Palestine Journal of Mathematics

Vol. 3(Spec 1) (2014) , 481–488

On Primal Compactly Packed Modules

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Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

Communicated by Ayman Badawi

MSC 2010 Classifications: 13A15, 13F05.

Keywords and phrases: primal submodule, compactly packed module, primal avoidance theorem for modules.

Abstract. Let R be a commutative ring with identity and let M be a unitary $R-$module. A proper submodule N of M is said to be primary compactly packed if for each family $\{P_\alpha\}_{\alpha\in\triangle}$ of primary submodules of M with $N\subseteq\bigcup_{\alpha\in\triangle}P_\alpha$, $N\subseteq P_\beta$ for some $\beta\in\triangle$. A module M is called primary compactly packed if every proper submodule of M is primary compactly packed. This concept was introduced in [11]. In this paper we generalize the concept of primary compactly packed modules to the concept of primal compactly packed modules. We say that a proper submodule N of M is priaml compactly packed if for each family $\{P_\alpha\}_{\alpha\in\triangle}$ of primal submodules of M with $N\subseteq\bigcup_{\alpha\in\triangle}P_\alpha$, $N\subseteq P_\beta$ for some $\beta\in\triangle$. A module M is called priaml compactly packed if every proper submodule of M is priaml compactly packed. We also generalize the Primary Avoidance Theorem for modules that was proved in [12] to the Primal Avoidance Theorem for modules.

1 Introduction

Let R be a commutative ring with identity and M be a unitary $R-$module. A proper submodule N of M is said to be prime (resp. primary) if $rm\in N$ for $r\in R$ and $m\in M$ implies that either $m\in N$ or $rM\subseteq N$ (resp. or $r^nM\subseteq N$ for some positive integer n).

A proper submodule N of M is said to be primal if the set $adj(N)=\{r\in R\mid rm\in N$ for some $m\in M-N\}$ forms an ideal of R.
It was known that every prime submodule is primary, and every primary submodule is primal.

Many studies were done on concepts related to prime submodule, one of these important concepts is the concept of compactly packed modules. A proper submodule N of M is compactly packed if for each family $\{P_\alpha\}_{\alpha\in\triangle}$ of prime submodules of M with $N\subseteq\bigcup_{\alpha\in\triangle}P_\alpha$, $N\subseteq P_\beta$ for some $\beta\in\triangle$. A module M is called compactly packed if every proper submodule of M is compactly packed. This concept was introduced in [23] and generalized to primary compactly packed modules by El-Atrash and Ashour (see [11]).
This paper is concerned with the properties of primal submodule, and generalizing the concept of primary compactly packed modules to the concept of primal compactly packed modules.
We introduce the concept of primal compactly packed modules as follows: A proper submodule N of M is said to be primal compactly packed if for each family $\{P_\alpha\}_{\alpha\in\triangle}$ of primal submodules of M with $N\subseteq\bigcup_{\alpha\in\triangle}P_\alpha$, $N\subseteq P_\beta$ for some $\beta\in\triangle$.
A module M is primal compactly packed if every proper submodule of M is primal compactly packed.
Then we prove that a module M is primal compactly packed if and only if every proper submodule of M is cyclic.

We also generalize the Primary Avoidance Theorem for modules to the Primal Avoidance Theorem for modules as follows: Let M be an R-module, $L_1,L_2,...,L_k$ a finite number of submodules of M and L a submodule of M such that $L\subseteq L_1\cup L_2\cup...\cup L_k$. Assume that at most two of the L_i's are not primal, and that $(L_i:M)\nsubseteq adj(L_j)$ whenever $i\neq j$, Then $L\subseteq L_m$ for some $m\in\{1,2,...,k\}$.

We assume throughout this paper that all rings will be commutative with identity and all modules will be unitary.

2 Preliminaries

Definition 2.1. [18] Let M be an $R-$module. A proper submodule N of M is prime if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $rM \subseteq N$.

Definition 2.2. [24] Let M be an $R-$module. A proper submodule N of M is primary if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r^n M \subseteq N$ for some positive integer n.

It is clear directly from the definitions that every prime submodule is primary.

Definition 2.3. [17] Let N be a submodule of an $R-$module M. The residual of N by M, denoted $(N : M)$, is the ideal $(N : M) = \{r \in R \mid rM \subseteq N\}$.

Definition 2.4. [6] Let M be an $R-$module and N a submodule of M. An element $a \in R$ is called prime to N if $am \in N$ $(m \in M)$ implies that $m \in N$. We denote $adj(N)$ to be the set of all elements of R that are not prime to N. A proper submodule N of M is said to be primal if $adj(N)$ forms an ideal of R, this ideal is called the adjoint ideal of N. Note that if N is a primal submodule of M, then $adj(N)$ is a prime ideal, for if $ab \in adj(N)$ with $a \notin adj(N)$, there exists $m \in M - N$ with $abm \in N$, so $bm \in N$ implies that $b \in adj(N)$.

If there exists an element $m_0 \in M - N$ such that $rm_0 \in N$ for all $r \in adj(N)$, then $adj(N)$ is an ideal of R and hence N is primal. In this case we say that N is a uniformly primal submodule.

Proposition 2.5. *[6] Let N be a proper submodule of an $R-$module M, then $(N : M) \subseteq adj(N)$.*

Definition 2.6. [17] Let M be an $R-$module. A proper submodule N of M is said to be irreducible if N is not the intersection of two submodules of M that properly contain it.

Definition 2.7. [2] Let M be an $R-$module. A proper submodule N of M is said to be completely irreducible (or strongly irreducible) if for any family $\{N_\alpha\}_{\alpha\in\triangle}$ of submodules of M with $N = \bigcap_{\alpha\in\triangle} N_\alpha$, $N = N_\beta$ for some $\beta \in \triangle$. On other words, N is not the intersection of any collection of submodules of M each properly containing N.

Proposition 2.8. *[10] Let N be a proper submodule of an $R-$module M. Then N is a prime submodule of M if and only if $adj(N) = (N : M)$.*

Proposition 2.9. *[10] Let M be an $R-$module. Then*

(i) Every prime submodule of M is uniformly primal.

(ii) Every irreducible submodule of M is primal.

(iii) Every completely irreducible submodule of M is uniformly primal.

Proposition 2.10. *[9] Let M be an $R-$module. Then every primary submodule of M is primal.*

Proposition 2.11. *Let N be a proper submodule of an $R-$module M. If $adj(N)$ is a principle ideal of R, then N is uniformly primal.*

Proof. By assumption $adj(N) = Ra$ for some $a \in R$, then a is not prime to N, so there exists an element $u \in M - N$ with $au \in N$, thus $adj(N)u = Rau \subseteq N$, and as a consequence, $adj(N)$ is uniformly not prime to N, and N is uniformly primal. □

Corollary 2.12. *Let R be a principal ideal ring, and M an $R-$module. Then every primal submodule is uniformly primal.*

Proposition 2.13. *Let R be a Boolean ring and M be an $R-$module. Then Every primal submodule of M is prime.*

Proof. Let N be a primal submodule of M. By Propositions 2.8 and 2.5, it suffices to show that $adj(N) \subseteq (N : M)$. Let $a \in adj(N)$, then $1 - a \notin adj(N)$, for otherwise, $1 = (1 - a) + a \in adj(N)$ which is a contradiction. That is $1 - a$ is prime to N, hence for all $m \in M$, $(1 - a)am = 0 \in N$ implies $am \in N$. Therefore $a \in (N : M)$. □

The following corollaries follows immediately from Proposition 2.13.

Corollary 2.14. *Let R be a Boolean ring and M be an $R-$module. Then Every primary submodule of M is prime.*

Corollary 2.15. *Every primal ideal of a Boolean ring is prime.*

Corollary 2.16. *Every primary ideal of a Boolean ring is prime.*

Definition 2.17. [11] Let N be a submodule of an $R-$module M. If there exist primary submodules which contain N, then the intersection of all primary submodules containing N is called the primary radical of N and denoted by $prad(N)$. If there is no primary submodule containing N, then $prad(N) = M$. In particular $prad(M) = M$. We say that a submodule N is a primary radical submodule if $prad(N) = N$.

Definition 2.18. [11] Let M be an R-module. A proper submodule N of M is primary compactly packed (PCP) if for each family $\{P_\alpha\}_{\alpha\in\triangle}$ of primary submodules of M with $N \subseteq \bigcup_{\alpha\in\triangle} P_\alpha$, $N \subseteq P_\beta$ for some $\beta \in \triangle$. A module M is called PCP if every proper submodule of M is PCP.

Theorem 2.19. *[11] Let M be an $R-$module. The following statements are equivalent:*

(i) M is PCP module.

(ii) For each proper submodule N of M there exists $a \in N$ such that $prad(N) = prad(Ra)$.

(iii) For each proper submodule N of M, if $\{N_\alpha\}_{\alpha\in\triangle}$ is a family of submodules of M and $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$, then $N \subseteq prad(N_\beta)$ for some $\beta \in \triangle$.

(iv) For each proper submodule N of M, if $\{N_\alpha\}_{\alpha\in\triangle}$ is a family of primary radical submodules of M and $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$, then $N \subseteq N_\beta$ for some $\beta \in \triangle$.

3 Primal Avoidance Theorem for Modules

The Primary Avoidance Theorem for modules in ref. [12] states as follows: Let M be an $R-$module, $L_1, L_2, \cdots, L_n$ a finite number of submodules of M and L a submodule of M such that $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$. Assume that at most two of the L_i's are not primary, and that $(L_j : M) \nsubseteq \sqrt{(L_k : M)}$ whenever $j \neq k$, Then $L \subseteq L_k$ for some $k \in \{1, 2, \cdots, n\}$. We consider a generalization of this theorem to the Primal Avoidance Theorem for modules.
Let $L, L_1, L_2, \ldots, L_n$ be a submodules of an $R-$module M. Following [14], The covering $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$ of L is called efficient if L is not contained in the union of any $n-1$ of the submodules L_i's. Analogously we shall say $L = L_1 \cup L_2 \cup \cdots \cup L_n$ is an efficient union if none of the L_i's may be excluded. Any cover or union consisting of submodules of M can be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms.
It is well-known that if L, L_1 and L_2 are submodules of an $R-$module M such that $L \subseteq L_1 \cup L_2$, then either $L \subseteq L_1$ or $L \subseteq L_2$. Consequently, a covering of a submodule by two submodules of a module is never efficient. Thus $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_m$ may be efficient cover only when $m > 2$ or $m = 1$.

Lemma 3.1. *[19] Let $L = L_1 \cup L_2 \cup \cdots \cup L_n$ be an efficient union of submodules of an R-module M where $n > 2$. Then $\bigcap_{j\neq k} L_j = \bigcap_{j=1}^{n} L_j$ for all $k \in \{1, 2, \ldots, n\}$.*

Corollary 3.2. *Let $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$ be an efficient cover of submodules of an R-module M where $n > 2$. Then $L \cap \bigcap_{j\neq k} L_j \subseteq L_k$ for all $k \in \{1, 2, \ldots, n\}$.*

Proof. Since $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$ is an efficient covering, $L = (L\cap L_1) \cup (L\cap L_2) \cup \cdots \cup (L\cap L_n)$ is an efficient union. Now apply the previous lemma to get, $L \cap \bigcap_{j\neq k} L_j = \bigcap_{j\neq k}(L \cap L_j) \subseteq (L \cap L_k) \subseteq L_k$. □

Proposition 3.3. *Let N and L be proper submodules of an $R-$module M, and I be an ideal of R. If $IL \subseteq N$ then either $L \subseteq N$ or $I \subseteq adj(N)$.*

Proof. Assume $L \nsubseteq N$, then there is $m \in L - N$. For each $a \in I$, $am \in IL \subseteq N$ while $m \notin N$, therefore $a \in adj(N)$. □

Lemma 3.4. *Let $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$ be an efficient covering of submodules of an $R-$module M where $n > 2$, then for all $j \in \{1, 2, \ldots, n\}$, $\bigcap_{i\neq j}(L_i : M) \subseteq adj(L_j)$.*

Proof. Let $j \in \{1, 2, \ldots, n\}$, put $I_j = \bigcap_{i \neq j}(L_i : M)$. Then $I_j = (\bigcap_{i \neq j} L_i : M)$, hence $I_j M \subseteq \bigcap_{i \neq j} L_i$, and in particular $I_j L \subseteq \bigcap_{i \neq j} L_i$. On other hand $I_j L \subseteq L$. Then $I_j L \subseteq L \cap (\bigcap_{i \neq j} L_i) \subseteq L_j$ by Corollary 3.2. Therefore either $L \subseteq L_j$ or $I_j \subseteq adj(L_j)$. But $L \nsubseteq L_j$, then we must have $I_j \subseteq adj(L_j)$. □

Theorem 3.5. *Let L be a submodule of an $R-$module M. If $L_1, L_2, \ldots, L_n$ are submodules of M such that $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$, and that $\bigcap_{i \neq j}(L_i : M) \nsubseteq adj(L_j)$ for all $j = 1, 2, \ldots, n$ except possibly for at most two of the j's, then $L \subseteq L_k$ for some $k \in \{1, 2, \ldots, n\}$.*

Proof. For the given covering $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$, let $L \subseteq L_{\alpha_1} \cup L_{\alpha_2} \cup \cdots \cup L_{\alpha_m}$ be its efficient reduction. Then $1 \leq m \leq n$ and $m \neq 2$. If $m > 2$, then there exists at least one L_{α_j} satisfying $\bigcap_{i \neq j}(L_{\alpha_i} : M) \nsubseteq adj(L_{\alpha_j})$. This is impossible in view of Lemma 3.4. Hence $m = 1$ and $L \subseteq L_{\alpha_1} = L_k$ for some $k \in \{1, 2, \ldots, n\}$. □

If A and B are ideals of a ring R, and P is a prime ideal of R such that $A \cap B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. This statement explains the following remark.

Remark 3.6. If L_j is a primal submodule (so $adj(L_j)$ is a prime ideal), then the following two conditions are equivalent:

(i) $\bigcap_{i \neq j}(L_i : M) \nsubseteq adj(L_j)$.

(ii) $(L_i : M) \nsubseteq adj(L_j)$ whenever $i \neq j$.

The following Theorem, which follows immediately from Theorem 3.5 and Remark 3.6, is a generalization of the Primary Avoidance Theorem for modules.

Theorem 3.7. *(Primal Avoidance Theorem for modules)*
Let L be a submodule of an $R-$module M and $L_1, L_2, \ldots, L_n$ are submodules of M such that $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$. Assume that at most two of the L_i's are not Primal, and $(L_i : M) \nsubseteq adj(L_j)$ whenever $i \neq j$, then $L \subseteq L_k$ for some $k \in \{1, 2, \ldots, n\}$.

Definition 3.8. Let M be an $R-$module. A proper submodule N of M is called primal weakly compactly packed ($P_L WCP$) if for each finite set $\{L_1, L_2, \ldots, L_n\}$ of primal submodules of M with $N \subseteq \bigcup_{i=1}^{n} L_i$, $N \subseteq L_j$ for some $j \in \{1, 2, \ldots, n\}$. A module M is called $P_L WCP$ if every proper submodule is $P_L WCP$.

The following theorem follows immediately from the Primal Avoidance Theorem for modules.

Theorem 3.9. *Let M be an $R-$module and assume that for any primal submodules L and K of M we have:*

$$L \nsubseteq K \text{ and } K \nsubseteq L \text{ implies } (L : M) \nsubseteq adj(K),$$

then M is $P_L WCP$.

4 Primal Compactly Packed Modules

In this section we introduce the concept of primal compactly packed modules and study various properties of primal compactly packed modules.

Definition 4.1. Let N be a proper submodule of an R-module M.

(i) N is said to be primal compactly packed ($P_L CP$) if for each family $\{P_\alpha\}_{\alpha \in \triangle}$ of primal submodules of M with $N \subseteq \bigcup_{\alpha \in \triangle} P_\alpha, N \subseteq P_\beta$ for some $\beta \in \triangle$.

(ii) N is said to be primal finitely compactly packed ($P_L FCP$) if for each family $\{P_\alpha\}_{\alpha \in \triangle}$ of primal submodules of M with $N \subseteq \bigcup_{\alpha \in \triangle} P_\alpha$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\triangle$ such that $N \subseteq \bigcup_{i=1}^{n} P_{\alpha_i}$.

(iii) A module M is called $P_L CP$ (resp. $P_L FCP$) if every proper submodule of M is $P_L CP$(resp. $P_L FCP$).

Evidently, a module M is P_LCP if and only if M is P_LFCP and P_LWCP. The next example shows that a P_LWCP module need not be P_LFCP.

Example 4.2. Let R be the set of all sequences (a_n) of elements of $\mathbb{Z}_2$, such that for some n_0, depending on the sequence, $a_n = a_{n_o}$ for all $n \geq n_0$. If we define operations on R by $(a_n) + (b_n) = (a_n + b_n)$ and $(a_n)(b_n) = (a_n b_n)$, then R is a Boolean ring. For each $k > 0$, let $P_k = \{(a_n) \in R \mid a_k = 0\}$, and let $P_0 = \{(a_n) \in R \mid$ for some n_0, depending on the sequence, $a_n = 0$ for all $n \geq n_0\}$. It is easily checked that P_k's $(k \geq 0)$ are prime ideals of R. By Corollary 2.15, An ideal of R is primal if and only if it is prime. By the Prime Avoidance Theorem for rings, R is P_LWCP. Since $P_0 \subseteq P_1 \cup P_2 \cup \ldots$, and $P_0 \nsubseteq P_k$ for all $k > 0$, then R is not P_LCP. Finally, R is not P_LFCP, for if R were P_LFCP, then R is P_LCP(as R is P_LWCP) which is a contradiction.

Since every primary submodule is primal, then we have the following proposition:

Proposition 4.3. *Every P_LFCP (resp. P_LCP) module is PFCP (resp. PCP).*

primal, uniformly primal, irreducible, completely irreducible submodules meets in the following theorem:

Theorem 4.4. *Let N be a submodule of an $R-$module M. The following statements are equivalent:*

(i) N is P_LCP.

(ii) If $\{N_\alpha\}_{\alpha\in\triangle}$ is a family of irreducible submodules of M and $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$, then $N \subseteq N_\beta$ for some $\beta \in \triangle$.

(iii) If $\{N_\alpha\}_{\alpha\in\triangle}$ is a family of uniformly primal submodules of M and $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$, then $N \subseteq N_\beta$ for some $\beta \in \triangle$.

(iv) If $\{N_\alpha\}_{\alpha\in\triangle}$ is a family of completely irreducible submodules of M and $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$, then $N \subseteq N_\beta$ for some $\beta \in \triangle$.

(v) N is cyclic.

(vi) If $\{N_\alpha\}_{\alpha\in\triangle}$ is a family of submodules of M and $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$, then $N \subseteq N_\beta$ for some $\beta \in \triangle$.

Proof. Since every irreducible submodule is primal, and every completely irreducible submodule is irreducible, then we have $(1 \to 2 \to 4)$.
Since every uniformly primal submodule is primal, and every completely irreducible submodule is uniformly primal, then we have $(1 \to 3 \to 4)$.
$(4 \to 5)$ It is clear that $Ra \subseteq N$ for each $a \in N$. Suppose that $N \nsubseteq Ra$ for each $a \in N$. By [10], every proper submodule of M is the intersection of all completely irreducible submodules containing it. Hence for each $a \in N$ there exists a completely irreducible submodule P_a for which $Ra \subseteq P_a$ and $N \nsubseteq P_a$. However, $N = \bigcup_{a\in N} Ra \subseteq \bigcup_{a\in N} P_a$, which contradicts (4).
$(5 \to 6)$ Let $\{N_\alpha\}_{\alpha\in\triangle}$ be a family of submodules of M such that $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$ by (5) there exists $a \in N$ such that $N = Ra$, then $a \in \bigcup_{\alpha\in\triangle} N_\alpha$ and hence $a \in N_\beta$ for some $\beta \in \triangle$, so that $Ra \subseteq N_\beta$ and hence $N \subseteq N_\beta$.
$(6 \to 1)$ is clear. □

Corollary 4.5. *Let M be an $R-$module. Then M is P_LCP if and only if every proper submodule of M is cyclic.*

Theorem 4.6. *If M is a P_LCP module which has at least one maximal submodule, then M satisfies the (ACC) on submodules.*

Proof. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of submodules of M. If $N_k = M$ for some k, then the result follows immediately, so assume that none of N_k's is M, and let $L = \bigcup_{i=1}^{\infty} N_i$. We claim that L is a proper submodule of M. Assume on contrary that $L = M$ and let H be a maximal submodule of M, then $H \subseteq \bigcup_{i=1}^{\infty} N_i$. Since M is P_LCP, by Theorem 4.4, there exist a positive integer m such that $H \subseteq N_m$. Therefore $H = N_m$, hence N_m is maximal, and $N_i = N_m$ for all $i \geq m$. Therefor $N_m = \bigcup_{i=1}^{\infty} N_i = M$ which is impossible, thus L is a proper submodule of M. Since M is P_LCP, $L \subseteq N_j$ for some j and hence $N_i \subseteq N_j$ for all i, thus $N_i = N_j$ for all $i \geq j$. Therefore the (ACC) is satisfied for submodules. □

Theorem 4.7. *If M is a P_LFCP which has at least one maximal submodule, then M satisfies the (ACC) on primal submodules.*

Proof. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of primal submodules of M, and $L = \bigcup_{i=1}^{\infty} N_i$. We claim that L is a proper submodule of M. Assume on contrary that $L = M$ and let H be a maximal submodule of M, then $H \subseteq \bigcup_{i=1}^{\infty} N_i$. Since M is P_LFCP, there exist $n_1, n_2, \ldots, n_k$ such that $H \subseteq \bigcup_{j=1}^{k} N_{n_j} = N_m$ where $m = max\{n_1, n_2, \ldots, n_k\}$. Therefore $H = N_m$, hence N_m is maximal, and $N_i = N_m$ for all $i \geq m$. Therefor $N_m = \bigcup_{i=1}^{\infty} N_i = M$ which contradicts N_m is primal, thus L is a proper submodule of M. Then, since M is P_LFCP, there exist $m_1, m_2, \ldots, m_s$ such that $L \subseteq \bigcup_{j=1}^{s} N_{m_j} = N_t$ where $t = max\{m_1, m_2, \ldots, m_s\}$. hence $N_i \subseteq N_t$ for all i, thus $N_i = N_t$ for all $i \geq t$. Therefore the (ACC) is satisfied for primal submodules. □

Now we recall the following definition:

Definition 4.8. [23] A module M is called a Bezout module if every finitely generated submodule of M is cyclic.

Proposition 4.9. *Let M be a Bezout $R-$module. If M satisfies the (ACC) on submodules, then M is P_LCP.*

Proof. Let N be a prober submodule of M. By [15], p. 375, Theorem 1.9, N is finitely generated submodule and hence it is cyclic, since M is Bezout module. By Corollary 4.5, M is P_LCP. □

Proposition 4.3, says every P_LCP module is PCP. We investigate when the converse is true.

Definition 4.10. [2] An $R-$module M is said to be with primary decomposition (resp. Laskerian) if each of its proper submodules is an intersection, possibly infinite, (resp. a finite intersection) of primary submodules of M.

By [17], p. 40, Theorem 2.7, every module satisfying the (ACC) on submodules is Laskerian. And every Laskerian module is a module with primary decomposition. The following result is trivial, and follows immediately from the definitions.

Proposition 4.11. *An $R-$module M is a module with primary decomposition if and only if $prad(N) = N$ for all submodules N of M.*

If we combine Theorem 2.19 and Proposition 4.11, we obtain the following result:

Theorem 4.12. *Let M be an R-module with primary decomposition. The following statements are equivalent:*

(i) M is PCP.

(ii) Every proper submodule of M is cyclic.

(iii) M is P_LCP.

Corollary 4.13. *Let M be an $R-$module satisfying the (ACC) on submodules. The following statements are equivalent:*

(i) M is PCP.

(ii) Every proper submodule of M is cyclic.

(iii) M is P_LCP.

In order to create another module which is PCP if and only if it is P_LCP, we need the following lemma:

Lemma 4.14. *[2] Let R be a Noetherian ring and M an $R-$module. Then every irreducible submodule of M is primary.*

Theorem 4.15. *Let R be a Noetherian ring and M an $R-$module. If M is PCP, then M is P_LCP.*

Proof. By Lemma 4.14, every irreducible submodule of M is primary. Hence if M is PCP, then M satisfies the statement 2 of Theorem 4.4 for all submodules N of M, which is equivalent to M is P_LCP. □

Corollary 4.16. *Let R be a Noetherian ring and M an $R-$module. Then M is PCP if and only if every proper submodule of M is cyclic.*

Theorem 4.17. *Let M be a PCP module which has at least one maximal submodule. Then M is P_LCP if and only if M satisfies the (ACC) on submodules.*

Proof. See Theorem 4.6 and Corollary 4.13. □

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Received: November 23, 2013.

Accepted: April 10, 2014.