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On Generalized Ricci-Recurrent (ϵ, δ)-trans-Sasakian manifolds

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Abstract A study of generalized Ricci-recurrent (ϵ, δ)-trans-Sasakian manifold has been made and it is shown that a generalized Ricci-recurrent cosymplectic manifold is always recurrent. Also, generalized Ricci-recurrent (ϵ, δ)-trans-Sasakian of dimension greater or equal to 5 are locally classified . Moreover, it is shown that if M is one of (ϵ)-Sasakian and (δ)-Kenmotsu manifold, then M is an Einstien manifold under certain conditions.

1 Introduction

In 1993, Bejancu and Duggal (cf.[2]) introduced the concept of (ϵ)-Sasakian manifolds which later on showed by Xufeng and Xiaolic [11] that these manifolds are real hypersurfaces of indefinite Kahlerian manifolds. (ϵ)-almost para contact manifolds were introduced by Tripathi et al [10]; while the concept of (ϵ)-Kenmotsu manifolds was introduced by De and A. Sarkar [3] who showed that the existence of new structure on indefinite metrics influences the curvarure. The curvature properties of (ϵ)-Sasakian manifolds were studied by Kumar et al [6]. In 2012, Nagaraja et al [8] generalized the notion of (ϵ)-Sasakian and (δ)-Kenmotsu manifolds by introducing the concept of (ϵ, δ)-trans-Sasakian manifolds. On the other hand Jeong-Sik-Kim et al [9] have studied the generalized Ricci-recurrent trans-Sasakian manifolds. Motivated by the above discussions, we plan to study generalized Ricci-recurrent (ϵ, δ)-trans-Sasakian manifolds. The paper is organised as follows. Section 2, contains necessary details about (ϵ, δ)-trans-Sasakian manifolds. Section 3, is concerned with the study of generalized Ricci-recurrent (ϵ, δ)-trans-Sasakian manifolds, where a relation among others, between the 1-forms A and B is establisshed. It is proved that a generalized Ricci-recurrent cosymplectic manifold is always Ricci-recurrent. Generalized Ricci-recurrent (ϵ, δ)-trans-Sasakian manifolds of dimension ≥ 5 are also classified. An expression for Ricci-recurrent of a generalized (ϵ, δ)-trans-Sasakian manifold with cyclic Ricci tensor is obtained in the final section. Here it is proved that if M is one of α-Sasakian or (ϵ)-Sasakian, β-Kenmotsu or (δ)-Kenmotsu manifolds which is generalized Ricci tensor and non zero $A(\xi)$ everywhere, then M is a an Einstein manifold. An example of (ϵ, δ)-trans-Sasakian manifolds has also been discussed in detail.

A non-flat Riemannian manifold M is called a generalized Ricci-recurrent manifold ([4]) if its Ricci tensor S satisfies the following condition.

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z), \tag{1.1}$$

where ∇ is Levi-Civita connection of the Riemannian metric g and A , B are 1-forms on M. In particular if 1-form B vanishes identically, then manifold M reduces to the Ricci-recurrent manifold.

Let $\bar{M}$ be an almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a compatiable Riemannian metric such that

$$\phi^2 = -I + \eta\otimes\xi, \;\; \phi\xi = 0, \;\; \eta o\phi = 0, \;\; \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.2}$$

$$g(\xi, \xi) = \epsilon \tag{2.3}$$

$$g(X, \phi Y) = -g(\phi X, Y), \;\; \epsilon g(X, \xi) = \eta(X) \tag{2.4}$$

for all vector fields X, Y on $T\bar{M}$, where $\epsilon = g(\xi, \xi) = \pm 1$. An (ϵ)- almost contact metric manifold is called an (ϵ, δ)-trans Sasakian manifold if

$$(2.5) \quad (\bar{\nabla}_X \phi)(Y) = \alpha(g(X, Y)\xi - \epsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X)$$

for some smooth functions α and β on $\bar{M}$ and $\epsilon = \pm 1$, $\delta = \pm 1$. For $\beta = 0$, $\alpha = 1$, an (ϵ, δ)-trans-Sasakian manifolds reduces to (ϵ)-Sasakian and for $\alpha = 0$, $\beta = 1$ it reduces to a (δ)-Kenmotsu manifolds.

2 (ϵ, δ)-trans Sasakian manifolds

Let $\bar{M}$ be an almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a compatiable Riemannian metric such that

$$(2.1) \qquad \phi^2 = -I + \eta \otimes \xi, \;\; \phi\xi = 0, \;\; \eta o \phi = 0, \;\; \eta(\xi) = 1,$$

$$(2.2) \qquad g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y),$$

$$(2.3) \qquad g(\xi, \xi) = \epsilon$$

$$(2.4) \qquad g(X, \phi Y) = -g(\phi X, Y), \;\; \epsilon g(X, \xi) = \eta(X)$$

for all vector fields X, Y on $T\bar{M}$, where $\epsilon = g(\xi, \xi) = \pm 1$. An (ϵ)- almost contact metric manifold is called an (ϵ, δ)-trans Sasakian manifold if

$$(2.5) \quad (\bar{\nabla}_X \phi)(Y) = \alpha(g(X, Y)\xi - \epsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X)$$

for some smooth functions α and β on $\bar{M}$ and $\epsilon = \pm 1$, $\delta = \pm 1$. For $\beta = 0$, $\alpha = 1$, an (ϵ, δ)-trans-Sasakian manifolds reduces to (ϵ)-Sasakian and for $\alpha = 0$, $\beta = 1$ it reduces to a (δ)-Kenmotsu manifolds.

3 3 Generalized Ricci-recurrent (ϵ, δ)-trans-Sasakian manifolds

Let M be a n-dimensional (ϵ, δ)-trans Sasakian manifold. From (2.5) it is easy to follows that

$$(3.1) \qquad \bar{\nabla}_X \xi = -\epsilon\alpha\phi X - \beta\delta\phi^2 X.$$

From (2.6), it follows that

$$(3.2) \qquad (\bar{\nabla}_X \eta)Y = \delta\beta(\epsilon g(X, Y) - \eta(X)\eta(Y)) - \alpha g(X, Y).$$

In view of (2.5), (3.1) and (3.2) we are able to state the following lemma.

Lemma 3.1. *([8]) In a n dimensional (ϵ, δ)-trans Sasakian manifold we have*

$$(3.3) \qquad R(X, Y)\xi = \epsilon((Y\alpha)\phi X - (X\alpha)\phi Y) + (\beta^2 - \alpha^2)(\eta(X)Y - \eta(Y)X)$$

$$- \delta((X\beta\phi^2 Y - (Y\beta\phi^2 X) + 2\delta\epsilon\alpha\beta(\eta(Y))(\phi X) - \eta(X)\phi Y)$$
$$+2\alpha\beta(\delta - \epsilon)g(\phi X, Y)\xi,$$

$$(3.4) \qquad S(X, \xi) = (n-1)\left\{(\epsilon\alpha^2 - \beta^2\delta - \xi\beta)\right\}\eta(X) - (n-2)(X\beta) - (\phi X)\alpha,$$

$$(3.5) \qquad Q\xi = (n-1)\left\{(\epsilon\alpha^2 - \beta^2\delta - \xi\beta)\right\}\xi - (n-2)grad\beta + \phi(grad\alpha),$$

where R and S are curvature and Ricci curvature, while Q is Ricci operator given by $S(X, Y) = g(QX, Y)$. In particular we have

$$(3.6) \qquad S(\xi, \xi) = (n-1)\left\{(\epsilon\alpha^2 - \beta^2\delta\right\} - (2n-3)(\xi\beta).$$

Now, we prove the following

Theorem 3.2. *(a) Let S be a semigroup, and let $x \in S$ be a nongenerator of Let M be a n dimensional generalized Ricci-recurrent (ϵ, δ)-trans Sasakian manifold. Then, the 1-froms A and B are related by*

$$B(X) = \frac{(n-1)}{\epsilon}\left\{X(\epsilon\alpha^2 - \beta^2\delta) - (\epsilon\alpha^2 - \beta^2\delta)A(X)\right\} \tag{3.7}$$

$$+ \frac{(2n-3)}{\epsilon}\left\{(\xi\beta)A(X) - X(\xi\beta)\right\}$$

$$+ \frac{2(n-2)}{\epsilon}\left\{(\alpha\phi X + \beta\phi^2 X)\beta\delta\right\} + \frac{2}{\epsilon}\ \beta\phi X - \alpha\phi^2 X)\epsilon\alpha\right\}.$$

In particular, we get

$$B(\xi) = \frac{(n-1)}{\epsilon}\left\{\xi(\epsilon\alpha^2 - \beta^2\delta) - (\epsilon\alpha^2 - \beta^2\delta)A(\xi)\right\} \tag{3.8}$$

$$+ \frac{(2n-3)}{\epsilon}\left\{(\xi\beta)A(\xi) - \xi(\xi\beta)\right\}.$$

Proof. Using equation(1.1) in the following equation

$$(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z) \tag{3.9}$$

we get

$$A(X)S(Y,Z) + B(X)g(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).$$

Putting $Y = Z = \xi$, in the above eqaution, we obtain

$$A(X)S(\xi,\xi) + B(X)\epsilon = XS(\xi,\xi) - 2S(\nabla_X \xi, \xi),$$

in view of (3.6), (2.4) and (3.1), we get (3.7). Then equation (3.8) is obvious from (3.7).
Let A^* and B^* be the associated vector fileds of A and B, that is $\epsilon g(X, A^*) = A(X)$ and $\epsilon g(X, B^*) = B(X)$, where $\epsilon = \pm 1$.

Corollary 3.3. *In a n-dimensional generalized Ricci-recurent α-Sasakian (resp. (ϵ)-Sasakian) manifold, we have*

$$B = -\frac{(n-1)}{\epsilon}\alpha^2\epsilon A \qquad (\text{resp. } \mathrm{B} = -(\mathrm{n}-1)\epsilon\mathrm{A}) \tag{3.8}$$

Thus, the associated vector fields A^* and B^* are in opposite direction.

Proof. An (ϵ, δ)-trans Sasakian manifold of type $(\alpha, 0)$ is (α)-Sasakian ([5]) (resp. (ϵ)-Sasakian ([8, 11])). In case $\alpha = 1$, then from the eqaution (3.7), the proof follows immediately.

Corollary 3.4. *In a n-dimensional generalized Ricci-recurrent normal indifnite almost cosymplectic f-structure (or f-Kenmotsu) manifold, we have*

$$B(X) = \frac{(n-1)}{\epsilon}\left\{(f^2\delta)A(X) - X(f^2\delta)\right\} \tag{3.9}$$

$$+ \frac{(2n-3)}{\epsilon}\left\{(\xi f)A(X) - X(\xi f)\right\} + \frac{2(n-2)}{\epsilon}(\phi^2 X)f\delta$$

Proof. An (ϵ, δ)-trans Sasakian structure with $\alpha = 0$ and $\beta = f$, is normal almost cosymplectic f-structure([7]) (or f-Kenmotsu structure). Thus putting $\alpha = 0$ and $\beta = f$ in (3.7), we get (3.9).

Corollary 3.5. *For a n-dimensional generalized Ricci-recurrent (β)-Kenmotsu (resp. (δ)-Kenmotsu) manifold, we have*

$$B = \frac{(n-1)}{\epsilon}\beta^2\delta A \qquad \left(resp.\ B = \frac{(n-1)}{\epsilon}\delta A\right) \tag{3.10}$$

Thus, the associated vector fields A^* and B^* are in same direction.

Proof. (ϵ, δ)-trans Sasakian structure reduces to (β)-Kenmotsu ([5]) (resp. (δ)-Kenmotsu ([8, 3])) if $\alpha = 0$ and $\beta = constant$. In particular 1-Kenmotsu structure is a Kenmotsu for $f = \beta = constant$ (resp. $f = 1$) from (3.9), we obtain (3.10).

An (ϵ, δ)-trans-Sasakian structures of type $(0, 0)$ is cosymplectic ([1]). Thus, putting $\alpha = 0 = \beta$ in (3.7), we get $B = 0$. Hence, we have the following

Theorem 3.6. *(a) A generalized Ricci-recurrent indefinite cosymplectic manifold M is always Ricci-recurrent.*

Now, we give the following classification for generalized Ricci-recurrent (ϵ, δ)- trans-Sasakian manifold of dimension≥ 5 locally.

Theorem 3.7. *Let M be generalized Ricci-recurrent (ϵ, δ)-trans-Sasakian manifold of dimemsion $n \geq 5$. Then*

(1) either $M\ is\ Ricci - recurrent,$

(2) $B + (n-1)\alpha^2 A = 0,$

(3) $B - \frac{(n-1)}{\epsilon}\beta^2 A = 0,$

where α and β are non-zero constants.

Proof. We know that locally an (ϵ, δ)- trans-Sasakian manifold of dimension≥ 5 is either cosymplectic or (α)-Sasakian (resp. (ϵ)-Sasakian) or β-Kenmotsu (resp. (δ)-Kenmotsu) manifolds. Hence in view of corollaries (3.3), (3.5) and theorem (3.6) the proof of the theorem completes.

4 4. Generalized Ricci-recurrent (ϵ, δ)- trans-Sasakian manifold with Cyclic Ricci tensor A Riemannian manifold is said to admit cyclic Ricci tensor if

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

(4.1)

$$Now, we prove the following.$$

Theorem 4.1. *In a n-dimensional generalized Ricci-recurrent (ϵ, δ)- trans-Sasakian manifold with Cyclic Ricci tensor, the Ricci tensor satisfies*

$$A(\xi)S(X,Y)(4.2)$$

$$\begin{aligned}
&= \frac{(n-1)}{\epsilon}\left\{(\epsilon\alpha^2 - \beta^2\delta)A(\xi) - \xi(\{(\epsilon\alpha^2 - \beta^2\delta)\})g(X,Y)\right. \\
&\quad + \frac{(2n-3)}{\epsilon}\{(\xi\beta)A(\xi) - \xi(\xi\beta)\}\, g(X,Y) \\
&\quad + (n-2)\{A(X)(Y\beta) + A(Y)(X\beta)\} \\
&\quad + (n-1)(\xi\beta)\{\eta(Y)A(X) + \eta(X)A(Y)\} \\
&\quad + \frac{(2n-3)}{\epsilon}\{(\xi\beta)\eta(Y)A(X) - X(\xi\beta)\eta(Y)\} \\
&\quad - \frac{(n-1)}{\epsilon}\left\{\eta(X)Y(\epsilon\alpha^2 - \beta^2\delta) + \eta(Y)X(\epsilon\alpha^2 - \beta^2\delta)\right\} \\
&\quad + \frac{2(n-2)}{\epsilon}\left\{\eta(X)(\alpha\phi Y + \beta\phi^2)\delta + \eta(Y)(\alpha\phi X + \beta\phi^2 X)\beta\delta\right\} \\
&\quad + \frac{2(n-2)}{\epsilon}\left\{\eta(X)(\phi^2 Y - \beta\phi Y)\epsilon\alpha + \eta(Y)(\alpha\phi^2 X - \phi X)\epsilon\alpha\right\}.
\end{aligned}$$

Proof. (Supppse that M is a generalized Ricci symmetric manifold admitting cyclic Ricci tensor. Then in view of (1.1) and (4.1), we get

$$\begin{aligned} 0 &= A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y) \\ &+ B(X)g(Y,Z) + B(Y)g(X,Z) + B(Z)g(X,Y). \end{aligned}$$

Moreover, if M is (ϵ, δ)- trans-Sasakian manifold, then by putting $Z = \xi$, in the above equation, we get

$$\begin{aligned} A(\xi)S(X,Y) &= -B(\xi)g(X,Y) - A(X)S(Y,\xi) \\ &- A(Y)S(\xi,X) - B(X)\eta(Y) - B(Y)\eta(X), \end{aligned}$$

which in view of (3.8) and (3.4) gives (4.2). □

Corollary 4.2. *For a n-dimensional generalized Ricci-recurrent manifold M with cyclic Ricci tensor, we have the following statements*
1. *If M is an (α)-Sasakian manifold, then*

$$A(\xi)S(X,Y) = \frac{(n-1)}{\epsilon}\epsilon\alpha^2 A(\xi)g(X,Y).$$

2. *If M is (ϵ)-Sasakian manifold, then*

$$A(\xi)S(X,Y) = \frac{(n-1)}{\epsilon}\epsilon A(\xi)g(X,Y).$$

3. *If M is a f-Kenmotsu manifold, then*

$$\begin{aligned} A(\xi)S(X,Y) &= \frac{(n-1)}{\epsilon}\left\{\xi(f^2\delta - (f^2\delta)A(\xi)\right\}g(X,Y) \\ &+ \frac{(2n-3)}{\epsilon}\left\{(\xi f)A(\xi) - \xi(\xi f)\right\}g(X,Y) \\ &+ (n+2)\left\{A(X)(Yf) + A(Y)(Xf)\right\} \\ &+ \frac{(2n-3)}{\epsilon}\left\{(\xi f)\eta(Y)A(X) - X(\xi f)\eta(Y)\right\} \\ &- \frac{(n-1)}{\epsilon}\left\{\eta(X)Y(f^2\delta) + \eta(Y)X(f^2\delta)\right\} \\ &- \frac{(2n-3)}{\epsilon}\left\{\eta(X)(f\phi^2 Y)f\delta + \eta(Y)(f\phi^2 X)f\delta)\right\}. \end{aligned}$$

4. *If M is β-Kenmotsu manifold, then*

$$A(\xi)S(X,Y) = -\frac{(n-1)}{\epsilon}\beta^2 A(\xi)g(X,Y).$$

5. *If M is an (δ)-Kenmotsu manifold, then*

$$A(\xi)S(X,Y) = -\frac{(n-1)}{\epsilon}\delta A(\xi)g(X,Y).$$

6. *If M is Cosymplectic manifolds, then*

$$A(\xi)S(X,Y) = 0.$$

A Riemannian manifold is an Einstein manifold if

$$S(X,Y) = \rho\epsilon g(X,Y).$$

Therefore, in view of corollary (4.2), we are able to state the following.

Theorem 4.3. *Let M be generalized Ricci-recurrent manifold with cyclic Ricci tensor and M is one of (ϵ)-Sasakian, α-Sasakian, (δ)-Kenmotsu and β-Kenmotsu manifolds with non-zero $A(\xi)$ everywhere, then M is Einstein manifold.*

Example: Consider the three dimensional manifold $M = \left\{(x,y,z) \in \boldsymbol{R}^3 \,|z \neq 0\right\}$, where (x,y,z) are the cartesian coordinates in $\boldsymbol{R}^3$ and let the vector fields are

$$e_1 = \frac{e^x}{z^2}\frac{\partial}{\partial x}, \quad e_2 = \frac{e^y}{z^2}\frac{\partial}{\partial y}, \quad e_3 = \frac{-(\epsilon+\delta)}{2}\frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent at each point of M. Let g be the Riemannain metric defined by
$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = \epsilon$, $g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0$,
where $\epsilon = \pm 1$.
Let η be the 1-form defined by $\eta(X) = \epsilon g(X,\xi)$ for any vector field X on M, let ϕ be the (1,1) tensor field defined by $\phi(e_1) = e_2,\ \phi(e_2) = -e_1,\quad \phi(e_3) = 0.$
Then by using the linearty of ϕ and g, we have $\phi^2 X = -X + \eta(X)\xi$, with $\xi = e_3$.

Further $g(\phi X, \phi Y) = g(X,Y) - \epsilon\eta(X)\eta(Y)$ for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an (ϵ)-almost contact structure in R^3. Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) \quad (4.3)$$
$$- g(Y,[X,Z]) + g(Z,[X,Y]),$$

which is know as Koszul's formula.
We, also have
$\nabla_{e1} e_3 = -\frac{(\epsilon+\delta)}{z} e_1, \quad \nabla_{e2} e_3 = -\frac{(\epsilon+\delta)}{z} e_2, \quad \nabla_{e1} e_2 = 0,$

using the above relation, for any vector X on M, we have
$\nabla_X \xi = -\epsilon\alpha\phi X - \beta\delta\phi^2 X$, where $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence (ϕ, ξ, η, g) structure defines the (ϵ, δ)-tran-Sasakian structure in R^3.

Here ∇ be the Levi-Civita connection with respect to the metric g , then we have
(4.4) $\quad [e_1,e_2] = 0,\ [e_1,e_3] = -\frac{(\epsilon+\delta)}{z} e_1,\ [e_2,e_3] = -\frac{(\epsilon+\delta)}{z} e_2.$
Using (4.3) and (4.4), we have

$$2g(\nabla_{e1} e_3, e_1) = 2g\left(-\frac{(\epsilon+\delta)}{z} e_1, e_1\right) + 2g(e_2, e_1)$$
$$= 2g\left(-\frac{(\epsilon+\delta)}{z} e_1 + e_2, e_1\right),$$

since $g(e_1,e_2) = 0$. Thus we have
(4.5) $\quad \nabla_{e1} e_3 = -\frac{(\epsilon+\delta)}{z} e_1 + e_2.$
Again using (4.3), we get

$$2g(\nabla_{e2} e_3, e_2) = 2g\left(-\frac{(\epsilon+\delta)}{z} e_2, e_2\right) - 2g(e_1, e_2)$$
$$= 2g\left(-\frac{(\epsilon+\delta)}{z} e_2 - e_1, e_2\right),$$

since $(e_1,e_2) = 0$. Therefore we have
(4.6) $\quad \nabla_{e2} e_3 = -\frac{(\epsilon+\delta)}{z} e_2 - e_1.$
Again from (4.3), we have
(4.7) $\quad \nabla_{e3} e_3 = 0, \quad \nabla_{e1} e_1 = -\frac{(\epsilon+\delta)}{z} e_1, \quad \nabla_{e1} e_2 = 0$

$\nabla_{e2} e_1 = 0, \quad \nabla_{e2} e_2 = -\frac{(\epsilon+\delta)}{z} e_2, \quad \nabla_{e2} e_3 = -\frac{(\epsilon+\delta)}{z} e_2 - e_1$

$\nabla_{e3} e_1 = 0, \quad \nabla_{e3} e_2 = 0, \quad \nabla_{e1} e_3 = -\frac{(\epsilon+\delta)}{z} e_1 + e_2.$

The manifold M satisfies (2.5) with $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence M is an (ϵ, δ)-trans-Sasakian manifolds. Using (4.5), (4.6) and (4.7) the non-vanishing components of the curvature tensor are computed as follows

(4.8) $\quad R(e_1,e_3)e_3 = \frac{(\epsilon+\delta)}{z^2} e_1, \quad R(e_3,e_1)e_3 = -\frac{(\epsilon+\delta)}{z^2} e_1,$

$R(e_2,e_3)e_3 = \frac{(\epsilon+\delta)}{z^2}e_1, \quad R(e_3,e_2)e_3 = -\frac{(\epsilon+\delta)}{z^2}e_1,$
The vectors e_1, e_2, e_3 form a basis of M and therefore any vector X can be written as $X = b_1e_1 + b_2e_2 + b_3e_3$, where $b_i \in R^3$, $i = 1,2,3$. Now, from (4.8), we have

$$(\nabla_X R)(e_1,e_3)e_3 = -2\frac{(\epsilon^2+\delta\epsilon)}{z^3}b_3e_1,$$

$$(\nabla_X R)(e_2,e_3)e_3 = -2\frac{(\epsilon^2+\delta\epsilon)}{z^3}b_3e_2,$$

where $A(X) = -2\frac{(\epsilon^2+\delta\epsilon)}{z^3}b_3$, is a non-vanishing 1-form.

From the above expresion of the curvature tensor we can also obtain

$$S(e_1,e_1) = g(R(e_1,e_3)e_3,e_1) = g\left(\frac{(\epsilon+\delta)}{z^2}e_1,e_1\right)$$

$$= g\left(\frac{(\epsilon+\delta)}{z^2}e_1,e_1\right)$$

$$S(e_1,e_1) = \frac{(\epsilon+\delta)}{z^2}\epsilon$$

or

$$S(e_1,e_1) = \frac{(\epsilon^2+\delta\epsilon)}{z^2}.$$

Similarly, we get

$$S(e_2,e_2) = S(e_3,e_3) = \frac{(\epsilon^2+\delta\epsilon)}{z^2}$$

also

$$(\nabla_X S)(e_1,e_3)e_3 = (\nabla_X S)(e_1,e_2)e_2 = 0,$$

since $g(e_1,e_3) = g(e_1,e_2) = 0$.
This implies that there exist a Ricci-recurrent (ϵ,δ)-trans-Sasakian manifold of dimension 3.
Since

$$S(e_i,e_i) = \frac{(\epsilon+\delta)}{z^2}\epsilon,$$

or

$$S(e_1,e_1) + S(e_2,e_3) + S(e_3,e_3) = -3\frac{(\epsilon^2+\delta\epsilon)}{z^2}.$$

Therefore, we have

$$S(e_i,e_i) = -\frac{(\epsilon+\delta)}{z^2}g(e_i,e_i),$$

for $i = 1,2,3$, and $\alpha = \frac{1}{z}$, $\beta = -\frac{1}{z}$. Hence from theorem (4.3), M is an $Einstein$ manifold.

References

[1] Blair, D. E., Contact manifolds in Riemannian geometry, *Lecture notes in Math ematics* **509**, Springer Verlag, Berlin-NewYork, 1976.

[2] Bejancu, A. Duggal, K. L., Real Hypersurfaces on indefinite Kahler manifolds. *,Int. Math. Math. Sci*,**16(3)**,(1993), 545-556.

[3] De, U. C., Sarkar, A., On $(\epsilon)-$Kenmotsu manifolds, *Hadronic journal*, **32** (2009), 231-242.

[4] De, U. C., Guha, N. and Kamilya, D., On generalized Ricci-recurrent manifolds, *Tensor (N.S)*, **56** (1995), no. 3, 312-317.

[5] Janssens, D. and Vanhecke, L., Almost contact structures and curvature tensors, *Kodai. Math. J,* **4** (1981), no. 1, 1-27.

[6] Kumar, R., Rani, R. and Nagaich, R. K., On sectional curvarture of (ϵ)-Sasakian manifolds, *Int. J. Math. Math. Sci.* (2007), ID. 93562.

[7] Matsomoto, K., Mihai, I. and Rosca, R., A certain locally conformal almost cosymplectic manifold and its application, *Tensor (N.S.)*, **51** (1992), no. 1, 91-102.

[8] Nagaraja, H. G., Premalatha, C. R. and Somashekara, G., On an (ϵ, δ)-trans-Sasakian manifolds, *Proced. Estonian Acced. Sci.* **1** (61), (2012), 20-28.

[9] Sik-Kim, J., Parsad, R. and Tripathi, M. M., On generalized Ricci-recurrent trans-Sasakian manifold, *J. Korean. Math. Soc.* **39** (2002), no. 6, 953-961.

[10] Tripathi, M. M., Erol, K. E., Perktas, S. Y. and Keles, S., Indefinite almost paracontact manifolds, **Int. J. Math. Math. Sci.** (2010), **Art. ID: 846195**, DOI, 10.1155/2010/846195.

[11] Xufeng, X. and Xiaolic, C., Two theorems on (ϵ)-Sasakian manifolds, *Int. J. Math. Math. Sci.*,**21 (2)**, 249-254, (1998).

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