

SOME COMMON FIXED POINT THEOREMS FOR (μ, ψ) -WEAKLY CONTRACTIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract Some common fixed point results satisfying a generalized weak contractive condition in the framework of convex metric spaces are derived. As applications, we also obtained some results on the set of best approximation for such class of mappings. The proved results generalize and extend various known results in the literature.

1 Introduction and Preliminaries

Fixed point theory is an old and rich branch of analysis and has a large number of applications. Fixed points satisfying some contractive or nonexpansive type conditions have been studied by many researchers and applied to various mathematical problems (see e.g. [1]- [26] and references cited therein). Alber and Guerre-Delabriere [2] introduced the concept of weakly contractive mappings and proved the existence of fixed points for single-valued weakly contractive mappings in Hilbert spaces. Thereafter, in 2001, Rhoades [24] proved the fixed point theorem which is one of the generalizations of Banach's Contraction Mapping Principle, because the weakly contractions contains contractions as a special case and he also showed that some results of [2] are true for any Banach space. In fact, weakly contractive mappings are closely related to the mappings of Boyd and Wong [5] and of Reich types [23]. The present work is also a step in this direction in the frame work of convex metric spaces.

First, we recall some basic definitions and notations.

For a metric space (X, d) , a continuous mapping $W : X \times X \times [0, 1] \rightarrow X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a *convex metric space* [26].

A subset K of a convex metric space (X, d) is said to be a *convex set* [26] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. The set K is said to be *p -starshaped* [18] where $p \in K$, provided $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$ i.e. the segment

$$[p, x] = \{W(x, p, \lambda) : \lambda \in [0, 1]\}$$

joining p to x is contained in K for all $x \in K$. K is said to be *starshaped* if it is p -starshaped for some $p \in K$.

Clearly, each convex set is starshaped but converse is not true.

A convex metric space (X, d) is said to satisfy *Property (I)* [18] if for all $x, y, q \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y)$$

holds.

A normed linear space X and each of its convex subsets are simple examples of convex metric spaces with W given by $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ for $x, y \in X$ and $0 \leq \lambda \leq 1$. There are many convex metric spaces which are not normed linear spaces (see [18], [26]). Property (I) is always satisfied in a normed linear space.

Example 1.1. [26] Let I be the unit interval $[0, 1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i, b_i]$, $I_j = [a_j, b_j]$ and λ ($0 \leq \lambda \leq 1$), define a mapping

W by $W(I_i, I_j, \lambda) = [\lambda a_i + (1 - \lambda)a_j, \lambda b_i + (1 - \lambda)b_j]$ and define a metric d in X by Hausdorff distance i.e.

$$d(I_i, I_j) = \sup_{a \in I} \left\{ \left| \inf_{b \in I_i} \{ |a - b| \} - \inf_{c \in I_j} \{ |a - c| \} \right| \right\}$$

The metric space (X, d) along with the convex structure W is a convex metric space.

Example 1.2. [1] Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\}$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in X , and $\alpha \in [0, 1]$, define a mapping $W : X \times X \times [0, 1] \rightarrow X$ by

$$W(x, y, \alpha) = \left(\alpha x_1 + (1 - \alpha)y_1, \frac{\alpha x_1 x_2 + (1 - \alpha)y_1 y_2}{\alpha x_1 + (1 - \alpha)y_1} \right),$$

and a metric $d : X \times X \rightarrow [0, \infty)$ by $d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|$. Then (X, d) is a convex metric space but not a normed linear space.

For a nonempty subset M of a metric space (X, d) and $x \in X$, an element $y \in M$ is said to be a *best approximation of x to M* or a *best M -approximant* if $d(x, y) = \text{dist}(x, M) \equiv \inf\{d(x, y) : y \in M\}$. The set of all such $y \in M$ is denoted by $P_M(x)$.

Let M be a nonempty subset of a metric space (X, d) , a point $x \in M$ is a *common fixed (coincidence) point* of f and T if $x = fx = Tx$ ($fx = Tx$). The set of fixed points (respectively, coincidence points) of f and T is denoted by $F(f, T)$ (respectively, $C(f, T)$). The mappings $T, f : M \rightarrow M$ are called *commuting* if $Tfx = fTx$ for all $x \in M$; *compatible* if $\lim_{n \rightarrow \infty} d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} fx_n = t$ for some t in M ; *weakly compatible* if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$.

The ordered pair (T, I) of two self maps of a metric space (X, d) is called a Banach operator pair [15], if the set $F(I)$ is T -invariant, i.e. $T(F(I)) \subseteq F(I)$. Obviously, a commuting pair (T, I) is a Banach operator pair but not conversely. If (T, I) is a Banach operator pair then (I, T) need not be Banach operator pair. If the self maps T and I of X satisfy $d(ITx, Tx) \leq kd(Ix, x)$, for all $x \in X$ and $k \geq 0$, then (T, I) is a Banach operator pair (see [6], [15], [17]).

Example 1.3. (see [17]) Consider $M = \mathbb{R}^2$ with usual metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$, $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Define T and I on M as $T(x, y) = (x^3 + x - 1, \frac{\sqrt[3]{x^2 + y^3 - 1}}{3})$ and $I(x, y) = (x^3 + x - 1, \sqrt[3]{x^2 + y^3 - 1})$. $F(T) = (1, 0)$, $F(I) = \{(1, y) : y \in \mathbb{R}\}$ and $C(I, T) = \{(x, y) : y = \sqrt[3]{1 - x^2}, x \in \mathbb{R}\}$. $T(F(I)) = \{T(1, y) : y \in \mathbb{R}\} = \{(1, \frac{y}{3}) : y \in \mathbb{R}\} \subseteq \{(1, y) : y \in \mathbb{R}\} = F(I)$. Thus (T, I) is a Banach operator pair, which is not weakly compatible as T and I do not commute on the set $C(I, T)$ and hence it is not compatible.

Khan et al. [20] initiated the use of a control function that alters distance between two points in a metric space, which they called an altering distance function.

A function $\mu : [0, \infty) \rightarrow [0, \infty)$ is called an *altering distance function* if the following properties are satisfied:

- (i) μ is monotone increasing and continuous;
- (ii) $\mu(t) = 0$ if and only if $t = 0$.

Suppose that T and f are self-mappings of a metric space (X, d) . A mapping T is said to be (μ, ψ) -generalized f -weakly contractive (see [8]) if for each $x, y \in X$,

$$\mu(d(Tx, Ty)) \leq \mu\left(\frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right) - \psi(d(fx, Ty), d(fy, Tx)), \quad (1.1)$$

holds, where $\mu : [0, \infty) \rightarrow [0, \infty)$ is an altering distance function and $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

Note. 1. If $f =$ identity mapping, then a (μ, ψ) -generalized f -weakly contractive mapping is a (μ, ψ) -generalized weakly contractive mapping.

2. If $\mu(t) = t$, then a (μ, ψ) -generalized f -weakly contractive mapping is a generalized f -weakly contractive mapping (see [7]) and further if $f =$ identity mapping, then a generalized f -weakly contractive mapping is a generalized weakly contractive mapping (see [16]).

3. If $\psi(x, y) = k(x + y)$, $0 < k < \frac{1}{2}$, then a generalized weakly contractive mapping is a Chatterjea mapping (see [14]).

The purpose of this work is to prove some common fixed point theorems for (μ, ψ) -generalized f -weakly contractive mappings in convex metric spaces. As applications, some results on the set of best approximation for this class of mappings are also obtained. The proved results generalize and extend some well known results in the literature.

2 Main Results

To start with we need the following result of Chandok [8].

Lemma 2.1. *Let M be a subset of a metric space (X, d) and f and T are self-mappings of M such that $cl T(F(f)) \subseteq F(f)$. If $cl T(M)$ is complete, $F(f)$ is nonempty and T is a (μ, ψ) -generalized f -weakly contractive mapping for all $x, y \in F(f)$, then $F(T) \cap F(f)$ is a singleton.*

Theorem 2.2. *Let M be a nonempty subset of a convex metric space (X, d) with property (I) and T, f are self mappings of M . Suppose that $F(f)$ is q -starshaped, $cl T(F(f)) \subseteq F(f)$, $cl T(M)$ is compact and T satisfies*

$$\begin{aligned} \mu(d(Tx, Ty)) &\leq \mu\left(\frac{1}{2}[dist(fx, [Ty, q]) + dist(fy, [Tx, q])] - \right. \\ &\quad \left. \psi(dist(fx, [Ty, q]), d(fy, [Tx, q]))\right), \end{aligned} \quad (2.1)$$

where $\mu : [0, \infty) \rightarrow [0, \infty)$ is continuous and monotonic increasing function with $\mu(at) \leq \mu(t)$, $0 < a \leq 1$, $t > 0$ and $\mu(t) = 0$ if and only if $t = 0$; and $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$, for all $x, y \in M$. Then $M \cap F(T) \cap F(f) \neq \emptyset$.

Proof. For each n , define $T_n : M \rightarrow M$ by $T_n(x) = W(Tx, q, k_n)$, $x \in M$ where (k_n) is a sequence in $(0, 1)$ such that $k_n \rightarrow 1$. Since $F(f)$ is q -starshaped and $cl T(F(f)) \subseteq F(f)$, we have

$$T_n(x) = W(Tx, q, k_n) = W(Tx, fq, k_n) \in F(f)$$

for all $x \in F(f)$ and so $cl T_n(F(f)) \subseteq F(f)$ (respectively, $wcl T_n(F(f)) \subseteq F(f)$) for each n . Consider

$$\begin{aligned} \mu(d(T_n x, T_n y)) &= \mu(d(W(Tx, q, k_n), W(Ty, q, k_n))) \\ &\leq \mu(k_n d(Tx, Ty)) \\ &\leq \mu(d(Tx, Ty)) \\ &\leq \mu\left(\frac{1}{2}[dist(fx, [Ty, q]) + dist(fy, [Tx, q])] - \right. \\ &\quad \left. \psi(dist(fx, [Ty, q]), dist(fy, [Tx, q]))\right), \end{aligned}$$

for all $x, y \in F(f)$. Thus T_n is a (μ, ψ) -generalized f -weakly contractive mapping. As $cl T(M)$ is compact, $cl T_n(M)$ is compact for each n and hence complete. Now by Lemma 2.1, there exists $x_n \in M$ such that x_n is a common fixed point of f and T_n for each n . The compactness of $cl T(M)$ implies there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \rightarrow z \in cl T(M)$. Since $\{Tx_n\}$ is a sequence in $T(F(f))$, $z \in cl T(F(f)) \subseteq F(f)$. Now, as $k_{n_i} \rightarrow 1$, we have

$$x_{n_i} = T_{n_i} x_{n_i} = W(Tx_{n_i}, q, k_{n_i}) \rightarrow z$$

and $d(fx_{n_i}, Tx_{n_i}) = d(x_{n_i}, Tx_{n_i}) \rightarrow 0$. Further, we have

$$\begin{aligned} \mu(d(Tx_{n_i}, Tz)) &\leq \mu\left(\frac{1}{2}[dist(fx_{n_i}, [Tz, q]) + dist(fz, [Tx_{n_i}, q])] - \right. \\ &\quad \left. \psi(dist(fx_{n_i}, [Tz, q]), dist(fz, [Tx_{n_i}, q]))\right) \\ &= \mu\left(\frac{1}{2}[dist(x_{n_i}, [Tz, q]) + dist(fz, [Tx_{n_i}, q])] - \right. \\ &\quad \left. \psi(dist(x_{n_i}, [Tz, q]), dist(fz, [Tx_{n_i}, q]))\right) \\ &= \mu\left(\frac{1}{2}[d(x_{n_i}, Tz) + d(fz, Tx_{n_i})] - \right. \\ &\quad \left. \psi(d(x_{n_i}, Tz), d(fz, Tx_{n_i}))\right), \end{aligned}$$

on taking limit, we get $z = Tz$ and so $M \cap F(T) \cap F(f) \neq \emptyset$. \square

Let M be a non-empty subset of a metric space (X, d) . Suppose that $C = P_M(u) \cap C_M^f(u)$, where $C_M^f(u) = \{x \in M : fx \in P_M(u)\}$.

Corollary 2.3. *Let T, f are self mappings of convex metric space (X, d) with property (I). If $u \in X$, $D \subseteq C$, $G = D \cap F(f)$ is q -starshaped, $cl T(G) \subseteq G$, $cl T(D)$ is compact and T satisfies the inequality (2.1) for all $x, y \in D$, then $P_M(u) \cap F(f, T)$ is nonempty.*

Corollary 2.4. Let T, f are self mappings of convex metric space (X, d) with property (I). If $u \in X, D \subseteq P_M(u), G = D \cap F(f)$ is q -starshaped, $cl T(G) \subseteq G, cl T(D)$ is compact and T satisfies the inequality (2.1) for all $x, y \in D$, then $P_M(u) \cap F(f, T)$ is nonempty.

If $\mu(t) = t$, we have the following results.

Corollary 2.5. Let M be a nonempty subset of a convex metric space (X, d) with property (I) and T, f are self mappings of M . Suppose that $F(f)$ is q -starshaped, $cl T(F(f)) \subseteq F(f), cl T(M)$ is compact and T satisfies

$$d(Tx, Ty) \leq \frac{1}{2}[dist(fx, [Ty, q]) + dist(fy, [Tx, q])] - \psi(dist(fx, [Ty, q]), dist(fy, [Tx, q])), \quad (2.2)$$

where $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$, for all $x, y \in M$. Then $M \cap F(T) \cap F(f) \neq \emptyset$.

Corollary 2.6. Let T, f are self mappings of a convex metric space (X, d) with property (I). If $u \in X, D \subseteq C, G = D \cap F(f)$ is q -starshaped, $cl T(G) \subseteq G, cl T(D)$ is compact and T satisfies the inequality (2.2) for all $x, y \in D$, then $P_M(u) \cap F(f, T)$ is nonempty.

Corollary 2.7. Let T, f are self mappings of a convex metric space (X, d) with property (I). If $u \in X, D \subseteq P_M(u), G = D \cap F(f)$ is q -starshaped, $cl T(G) \subseteq G, cl T(D)$ is compact and T satisfies the inequality (2.2) for all $x, y \in D$, then $P_M(u) \cap F(f, T)$ is nonempty.

Remark 2.8. Theorem 2.2 extends and generalizes the corresponding results of [4], [6], [7], [8], [14], [15], [16], [19], and [21].

Let G_0 denote the class of closed convex subsets containing a point x_0 of a convex metric space (X, d) . For $M \in G_0$ and $p \in X$, let $M_p = \{x \in M : d(x, x_0) \leq 2d(p, x_0)\}$; and $P_M(p) = \{x \in M : d(p, x) = dist(p, M)\}$ be the set of best approximants to p in M .

Note $P_M(p) \subseteq M_p$ since if $x \in P_M(p)$ then

$$d(x, x_0) \leq d(x, p) + d(p, x_0) = dist(p, M) + d(p, x_0) \leq 2d(p, x_0).$$

Proposition 2.9. If M is a closed convex subset of a convex metric space (X, d) and $x \in X$, then $P_M(x)$ is closed and convex.

Proof. Let $y, z \in P_M(x)$ and $\lambda \in [0, 1]$. Consider

$$\begin{aligned} d(x, W(y, z, \lambda)) &\leq \lambda d(x, y) + (1 - \lambda)d(x, z) \\ &= \lambda dist(x, M) + (1 - \lambda)dist(x, M) \\ &= dist(x, M) \\ &\leq d(x, W(y, z, \lambda)) \text{ as } W(y, z, \lambda) \in M. \end{aligned}$$

Therefore, $d(x, W(y, z, \lambda)) = dist(x, M)$ and so $W(y, z, \lambda) \in P_M(x)$. Thus $P_M(x)$ is convex and it is easy to see its closedness. \square

Theorem 2.10. Let T, g are self mappings of a convex metric space (X, d) with property (I). If $p \in X$ and $M \in G_0$ such that $T(M_p) \subseteq M, cl T(M_p)$ is compact, and $d(Tx, p) \leq d(x, p)$ for all $x \in M_p$, then $P_M(p)$ is nonempty, closed and convex with $T(P_M(p)) \subseteq P_M(p)$. If, in addition, D is a subset of $P_M(p), G = D \cap F(g)$ is q -starshaped, $cl T(G) \subseteq G$, and T satisfies inequality (2.1) for all $x, y \in D$, then $P_M(p) \cap F(g, T)$ is nonempty.

Proof. If $p \in M$ then the result is clear. So assume that $p \notin M$. If $x \in M \setminus M_p$, then

$$d(x, x_0) > 2d(p, x_0)$$

and so

$$d(p, x) \geq d(x, x_0) - d(p, x_0) > d(p, x_0) \geq dist(p, M).$$

Thus

$$\alpha = dist(p, M_p) = dist(p, M) \leq d(p, x_0).$$

Since $cl(T(M_p))$ is compact, and the distance function is continuous, there exists $z \in cl(T(M_p))$ such that

$$\beta = dist(p, cl(T(M_p))) = d(p, z).$$

Hence

$$\begin{aligned} \alpha &= \text{dist}(p, M_p) \leq \text{dist}(p, \text{cl}(T(M_p))) \text{ as } T(M_p) \subseteq M \\ &= \beta \\ &\leq \text{dist}(p, T(M_p)) \\ &\leq d(p, Sx) \\ &\leq d(p, x) \end{aligned}$$

for all $x \in M_p$. Therefore, $\alpha \leq \beta \leq \text{dist}(p, M_p) = \text{dist}(p, M) = \alpha$, $\alpha = \beta = \text{dist}(p, M)$, i.e.

$$\text{dist}(p, M) = \text{dist}(u, \text{cl}(T(M_p))) = d(p, z),$$

i.e. $z \in P_M(p)$ and so $P_M(p)$ is nonempty. The closedness and convexity of $P_M(p)$ follows from that of M .

Now to prove $T(P_M(p)) \subseteq P_M(p)$, let $y \in T(P_M(p))$. Then $y = Tx$ for $x \in P_M(p)$. Consider

$$d(p, y) = d(p, Tx) \leq d(p, x) = \text{dist}(p, M)$$

and so $y \in P_M(p)$ as $P_M(p) \subset M_p \Rightarrow T(P_M(p)) \subset M$ i.e. $y \in M$.

The compactness of $\text{cl } T(M_p)$ implies that $\text{cl } T(D)$ compact. Hence the result follows from Corollary 2.4. \square

Corollary 2.11. *Let T, g are self mappings of a convex metric space (X, d) with property (I). If $p \in X$ and $M \in G_0$ such that $T(M_p) \subseteq M$, $\text{cl } T(M_p)$ is compact, and $d(Tx, p) \leq d(x, p)$ for all $x \in M_p$, then $P_M(p)$ is nonempty, closed and convex with $T(P_M(p)) \subseteq P_M(p)$. If, in addition, D is a subset of $P_M(p)$, $G = D \cap F(g)$ is q -starshaped, and closed, (T, g) is a Banach operator pair on D , and T satisfies inequality (2.1) for all $x, y \in D$, then $P_M(p) \cap F(g, T)$ is nonempty.*

If $\mu(t) = t$, we have the following results.

Corollary 2.12. *Let T, g are self mappings of a convex metric space (X, d) with property (I). If $p \in X$ and $M \in G_0$ such that $T(M_p) \subseteq M$, $\text{cl } T(M_p)$ is compact, and $d(Tx, p) \leq d(x, p)$ for all $x \in M_p$, then $P_M(p)$ is nonempty, closed and convex with $T(P_M(p)) \subseteq P_M(p)$. If, in addition, D is a subset of $P_M(p)$, $G = D \cap F(g)$ is q -starshaped, $\text{cl } T(G) \subseteq G$, and T satisfies inequality (2.2) for all $x, y \in D$, then $P_M(p) \cap F(g, T)$ is nonempty.*

Corollary 2.13. *Let T, g are self mappings of a convex metric space (X, d) with property (I). If $p \in X$ and $M \in G_0$ such that $T(M_p) \subseteq M$, $\text{cl } T(M_p)$ is compact, and $d(Tx, p) \leq d(x, p)$ for all $x \in M_p$, then $P_M(p)$ is nonempty, closed and convex with $T(P_M(p)) \subseteq P_M(p)$. If, in addition, D is a subset of $P_M(p)$, $G = D \cap F(g)$ is q -starshaped, and closed, (T, g) is a Banach operator pair on D , and T satisfies inequality (2.2) for all $x, y \in D$, then $P_M(p) \cap F(g, T)$ is nonempty.*

Remark 2.14. Theorem 2.10 extends and generalizes the corresponding results of [3], [4], [8], [19], [22] and [25].

Remark 2.15. It may be noted that the assumption of linearity or affinity for I is necessary in almost all known results about common fixed points of maps T, I such that T is I -nonexpansive under the conditions of commuting, weakly commuting, R -subweakly commuting or compatibility (see [3], [6], [15], [22], [25] and the literature cited therein), but our results in this paper are independent of the linearity or affinity.

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