\mathcal{H})$. *Moreover, if* $\boldsymbol{\sigma}_i$, $\boldsymbol{u}_i$ *and* $\zeta_i$ *represent the solutions of problems* $\boldsymbol{PV}^{\sigma}_{\eta_i\theta_i}$, $\boldsymbol{PV}^{u}_{\eta_i}$ *and* $\boldsymbol{PV}^{\zeta}_{\theta_i}$ *respectively, for* $(\boldsymbol{\eta}_i, \theta_i) \in L^2(0,T;\boldsymbol{V}' \times E_0)$, $i = 1,2$, *then there exists* $c > 0$ *such that*

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}} \le c\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}} + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_{\boldsymbol{V}}\, ds$$
$$+ \int_0^t \|\zeta_1(s) - \zeta_2(s)\|^2_{E_0}\, ds\Big) \quad \forall t \in [0,T]. \tag{4.24}$$

*Proof.* Let $\Lambda_{\eta\theta} = (\Lambda^1_{\eta\theta}, \Lambda^2_{\eta\theta}) : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ be the operator given by

$$\Lambda^{\ell}_{\eta\theta}\boldsymbol{\sigma}(t) = \mathcal{G}^{\ell}\boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell}_{\eta}(t)) + \int_0^t \mathcal{F}^{\ell}\big(\boldsymbol{\sigma}^{\ell}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell}_{\eta}(s)), \zeta^{\ell}_{\theta}\big)\, ds, \quad \ell = 1,2 \tag{4.25}$$

for all $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \in L^2(0,T;\mathcal{H})$ and $t \in [0,T]$. For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in L^2(0,T;\mathcal{H})$ we use (4.25) and (3.7) to obtain

$$\|\Lambda_{\eta\theta}\, \boldsymbol{\sigma}_1(t) - \Lambda_{\eta\theta}\, \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le \max(L_{\mathcal{F}^1}, L_{\mathcal{F}^2}) \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\, ds$$

for all $t \in [0,T]$. It follows from this inequality that for $p$ large enough, a power $\Lambda^p_{\eta\theta}$ of the operator $\Lambda_{\eta\theta}$ is a contraction on the Banach space $L^2(0,T;\mathcal{H})$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_{\eta\theta} \in L^2(0,T;\mathcal{H})$ such that $\Lambda_{\eta\theta}\boldsymbol{\sigma}_{\eta\theta} = \boldsymbol{\sigma}_{\eta\theta}$. Moreover, $\boldsymbol{\sigma}_{\eta\theta}$ is the unique solution of Problem $\mathbf{PV}^{\sigma}_{\eta\theta}$.

Consider now $(\boldsymbol{\eta}_1, \theta_1)$, $(\boldsymbol{\eta}_2, \theta_2), \in L^2(0,T;\boldsymbol{V}' \times E_0)$ and, for $i = 1,2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i\theta_i} = \boldsymbol{\sigma}_i$ and $\zeta_{\theta_i} = \zeta_i$. We have

$$\boldsymbol{\sigma}^{\ell}_i(t) = \mathcal{G}^{\ell}\boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell}_i(t)) + \int_0^t \mathcal{F}^{\ell}\big(\boldsymbol{\sigma}^{\ell}_i(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell}_i(s)), \zeta^{\ell}_i(s)\big)\, ds, \quad \ell = 1,2 \;\; \forall t \in [0,T],$$

and, using the properties (3.6) and (3.7) of $\mathcal{F}^{\ell}$, and $\mathcal{G}^{\ell}$ we find

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}} \le c\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\mathbf{V}} + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\, ds$$
$$+ \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_{V}\, ds + \int_0^t \|\zeta_1(s) - \zeta_2(s)\|^2_{E_0}\, ds\Big) \quad \forall t \in [0,T].$$

Using now a Gronwall argument in the previous inequality we deduce (4.24), which concludes the proof. □

Finally as a consequence of these results and using the properties of the operator $\mathcal{G}^{\ell}$, the operator $\mathcal{E}^{\ell}$, the operator $\mathcal{F}^{\ell}$, the functional $j_{ad}$, the function $j_{\nu c}$ and the function $\phi^{\ell}$ for $t \in [0,T]$, we consider the element

$$\Lambda(\eta,\theta)(t) = \big(\Lambda^1(\eta,\theta)(t),\; \Lambda^2(\eta,\theta)(t)\big) \in \boldsymbol{V}' \times E_0, \tag{4.26}$$

defined by the equations

$$(\Lambda^1(\eta,\theta)(t), \boldsymbol{v})_{\boldsymbol{V}' \times \boldsymbol{V}} = \sum_{\ell=1}^{2} \big(\mathcal{G}^{\ell}\varepsilon(\boldsymbol{u}^{\ell}_{\eta}(t)),\; \varepsilon(\boldsymbol{v}^{\ell})\big)_{\mathcal{H}^{\ell}}$$
$$+ \sum_{\ell=1}^{2} \big((\mathcal{E}^{\ell})^*\nabla\varphi^{\ell}_{\eta},\; \varepsilon(\boldsymbol{v}^{\ell})\big)_{\mathcal{H}^{\ell}} + \sum_{\ell=1}^{2} \Big(\int_0^t \mathcal{F}^{\ell}\big(\boldsymbol{\sigma}^{\ell}_{\eta\theta}, \varepsilon(\boldsymbol{u}^{\ell}_{\eta}(s)), \zeta_{\theta}(s)\big)\, ds,\; \varepsilon(\boldsymbol{v}^{\ell})\Big)_{\mathcal{H}^{\ell}}$$
$$+ j_{ad}(\beta_{\eta}(t), \boldsymbol{u}_{\eta}(t), \boldsymbol{v}) + j_{\nu c}(\boldsymbol{u}_{\eta}(t), \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{4.27}$$

$$\Lambda^2(\eta,\theta)(t) = \Big(\phi^1\big(\boldsymbol{\sigma}^1_{\eta\theta}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}^1_\eta(t)), \zeta^1_\theta(t)\big),\ \phi^2\big(\boldsymbol{\sigma}^2_{\eta\theta}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}^2_\eta(t)), \zeta^2_\theta(t)\big)\Big). \tag{4.28}$$

Here, for every $(\eta,\theta) \in L^2(0,T;\boldsymbol{V}' \times E_0)$, $\boldsymbol{u}_\eta$, $\varphi_\eta$, $\beta_\eta$ $\zeta_\theta$, and $\boldsymbol{\sigma}_{\eta\theta}$ represent the displacement field, the stress field, the the potential electric field and bonding field obtained in Lemmas 4.3, 4.4, 4.5, 4.7 and 4.8 respectively. We have the following result.

**Lemma 4.9.** *The operator $\Lambda$ has a unique fixed point $(\eta^*,\theta^*) \in L^2(0,T;\boldsymbol{V}' \times E_0)$.*

*Proof.* We show that, for a positive integer $n$, the mapping $\Lambda^n$ is a contraction on $L^2(0,T;\boldsymbol{V}' \times E_0)$. To this end, we suppose that $(\eta_1,\theta_1)$ and $(\eta_2,\theta_2)$ are two functions in $L^2(0,T;\boldsymbol{V}' \times E_0)$ and denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i} = \boldsymbol{v}_i$, $\boldsymbol{\sigma}_{\eta_i\theta_i} = \boldsymbol{\sigma}_i$, $\varphi_{\eta_i} = \varphi_i$, $\zeta_{\theta_i} = \zeta_i$ and $\beta_{\eta_i} = \beta_i$ for $i = 1,2$. We use (3.6), (3.7), (3.9), (3.11) and (3.12), the definition of $R_\nu$, $\boldsymbol{R}_\tau$ and Remark 3.1, we have

$$\|\Lambda^1(\eta_1,\theta_1)(t) - \Lambda^1(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'} \le \sum_{\ell=1}^{2} \|\mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_1(t)) - \mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_2(t))\|^2_{\mathcal{H}^\ell}$$

$$+\sum_{\ell=1}^{2}\int_0^t \big\|\mathcal{F}^\ell\big(\boldsymbol{\sigma}^\ell_1(s), \varepsilon(\boldsymbol{u}^\ell_1(s)), \zeta^\ell_1(s)\big) - \mathcal{F}^\ell\big(\boldsymbol{\sigma}^\ell_2(s), \varepsilon(\boldsymbol{u}^\ell_2(s)), \zeta^\ell_2(s)\big)\big\|^2_{\mathcal{H}^\ell}\,ds$$

$$+\sum_{\ell=1}^{2} \|(\mathcal{E}^\ell)^*\nabla\varphi^\ell_1(t) - (\mathcal{E}^\ell)^*\nabla\varphi^\ell_2(t)\|^2_{\mathcal{H}^\ell}$$

$$+C\|p_\nu([u_{1\nu}(t)]) - p_\nu([u_{2\nu}(t)])\|^2_{L^2(\Gamma_3)}$$

$$+C\|\beta^2_1(t)R_\nu([u_{1\nu}(t)]) - \beta^2_2(t)R_\nu([u_{2\nu}(t)])\|^2_{L^2(\Gamma_3)}$$

$$+C\|p_\tau(\beta_1(t))\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(t)]) - p_\tau(\beta_2(t))\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(t)])\|^2_{L^2(\Gamma_3)}.$$

Therefore,

$$\|\Lambda^1(\eta_1,\theta_1)(t) - \Lambda^1(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'} \le C\bigg(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$

$$+\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s))\|^2_{\mathcal{H}}\,ds +$$

$$\int_0^t \|\zeta_1(s) - \zeta_2(s))\|^2_{E_0}\,ds + \|\varphi_1(t) - \varphi_2(t)\|^2_W + \|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)}\bigg).$$

We use estimate (4.24) to obtain

$$\|\Lambda^1(\eta_1,\theta_1)(t) - \Lambda^1(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'} \le C\bigg(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$

$$+\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds + \int_0^t \|\zeta_1(s) - \zeta_2(s))\|^2_{E_0}\,ds$$

$$+\|\varphi_1(t) - \varphi_2(t)\|^2_W + \|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)}\bigg).$$

Recall that above $u^\ell_{\eta\nu}$ and $\boldsymbol{u}^\ell_{\eta\tau}$ denote the normal and the tangential component of the function $\boldsymbol{u}^\ell_\eta$ respectively. By similar arguments, from (4.24), (4.28) and (3.8) it follows that

$$\|\Lambda^2(\eta_1,\theta_1)(t) - \Lambda^2(\eta_2,\theta_1)(t)\|^2_{E_0} \le C\bigg(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$

$$+\|\varphi_1(t) - \varphi_2(t)\|^2_W + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds + \|\zeta_1(t) - \zeta_2(t))\|^2_{E_0} +$$

$$\int_0^t \|\zeta_1(s) - \zeta_2(s))\|^2_{E_0}\,ds\bigg).$$

Also, since

$$\boldsymbol{u}_i^\ell(t)=\int_0^t \boldsymbol{v}_i^\ell(s)ds+\boldsymbol{u}_0^\ell,\quad t\in[0,T],\ \ \ell=1,2,$$

we have

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}}\le\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|_{\boldsymbol{V}}\,ds$$

which implies

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}}^2+\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|_{\boldsymbol{V}}^2\,ds\le c\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|_{\boldsymbol{V}}^2\,ds. \tag{4.29}$$

Therefore

$$\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_1)(t)\|_{\boldsymbol{V}'\times E_0}^2\le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}}^2$$

$$+\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|_{\boldsymbol{V}}^2\,ds+\|\zeta_1(t)-\zeta_2(t))\|_{E_0}^2+\int_0^t \|\zeta_1(s)-\zeta_2(s))\|_{E_0}^2\,ds$$

$$+\|\varphi_1(t)-\varphi_2(t)\|_W^2+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}^2\Big). \tag{4.30}$$

Moreover, from (4.7) we obtain

$$(\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\boldsymbol{v}_1^\ell)-\mathcal{A}^\ell\varepsilon(\boldsymbol{v}_2^\ell),,\varepsilon(\boldsymbol{v}_1^\ell-\boldsymbol{v}_2^\ell))_{\mathcal{H}^\ell}$$

$$+(\eta_1-\eta_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}=0.$$

We integrate this equality with respect to time, use the initial conditions $\boldsymbol{v}_1(0)=\boldsymbol{v}_2(0)=\boldsymbol{v}_0$ and condition (3.5) to find

$$\min(m_{\mathcal{A}^1},m_{\mathcal{A}^2})\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|_{\boldsymbol{V}}^2\,ds\le$$

$$-\int_0^t (\eta_1(s)-\eta_2(s),\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))_{\boldsymbol{V}'\times\boldsymbol{V}}\,ds,$$

for all $t\in[0,T]$. Then, using the inequality $2ab\le\frac{a^2}{m}+mb^2$ we obtain

$$\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|_{\boldsymbol{V}}^2\,ds\le C\int_0^t \|\eta_1(s)-\eta_2(s)\|_{\boldsymbol{V}'}^2\,ds\quad\forall t\in[0,T]. \tag{4.31}$$

On the other hand, from the Cauchy problem (4.20)–(4.21) we can write

$$\beta_i(t)=\beta_0-\int_0^t\Big(\beta_i(s)\big(\gamma_\nu(R_\nu([u_{i\nu}(s)]))^2+\gamma_\tau\left|\boldsymbol{R}_\tau([\boldsymbol{u}_{i\tau}(s)])\right|^2\big)-\varepsilon_a\Big)_+ds$$

and then

$$\big\|\beta_1(t)-\beta_2(t)\big\|_{L^2(\Gamma_3)}\le C\int_0^t\big\|\beta_1(s)R_\nu([u_{1\nu}(s)])^2-\beta_2(s)R_\nu([u_{2\nu}(s)])^2\big\|_{L^2(\Gamma_3)}ds$$

$$+C\int_0^t\big\|\beta_1(s)\left|\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(s)])\right|^2-\beta_2(s)\left|\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(s)])\right|^2\big\|_{L^2(\Gamma_3)}ds.$$

Using the definition of $R_\nu$ and $\boldsymbol{R}_\tau$ and writing $\beta_1=\beta_1-\beta_2+\beta_2$, we get

$$\big\|\beta_1(t)-\beta_2(t)\big\|_{L^2(\Gamma_3)}\le$$

$$C\Big(\int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}ds+\int_0^t \big\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\big\|_{L^2(\Gamma_3)^d}ds\Big).$$

Next, we apply Gronwall's inequality to deduce

$$\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)} \le C\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_{L^2(\Gamma_3)^d}\,ds.$$

and from the relation (3.3) we obtain

$$\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)} \le C\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_V\,ds. \tag{4.32}$$

We use now (4.17), (3.4), (3.9) and (3.10) to find

$$\|\varphi_1(t)-\varphi_2(t)\|^2_W \le C\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_V. \tag{4.33}$$

We substitute (4.29), (4.32) and (4.33) in (4.30) to obtain

$$\begin{aligned} &\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'\times E_0} \\ &\le C\Big(\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\,ds + \|\zeta_1(t)-\zeta_2(t))\|^2_{E_0} + \int_0^t \|\zeta_1(s)-\zeta_2(s))\|^2_{E_0}\,ds\Big). \end{aligned} \tag{4.34}$$

On the other hand, from (4.22) we deduce that

$$\begin{aligned} &(\dot\zeta_1-\dot\zeta_2,\zeta_1-\zeta_2)_{E_0} + a(\zeta_1-\zeta_2,\zeta_1-\zeta_2) \le \\ &\big(\theta_1-\theta_2,\zeta_1-\zeta_2\big)_{E_0},\ \text{a.e. } t\in(0,T). \end{aligned}$$

Integrating the previous inequality with respect to time, using the initial conditions $\zeta_1(0)=\zeta_2(0)=\zeta_0$ and inequality $a(\zeta_1-\zeta_2,\zeta_1-\zeta_2)\ge 0$, to find

$$\frac{1}{2}\|\zeta_1(t)-\zeta_2(t)\|^2_{E_0} \le \int_0^t \big(\theta_1(s)-\theta_2(s),\zeta_1(s)-\zeta_2(s)\big)_{E_0}\,ds,$$

which implies that

$$\|\zeta_1(t)-\zeta_2(t)\|^2_{E_0} \le \int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{E_0}\,ds + \int_0^t \|\zeta_1(s)-\zeta_2(s)\|^2_{E_0}\,ds.$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\zeta_1(t)-\zeta_2(t)\|^2_{E_0} \le C\int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{E_0}\,ds \quad \forall t\in[0,T]. \tag{4.35}$$

We substitute (4.31) and (4.35) in (4.34) to obtain

$$\begin{aligned} &\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'\times E_0} \\ &\le C\int_0^t \|(\eta_1,\theta_1)(s)-(\eta_2,\theta_1)(s)\|^2_{\boldsymbol{V}'\times E_0}\,ds. \end{aligned}$$

Reiterating this inequality $n$ times we obtain

$$\|\Lambda^n(\eta_1,\theta_1)-\Lambda^n(\eta_2,\theta_1)\|^2_{L^2(0,T;\boldsymbol{V}'\times E_0)} \le \frac{C^nT^n}{n!}\|(\eta_1,\theta_1)-(\eta_2,\theta_1)\|^2_{L^2(0,T;\boldsymbol{V}'\times E_0)}.$$

Thus, for $n$ sufficiently large, $\Lambda^n$ is a contraction on the Banach space $L^2(0,T;\boldsymbol{V}'\times E_0)$, and so $\Lambda$ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 4.1.

*Proof. Existence.* Let $(\eta^*, \theta^*) \in L^2(0,T; \boldsymbol{V}' \times E_0)$ be the fixed point of $\Lambda$ defined by (4.26)–(4.28) and denote by

$$\boldsymbol{u}_* = \boldsymbol{u}_{\eta^*}, \quad \varphi_* = \varphi_{\eta^*}, \quad \zeta_* = \zeta_{\theta^*}, \quad \beta_* = \beta_{\eta^*}. \tag{4.36}$$

Let by $\boldsymbol{\sigma}_* = (\boldsymbol{\sigma}_*^1, \boldsymbol{\sigma}_*^2) : [0,T] \to \mathcal{H}$ and $\boldsymbol{D}_* = (\boldsymbol{D}_*^1, \boldsymbol{D}_*^2) : [0,T] \to H$ the functions defined by

$$\boldsymbol{\sigma}_*^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi_*^\ell + \boldsymbol{\sigma}_{\eta^*\theta^*}^\ell, \quad \ell = 1, 2, \tag{4.37}$$

$$\boldsymbol{D}_*^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}_*^\ell) - \mathcal{B}^\ell \nabla \varphi_*^\ell, \quad \ell = 1, 2. \tag{4.38}$$

We prove that the $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \zeta_*, \beta_*, \boldsymbol{D}_*\}$ satisfies (3.25)–(3.31) and the regularites (4.1)–(4.6). Indeed, we write (4.7) for $\eta = \eta^*$ and use (4.36) to find

$$\begin{aligned}(\ddot{u}_*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\dot{\boldsymbol{u}}_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + (\eta^*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}}\\ = (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, a.e.\ t \in [0,T].\end{aligned} \tag{4.39}$$

We use equalities $\Lambda^1(\eta^*, \theta^*) = \eta^*$ and $\Lambda^2(\eta^*, \theta^*) = \theta^*$ it follows that

$$\begin{aligned}(\eta^*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\ell=1}^{2} \left(\mathcal{G}^\ell \varepsilon(\boldsymbol{u}_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2} \left((\mathcal{E}^\ell)^* \nabla \varphi_*^\ell,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell}\\ + \sum_{\ell=1}^{2} \left( \int_0^t \mathcal{F}^\ell \Big(\boldsymbol{\sigma}_*^\ell(s) - \mathcal{A}^\ell \varepsilon(\dot{\boldsymbol{u}}_*^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \varphi_*^\ell(s), \varepsilon(\boldsymbol{u}_*^\ell(s)), \zeta_*^\ell(s)\Big)\, ds,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell}\\ + j_{ad}(\beta_*(t), \boldsymbol{u}_*(t), \boldsymbol{v}) + j_{\nu c}(\boldsymbol{u}_*(t), \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}.\end{aligned} \tag{4.40}$$

$$\theta_*^\ell(t) = \phi^\ell\big(\boldsymbol{\sigma}_*^\ell(t) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t)) - (\mathcal{E}^\ell)^* \nabla \varphi_*^\ell(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell(t)), \zeta_*^\ell(t)\big), \ \ell = 1, 2. \tag{4.41}$$

We now substitute (4.40) in (4.39) to obtain

$$\begin{aligned}(\ddot{\boldsymbol{u}}_*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\dot{\boldsymbol{u}}_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2} \left(\mathcal{G}^\ell \varepsilon(\boldsymbol{u}_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell}\\ + \sum_{\ell=1}^{2} \left((\mathcal{E}^\ell)^* \nabla \varphi_*^\ell,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell}\\ + \sum_{\ell=1}^{2} \left( \int_0^t \mathcal{F}^\ell (\boldsymbol{\sigma}_*^\ell(s) - \mathcal{A}^\ell \varepsilon(\dot{\boldsymbol{u}}_*^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \varphi_*^\ell(s), \varepsilon(\boldsymbol{u}_*^\ell(s)), \zeta_*^\ell(s))\, ds,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell}\\ + j_{ad}(\beta_*(t), \boldsymbol{u}_*(t), \boldsymbol{v}) + j_{\nu c}(\boldsymbol{u}_*(t), \boldsymbol{v}) = (\mathbf{f}(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}}, \quad \forall \boldsymbol{v} \in \boldsymbol{V},\end{aligned} \tag{4.42}$$

and we substitute (4.41) in (4.22) to have

$$\begin{aligned}\zeta_*(t) \in K, \qquad \sum_{\ell=1}^{2} (\dot{\zeta}_*^\ell(t), \xi^\ell - \zeta_*^\ell(t))_{L^2(\Omega^\ell)} + a(\zeta(t), \xi - \zeta(t)) \geq\\ \sum_{\ell=1}^{2} \Big(\phi^\ell\big(\boldsymbol{\sigma}_*^\ell(t) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t)) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell(t)), \zeta_*^\ell(t)\big), \xi^\ell - \zeta_*^\ell(t)\Big)_{L^2(\Omega^\ell)},\\ \forall \xi \in K, \ \text{a.e. } t \in [0,T].\end{aligned} \tag{4.43}$$

We write now (4.17) for $\eta = \eta^*$ and use (4.36) to see that

$$\begin{aligned}\sum_{\ell=1}^{2} (\mathcal{B}^\ell \nabla \varphi_*^\ell(t), \nabla \phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}_*^\ell(t)), \nabla \phi^\ell)_{H^\ell} = (q(t), \phi)_W,\\ \forall \phi \in W, \ \text{ a.e. } t \in [0,T].\end{aligned} \tag{4.44}$$

Additionally, we use $\boldsymbol{u}_{\eta^*}$ in (4.20) and (4.36) to find

$$\dot{\beta}_*(t) = -\Big(\beta_*(t)\big(\gamma_\nu(R_\nu([u_{*\nu}(t)]))^2 + \gamma_\tau \left|\boldsymbol{R}_\tau([\boldsymbol{u}_{*\tau}(t)])\right|^2\big) - \varepsilon_a\Big)_+, \text{ a.e. } t \in [0,T]. \tag{4.45}$$

The relations (4.36), (4.37), (4.38), (4.42), (4.43), (4.44) and (4.45) allow us to conclude now that $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \zeta_*, \beta_*, \boldsymbol{D}_*\}$ satisfies (3.25)–(3.30). Next, (3.31) and the regularity (4.1), (4.3), (4.4) and (4.5) follow from Lemmas 4.3, 4.4, 4.7 and 4.5. Since $\boldsymbol{u}_*$ and $\varphi_*$ satisfy (4.1) and (4.5), it follows from lemma 4.8 and (4.37) that

$$\sigma_* \in L^2(0,T;\mathcal{H}). \tag{4.46}$$

We choose $v = (v^1, v^2)$ with $v^\ell = \omega^\ell \in D(\Omega^\ell)^d$ and $v^{3-\ell} = 0$ in (4.42) and by (4.36) and (3.20):

$$\rho^\ell \ddot{\boldsymbol{u}}^\ell_* = \operatorname{Div} \boldsymbol{\sigma}^\ell_* + \boldsymbol{f}^\ell_0, \text{ a.e. } t \in [0,T], \quad \ell = 1,2.$$

Also, by (3.13), (3.14), (4.1) and (4.46) we have:

$$(\operatorname{Div} \boldsymbol{\sigma}^1_*, \operatorname{Div} \boldsymbol{\sigma}^2_*) \in L^2(0,T;\boldsymbol{V}')$$

Let $t_1, t_2 \in [0,T]$, by (3.4), (3.9), (3.10) and (4.38), we deduce that

$$\|\boldsymbol{D}_*(t_1) - \boldsymbol{D}_*(t_2)\|_H \le C\left(\|\varphi_*(t_1) - \varphi_*(t_2)\|_W + \|\boldsymbol{u}_*(t_1) - \boldsymbol{u}_*(t_2)\|_{\boldsymbol{V}}\right).$$

The regularity of $\boldsymbol{u}_*$ and $\varphi_*$ given by (4.1) and (4.3) implies

$$\boldsymbol{D}_* \in C(0,T;H). \tag{4.47}$$

We choose $\phi = (\phi^1, \phi^2)$ with $\phi^\ell \in D(\Omega^\ell)^d$ and $\phi^{3-\ell} = 0$ in (4.44) and using (3.21), (4.38) we find

$$div \boldsymbol{D}^\ell_*(t) = q^\ell_0(t) \quad \forall t \in [0,T], \quad \ell = 1,2,$$

and, by (3.14), (4.47), we obtain

$$\boldsymbol{D}_* \in C(0,T;\mathcal{W}).$$

Finally we conclude that the weak solution $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \zeta_*, \beta_*, \boldsymbol{D}_*\}$ of the piezoelectric contact problem **PV** has the regularity (4.1)–(4.6), which concludes the existence part of Theorem 4.1.

*Uniqueness.* The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator $\Lambda$ defined by (4.27)-(4.28) and the unique solvability of the Problems $\mathrm{PV}^u_\eta$, $\mathrm{PV}^\varphi_\eta$, $\mathrm{PV}^\beta_\eta$, $\mathrm{PV}^\zeta_\theta$ and $\mathrm{PV}^\sigma_{\eta\theta}$. □

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Received: August 4, 2015.

Accepted: November 23, 2015