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Strongly FI-δ-Lifting Modules

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Abstract Let R be a ring and M a right R-module. We call a module M is FI-δ-lifting if every fully invariant submodule A of M contains a direct summand B of M such that $A/B \ll_\delta M/B$. In this paper several properties of these modules are studied. We show that a ring R is FI-δ-lifting as an R-module if and only if R/I has a projective δ-cover for every two sided ideal I of R.

1 Introduction

Throughout this paper R is an associative ring with unity and all modules are unital right R-modules. A submodule K of a module M is denoted by $K \leq M$. Let M be an R-module and $S \leq M$. S is called a $small$ submodule of M (denoted by $S \ll M$) if for every submodule T of M with $M = S + T$, then $M = T$. Let M be an R-module and $N \leq M$. If any submodule K of M is minimal with the property that $M = N + K$, then the submodule K is called a $supplement$ of N in M. K is a supplement (weak supplement) of N in M if and only if $M = N + K$ and $N \cap K \ll K$ ($N \cap K \ll M$). The module M is said to be a *lifting* module if for any submodule N of M there exists $A \leq N$ such that $M = A \oplus B$ and $N \cap B \ll B$. As a generalization of small submodules, a submodule N of M is called δ-small in M, denoted by $N \ll_\delta M$, if $M = N + K$ with M/K singular implies $M = K$. Every small submodule of M is δ-small in M and the converse is true whenever M is singular. The sum of all δ-small submodules of a module M is denoted by $\delta(M)$, which defines a preradical on the category of R-modules, $\delta(M) = \Sigma\{L \leq M \mid L \ll_\delta M\}$ (See [12]). A module M is called *δ-lifting* if for any $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is δ-small in M. An epimorphism $f : P \longrightarrow N$ is called a δ-cover of N if $Ker(f) \ll_\delta P$ and if moreover P is projective, then it is called a projective δ-cover. A submodule K of M is called *fully invariant* if $\lambda(K) \subseteq K$ for all $\lambda \in End_R(M)$. In [2] and [4] FI-lifting defined and studied and also several properties of FI-lifting investigated. We say that a module M is *FI-δ-lifting* if every fully invariant submodule A of M contains a direct summand B of M such that $A/B \ll_\delta M/B$. Also, for the other definition and notation in this paper we refer to [1], [3].

2 FI-δ-Lifting Modules

In this section we define FI-δ-lifting modules. We show that this class of modules is closed under finite direct sums. We prove that ring R is FI-δ-lifting module as an R-module if and only if R/I has a projective cover for every two sided ideal I of R (Corollary 2.7).

Lemma 2.1. *Let M be a module. Then:*

(1) *Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule of M (in fact the fully invariant submodules form a complete modular sublattice of the lattice of submodules of M).*

(2) *If $X \subseteq Y \subseteq M$ such that Y is a fully invariant submodule of M and X is a fully invariant submodule of Y, then X is a fully invariant submodule of M.*

(3) *If $M = \oplus_{i \in I} X_i$ and S is a fully invariant submodule of M, then $S = \oplus_{i\in I}\pi_i(S) = \oplus_{i\in I}(X_i \cap S)$, where π_i is the i-th projection homomorphism of M.*

Proof. See [1, Lemma 1.1].

We note that if $M = \oplus_{i=1}^n M_i$ and N is a fully invariant submodule of M, then $N = \oplus_{i=1}^n (N \cap M_i)$ and $N \cap M_i$ is a fully invariant submodule of M_i. □

Example 2.2. Let R be a right semisimple ring and M a nonzero right R-module. Then M is semisimple and nonsingular. For any nonzero $N \leq M$, N is a direct summand of M and hence is not small in M, but every submodule of M(even M itself) is δ-small in M. So M is δ-lifting but is not lifting

The following Proposition introduces an equivalent condition for a FI-δ-lifting module.

Proposition 2.3. *Let M be an R-module. Then the following are equivalent:*
(1) *M is FI-δ-lifting module;*
(2) *For every fully invariant submodule A of M there is a decomposition $A = N \oplus S$ where N is a dirct summand of M and S, δ-small in M.*

Proof. (1) $\Longrightarrow$ (2) Let A be a fully invariant submodule of M. Since M is FI-δ-lifting, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A$ δ-small in M_2. Therefore $A = M_1 \oplus (A \cap M_2)$.
(2) $\Longrightarrow$ (1) Assume that every fully invariant submodule has the stated decomposition. Let A be a fully invariant submodule of M. By hypothesis, there exists a direct summand N of M and a δ-small submodule S of M such that $A = N \oplus S$. Now let $M = N \oplus N'$ for some submodule N' of M. Consider the natural epimorphism $\pi : M \longrightarrow M/N$. Then $\pi(S) = (S + N)/N = A/N \ll_\delta M/N$. Therefore, M is FI-δ-lifting □

Theorem 2.4. *Let $M = \bigoplus_{i=1}^n M_i$ be a finite direct sum of FI-δ-lifting modules. Then M is FI-δ-lifting.*

Proof. Let N be a fully invariant submodule of M. Then $N = \bigoplus_{i=1}^n (N \cap M_i)$ and $N \cap M_i$ is a fully invariant submodule of M_i. Since each M_i is FI-δ-lifting, by Proposition 2.3, $N \cap M_i = L_i \oplus S_i$ where L_i is a direct summand of M_i and $S_i \ll_\delta M_i$. Set $L = \bigoplus_{i=1}^n L_i$ and $S = \bigoplus_{i=1}^n S_i$. Then $N = L \oplus S$ where L is a direct summand of M and $S \ll_\delta M$. □

Corollary 2.5. *If M is a finite direct sum of δ-lifting modules, then M is FI-δ-lifting.*

Proposition 2.6. *Let P be a projective module. Then P is FI-δ-lifting if and only if P/A has a projective δ-cover for every fully invariant submodule A of P.*

Proof. Suppose P is a projective FI-δ-lifting module and A a fully invariant submodule of P. Then $A = X \oplus S$ where X is a direct summand of P and $S \ll_\delta P$. Let $P = X \oplus Y$ for some $Y \leq P$. As $S \ll_\delta P$, $(X + S)/X \ll_\delta P/X$. Hence the natural map $f : P/X \to P/(X + S) = P/A$ is a projective δ-cover.

Conversely, suppose for every fully invarient submodule A of P, P/A has a projective δ-cover. Let $f : Q \to P/A$ be a projective δ-cover of P/A. Then there exists a map $h : P \to Q$ such that $fh = \eta$ where $\eta : P \to P/A$ is the natural map. As $Kerf \ll_\delta Q$ and η is an epimorphism, h is an epimorphism and hence h splits. Suppose $P = Kerh \oplus B$ for some submodule B of P. Then $A = Kerh \oplus (A \cap B)$ and $A \cap B \ll_\delta P$. Thus P is FI-δ-lifting. □

Corollary 2.7. *Let R be a ring. The module R_R is FI-δ-lifting if and only if R/I has a projective δ-cover for every two sided ideal I of R.*

Proposition 2.8. *Let R be a ring and M FI-δ-lifting. Then every fully invariant submodule of the module $M/\delta(M)$ is a direct summand.*

Proof. Let $N/\delta(M)$ be a fully invariant submodule of $M/\delta(M)$. Then N is fully invariant submodule of M by [7, Lemma 3.2]. By hypothesis, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll_\delta M_2$. Since $M_2 \cap N$ is also δ-small in M, $N \cap M_2 \leq \delta(M)$. Thus $M/\delta(M) = (N/\delta(M)) \oplus ((M_2 + \delta(M))/\delta(M))$. □

Lemma 2.9. *[11, 41.14] The following are equivalent for a module $M = M' \oplus M''$:*
(1) *M' is M''-projective.*
(2) *For each submodule N of M with $M = N + M''$, there exists a submodule $N' \leq N$ such that $M = N' \oplus M''$.*

Theorem 2.10. *Let M_1 and M_2 be two modules such that M_1 is semisimple and M_2 is FI-δ-lifting. If M_1 and M_2 be relatively projective, then $M = M_1 \oplus M_2$ is FI-δ-lifting.*

Proof. Let $0 \neq N \leq M$ be fully invariant. Let $K = M_1 \cap (N + M_2)$. We divide the proof into two cases:

Case (1): Let $K \neq 0$. Then $M_1 = K \oplus K_1$ for some submodule K_1 of M_1 and so $M = K \oplus K_1 \oplus M_2 = N + (M_2 \oplus K_1)$. Hence, K is $(M_2 \oplus K_1)$-projective. By Lemma 2.9, there exists a submodule N_1 of N such that $M = N_1 \oplus (M_2 \oplus K_1)$. We may assume $N \cap (M_2 \oplus K_1) \neq 0$.

Then $N \cap (L + K_1) = L \cap (N + K_1)$ for any submodule L of M_2. Since M_2 is FI-δ-lifting, there exists a submodule X of $M_2 \cap (N + K_1) = N \cap (M_2 \oplus K_1)$ such that $M_2 = X \oplus Y$ and $Y \cap (N + K_1)$ is δ-small in M_2. Hence, $M = (N_1 \oplus X) \oplus (Y \oplus K_1)$. Since $N_1 \oplus X \leq N$ and $N \cap (Y \oplus K_1) = Y \cap (N + K_1)$, $N \cap (Y \oplus K_1) = Y \cap (N + K_1)$ is δ-small in $Y \oplus K_1$ by [12, Lemma 1.3]. So M is FI-δ-lifting.

Case (2): Let $K = 0$. Then $N \leq M_2$. Since M_2 is FI-δ-lifting, there exists a submodule X of N such that $M_2 = X \oplus Y$ and $N \cap Y$ is δ-small in Y for some submodule Y of M_2. Hence, $M = X \oplus (M_1 \oplus Y)$ and $N \cap (M_1 \oplus Y) = N \cap Y$ is δ-small in Y . By [12, Lemma 1.3], $N \cap (M_1 \oplus Y) \ll_\delta M_1 \oplus Y$. □

Example 2.11. Let $M_\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Then $M_\mathbb{Z}$ is FI-δ-lifting by Corollary 2.5. But $M_\mathbb{Z}$ is not δ-lifting by [8, Example 2.8].

Proposition 2.12. *Let M be a module. The following are equivalent:*

(1) *M is FI-δ-lifting;*

(2) *every fully invariant submodule of M has a direct summand δ-supplement;*

(3) *for each fully invariant submodule X of M, there is a coclosed submodule K of M and a direct summand δ-supplement L of K such that $K \subseteq X$, $X/K \ll_\delta M/K$ and every homomorphism $f : M \longrightarrow M/L \cap K$ can be lifted to an endomorphism $g : M \longrightarrow M$, that is, such that $g(m) + (L \cap K) = f(m)$ for all $m \in M$.*

Proof. (1) $\iff$ (2) Let X be a fully invariant submodule of M. First assume that M is FI-δ-lifting. Then there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq X$ and $M_2 \cap X \ll_\delta M_2$. Then $M = X + M_2$ and M_2 is a direct summand δ-supplement of X. Conversely, let K be a direct summand δ-supplement of X in M. Then $M = K + X = K \oplus K'$ and $K \cap X \ll_\delta K$ for some submodule K' of M. Consider the natural projection map $\phi : M \longrightarrow K'$. Since X is fully invariant, $\phi(X) = (X + K) \cap K' = M \cap K' = K' \leq X$. Thus, M is FI-δ-lifting.

(2) $\Longrightarrow$ (3) Let X be a fully invariant submodule of M. Since M is FI-δ-lifting, there exists a decomposition $M = L \oplus K$ such that $K \leq X$ and $X/K \ll_\delta M/K$. Since $L \cap K = 0$, clearly any homomorphism $f : M \longrightarrow M/(L \cap K)$ lifts to a $g : M \longrightarrow M$.

(3) $\Longrightarrow$ (1) Let X be a fully invariant submodule of M. By (3), there is a coclosed submodule K of M and a direct summand δ-supplement L of K such that $K \leq X$ and $X/K \ll_\delta M/K$. Since K is a δ-supplement in M by [4, Proposition 3], it follows from [5, Lemma 2.2] that K is a direct summand of M. Thus, M is FI-δ-lifting. □

A module M is called a *duo* module provided that every submodule of M is fully invariant.

Proposition 2.13. *Let M be a module. Consider the following statements:*

(1) *M is δ-lifting;*

(2) *M is $\oplus$-δ-supplemented;*

(3) *M is FI-δ-lifting. Then* (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). *If M is a duo module, then* (3) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2) This is clear.

(2) $\Longrightarrow$ (3) This is clear by Proposition 2.12. □

Proposition 2.14. *Let $M = M_1 \oplus M_2$. Then M_2 is FI-δ-lifting if and only if for every fully invariant submodule N/M_1 of M/M_1, there exists a direct summand K of M such that $K \leq M_2$, $M = K + N$ and $N \cap K \ll_\delta M$.*

Proof. Suppose that M_2 is FI-δ-lifting. Let N/M_1 be any fully invariant submodule of M/M_1. It is easy to see that $N \cap M_2$ is fully invariant in M_2. Since M_2 is FI-δ-lifting, there exists a decomposition $M_2 = K \oplus K'$ such that $M_2 = (N \cap M_2) + K$ and $N \cap K \ll_\delta K$. Clearly, $M = N + K$.

Conversely, suppose that M/M_1 has the stated property. Let H be a fully invariant submodule of M_2. It is easy to see that $(H \oplus M_1)/M_1$ is fully invariant in M/M_1. By hypothesis, there exists a direct summand L of M such that $L \leq M_2$, $M = L + H + M_1$ and $L \cap (H + M_1) \ll_\delta M$. By modularity, $M_2 = L + H$. It follows easily that L is a δ-supplement of H in M_2. Therefore, M_2 is FI-δ-lifting by Proposition 2.12. □

We define a module to be H-δ-supplemented if for every submodule N of M there exists a direct summand D of M such that $(N + D)/N \ll_\delta M/N$ and $(N + D)/D \ll_\delta M/D$.

Proposition 2.15. *Let $M = M_1 \oplus M_2$, where M_1 is a fully invariant submodule of M. If M is H-δ-supplemented, then M_1 and M_2 are H-δ-supplemented.*

Proof. Similar to [9, Corollary 2.4], M_2 is H-δ-supplemented. Next we show that M_1 is H-δ-supplemented. Let K be a submodule of M_1. Since M is H-δ-supplemented, there exists a direct summand D of M such that $(K+D)/K \ll_\delta M/K$ and $(K+D)/D \ll_\delta M/D$. Write $M = D \oplus D'$, $D' \leq M$. Then $M = K + D'$. Since M_1 is a fully invariant submodule of M, $M_1 = (M_1 \cap D) \oplus (M_1 \cap D')$. Hence $M = M_1 + D' = (M_1 \cap D) \oplus D'$. Thus $D = D \cap M_1$ and so $D \leq M_1$. Now $(K+D)/K \ll_\delta M_1/K$ and $(K+D)/D \ll_\delta M_1/D$. Therefore, M_1 is H-δ-supplemented. □

Theorem 2.16. *Let $M = M_1 \oplus M_2$ be a direct sum of modules. If M_1 is M_2-projective and M is H-δ-supplemented, then M_2 is H-δ-supplemented.*

Proof. Let Y be a submodule of M_2. Considering the submodule $Y \oplus M_1$ of M. Since M is H-δ-supplemented, there exists a submodule X of M and a direct summand D of M such that $X/(Y \oplus M_1) \ll_\delta M/(Y \oplus M_1)$ and $X/D \ll_\delta M/D$. Then $M = X + M_2$ and so $M = D + M_2$. Since M_1 is M_2-projective, there exists a submodule D' of D such that $M = D' \oplus M_2$. Hence $D = D' \oplus (D \cap M_2)$, $(X \cap M_2)/(D \cap M_2) \ll_\delta M/(D \cap M_2)$, and $(X \cap M_2)/Y \ll_\delta M/Y$. Thus M_2 is H-δ-supplemented. □

In [10], Özcan defined the submodule $\overline{Z}_\delta(M)$ of M as $\overline{Z}_\delta(M) = \bigcap\{Ker g \mid g : M \to N, N \text{ is a } \delta\text{-small module}\}$. Any module M is called a *δ-cosingular* (*non-δ-cosingular*) module if $\overline{Z}_\delta(M) = 0$ ($\overline{Z}_\delta(M) = M$).

Proposition 2.17. *Let M be an amply supplemented module. Then M is H-δ-supplemented if and only if $M = \overline{Z}^2_\delta(M) \oplus M'$, where $\overline{Z}^2_\delta(M)$ and M' are H-δ-supplemented, where $\overline{Z}^2_\delta(M) = \overline{Z}_\delta(\overline{Z}_\delta(M))$.*

Proof. Let M be an H-δ-supplemented module. Note that $\overline{Z}^2_\delta(M)$ is a fully invariant coclosed submodule of M. Since M is FI-δ-lifting, $M = \overline{Z}^2_\delta(M) \oplus M'$, where $\overline{Z}^2_\delta(M)$ and M' are H-δ-supplemented by Proposition 2.15. Conversely, let M be an amply supplemented module. Then $\overline{Z}^2_\delta(M)$ is M'-projective by the proof [10, Theorem 2.19]. Therefore, M is H-δ-supplemented by Theorem 2.16. □

Proposition 2.18. *Let M be an indecomposable R-module. If M is FI-δ-lifting, then for every fully invariant submodule A of M, $\delta(A) \ll_\delta M$.*

Proof. Let A be a fully invariant submodule of M. Since $\delta(A)$ is a fully invariant submodule of A, then $\delta(A)$ is a fully invariant submodule of M, by [2, lemma 2.1]. Hence $\delta(A) = B \oplus L$, where B is a direct summand of M and $L \ll_\delta M$. But M is an indecomposable, therefore $B = 0$. Thus $\delta(A) = L$ and hence $\delta(A) \ll_\delta M$. □

3 Strongly FI-δ-lifting Modules

In this section we define strongly FI-δ-lifting. This class of modules is properly contained in the class of FI-δ-lifting. We show that a finite direct sum of copies of a strongly FI-δ-lifting module is strongly FI-δ-lifting and if M is $\mathcal{T}$-δ-noncosingular, then FI-δ-lifting and strongly FI-δ-lifting are same (Proposition 3.6).

We say that a module M is *strongly FI-δ-lifting* if every fully invariant submodule A of M contains a fully invariant direct summand B of M such that $A/B \ll_\delta M/B$.

Proposition 3.1. *The following are equivalent for an R-module M:*

(1) *M is a strongly FI-δ-lifting module;*

(2) *Every fully invariant submodule A of M can be written as $A = B \oplus S$, where B is a fully invariant direct summand of M and $S \ll_\delta M$.*

Proposition 3.2. *Let M be an FI-δ-lifting with $\delta(M) = 0$. Then every fully invariant submodule (in particular M) is strongly FI-δ-lifting.*

Proof. Let $A \leq N \leq M$ such that A is fully invariant in N and N is fully invariant in M . Then A is fully invariant in M [7, Lemma 3.2]. As M is FI-δ-lifting, $A = B \oplus S$ where B is a direct summand of M and $S \ll_\delta M$ by Proposition 2.3. Since $\delta(M) = 0$, $S = 0$ and so A is a direct summand of M and hence of N. Thus, N is strongly FI-δ-lifting. □

Theorem 3.3. *Let $M = \bigoplus_{i=1}^n M_i$ be a finite direct sum of strongly FI-δ-lifting modules such that $M_i \cong M_j$ for all i and j. Then M is a strongly FI-δ-lifting module.*

Proof. There exist isomorphisms $f_i : M_1 \to M_i$ for $i = 2, \cdots, n$. If A is a fully invariant submodule of M, then it is easy to see that $A = A_1 \oplus f_2(A_1) \oplus \cdots \oplus f_n(A_1)$ where $A_1 = M_1 \cap A$.

As M_1 is strongly FI-δ-lifting and A_1 is a fully invariant submodule of M_1, we have $A_1 = L_1 \oplus S_1$ where L_1 is a fully invariant direct summand of M_1 and $S_1 \ll_\delta M_1$ by proposition 3.1. Put $L := L_1 \oplus f_2(L_1) \oplus \cdots \oplus f_n(L_1)$ and $S := S_1 \oplus f_2(S_1) \oplus \cdots \oplus f_n(S_1)$. Then $A = L \oplus S$ such that L is a fully invariant direct summand of M and $S \ll_\delta M$. Hence, M is strongly FI-δ-lifting. □

Example 3.4. (1) Consider the module M given in Example 2.11. Consider the submodule $N = \mathbb{Z}/p\mathbb{Z} \oplus p^2\mathbb{Z}/p^3\mathbb{Z}$ of M. N is not small in M and since N is singular, N is not δ-small in M and contains no nonzero fully invariant direct summand of M. Hence M is not strongly FI-δ-lifting. But M is FI-δ-lifting.

(2) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$. M is lifting by [3, 23.20] and so δ-lifting. For, consider the fully invariant submodule $N = \mathbb{Z}/p\mathbb{Z} \oplus p\mathbb{Z}/p^2\mathbb{Z}$ of M. N is a fully invariant submodule of M which is not δ-small in M. But N does not contain any nonzero fully invariant direct summand of M. Thus M is not strongly FI-δ-lifting.

(3) The only fully invariant submodules of $\mathbb{Q}$ are 0 and $\mathbb{Q}$. Therefore, $\mathbb{Q}$ is strongly FI-δ-lifting. On the other hand, $\mathbb{Q}$ is not $\oplus$-supplemented and since every torsion $\mathbb{Z}$-module is singular, so $\mathbb{Q}$ is not $\oplus$-δ-supplemented.

Following [6], the module M is called $\mathcal{T}$-noncosingular if, for every nonzero endomorphism φ of M, $Im\varphi$ is not small in M. We define $\mathcal{T}$-δ-noncosingular, we say M is $\mathcal{T}$-δ-noncosingular if,for every nonzero endomorphism φ of M, $Im\varphi$ is not δ-small in M.

Proposition 3.5. *Let M be a $\mathcal{T}$-δ-noncosingular module and X fully invariant in M. Let $N \leq X$ such that $X/N \ll_\delta M/N$ and N a direct summand of M. Then N is (unique) fully invariant in M.*

Proof. Let P be a submodule of M such that $M = N \oplus P$. Assume that N is not fully invariant in M. Then there exist an endomorphism φ of M and $x \in N$ such that $\varphi(x) in N$. Let $\psi = \pi_P \varphi \pi_N : M \longrightarrow P$, where $\pi_N : M \longrightarrow N$ and $\pi_N : M \longrightarrow P$ are the projections. Note that $\psi \neq 0 (\varphi(x) \in N)$ and $Im\psi \subseteq X \cap P \ll_\delta M$. This contradicts the fact that M is $\mathcal{T}$-δ-noncosingular. Thus, N is fully invariant in M. □

Proposition 3.6. *Let M be a $\mathcal{T}$-δ-noncosingular module. Then M is FI-δ-lifting if and only if M is strongly FI-δ-lifting.*

Proof. Let M be FI-δ-lifting and X a fully invariant submodule of M. Then there exists a direct summand N of M such that $X/N \ll_\delta M/N$. By Proposition 3.5, N is fully invariant in M. Thus, M is strongly FI-δ-lifting. The converse is clear. □

Corollary 3.7. *Let M be a δ-noncosingular module. Then M is FI-δ-lifting if and only if M is strongly FI-δ-lifting.*

Proposition 3.8. *Let M be an FI-δ-lifting module and X a fully invariant submodule of M. If one of the following conditions is satisfied, then M/X is strongly FI-δ-lifting:*

(1) *M/X is indecomposable;*

(2) *M/X is $\mathcal{T}$-δ-noncosingular.*

Proof. By [7, Proposition 3.3], M/X is FI-δ-lifting. (1) Clearly, indecomposable FI-δ-lifting modules are strongly FI-δ-lifting. (2) This follows from Proposition 3.5. □

Proposition 3.9. *Let M be a δ-lifting (respectively δ-noncosingular weakly δ-supplemented FI-δ–lifting) module such that every δ-small submodule is fully invariant. Then every factor module of M is δ-lifting (respectively strongly FI-δ-lifting).*

Proof. Let X, Y be submodules of M such that $M = X + Y$ and $X \cap Y \ll_\delta M$. Note that $M/(X \cap Y) = X/(X \cap Y) \oplus Y/(X \cap Y)$. By hypothesis, $X \cap Y$ is fully invariant in M. If M is δ-lifting, then $M/(X \cap Y)$ is δ-lifting by [1, 22.2]. Since the δ-lifting property is inherited by direct summands, M/X is δ-lifting. Now assume that M is a δ-noncosingular weakly δ-supplemented FI-δ-lifting module. Then the result follows from [7, Proposition 3.3], Corollary 3.7 and the fact that any direct summand of a strongly FI-δ-lifting module is strongly FI-δ-lifting. □

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