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On Approximative Frames in Hilbert Spaces

S.K. Sharma, A. Zothansanga and S.K. Kaushik

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Abstract A generalization of frames called approximative frames has been introduced and studied. Results on the existence of approximative frames have been proved. Also, a characterization of approximative frame in terms of approximative frame operator is given. Finally, a Paley-Wiener type perturbation result for approximative frames has been obtained.

1 Introduction

Frames for Hilbert spaces were formally introduced in 1952 by Duffin and Schaeffer [17] who used frames as a tool in the study of non-harmonic Fourier series. Daubechies, Grossmann and Meyer [16], in 1986, reintroduced frames and observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$. Frames are generalizations of orthonormal bases in Hilbert spaces. The main property of frames which makes them so useful is their redundancy. Representation of signals using frames, is advantageous over basis expansions in a variety of practical applications. Many properties of frames make them very useful in characterization of function spaces and other fields of applications such as signal and image processing [8], filter bank theory [7], wireless communications [19] and sigma - delta quantization [6]. For more literature on frame theory, one may refer to [5, 9, 10, 14].

Recently, many generalizations of frames have been introduced and studied. Casazza and Kutyniok [12] introduced the notion of frames of subspaces or fusion frames to have many more applications of frames in sensor networks and packet encoding. Li and Ogawa [21] introduced the notion of pseudo-frames in Hilbert spaces using two Bessel sequences. Christensen and Eldar [15] gave another generalization of frames called oblique frames. Later on, Sun [22] introduced a more general concept, called generalized frames or $g-$frames for Hilbert spaces and proved that oblique frames, pseudo frames and fusion frames are special cases of $g-$frames. Fusion frames and $g-$frames are also studied in [1, 3, 20, 22]. Very recently, Găvrua [18] introduced another generalization of frames called frames for operators (K-frames).

Aldroubi [2] introduced two methods for generating frames for a Hilbert space $\mathcal{H}$. The first method uses bounded operators on $\mathcal{H}$ and the other one uses bounded linear operators on ℓ^2 to generate frames for $\mathcal{H}$. He also characterized all the mappings that transform frames into other frames.

Recently, in various areas of applied mathematics, computer science, and electrical engineering, sparsity has played a key role. Choosing a suitable basis (or a frame), many types of signals can be represented by only a few non-zero coefficients using sparse signal processing methods.

In this paper, we introduce notion of approximative frame, as another generalizations of frames, which is constructed keeping in mind that it has sparsity in its nature and with the help of examples it has been shown that the class of approximative frames is larger than the class of frames in the sense that a non-frame can be made an approximative frame. Also, we generalize some basic results on existence and perturbation of approximative frames.

2 Preliminaries

Throughout the paper, $\mathcal{H}$ will denote an infinite dimensional Hilbert space, $\{m_n\}$ an infinite increasing sequence in $\mathbb{N}$ and $\mathbb{K}$ will denote the scalar field $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. A sequence $\{x_n\}$ in a Hilbert space $\mathcal{H}$ is said to be a *frame* for $\mathcal{H}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2,\ x \in \mathcal{H}. \tag{2.1}$$

The positive constants A and B, respectively, are called lower and upper frame bounds for the frame $\{x_n\}$. The inequality (2.1) is called the *frame inequality* for the frame $\{x_n\}$.

A frame $\{x_n\}$ is said to be

- *tight* if it is possible to choose $A,\ B$ satisfying inequality (2.1) with $A = B$ as frame bounds.
- *Parseval* if $A = B = 1$ as frame bounds.
- *exact* if removal of any arbitrary x_n renders the collection $\{x_n\}_{i\neq n}$ no longer a frame for $\mathcal{H}$.

A sequence $\{x_n\} \subset \mathcal{H}$ is called a *Bessel sequence* if it satisfies upper frame inequality in (2.1).

If $\{x_n\}$ is a frame for $\mathcal{H}$, then the bounded linear operator

$$T : \ell^2(\mathbb{N}) \to \mathcal{H}, \quad T\big(\{\alpha_n\}_{n=1}^{\infty}\big) = \sum_{n=1}^{\infty} \alpha_n x_n, \ \ \{\alpha_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$$

is called the *pre-frame operator* or the *synthesis operator*. The adjoint operator is given by

$$T^* : \mathcal{H} \to \ell^2(\mathbb{N}), \quad T^*(x) = \{\langle x, x_n\rangle\}_{n=1}^{\infty}, \ \ x \in \mathcal{H}$$

T^* is called the *analysis operator*. By composing T and T^*, we obtain *frame operator*

$$S : \mathcal{H} \to \mathcal{H}, \quad S(x) = TT^*(x) = \sum_{n=1}^{\infty} \langle x, x_n\rangle x_n, \ \ x \in \mathcal{H}.$$

Definition 2.2. ([14]) The *pseudo-inverse* of an operator U with closed range is defined as the unique operator $U^\dagger$ satisfying

$$\mathcal{N}_{U^\dagger} = \mathcal{R}_U^{\perp}, \ \ \mathcal{R}_{U^\dagger} = \mathcal{N}_U^{\perp} \ \text{ and } \ UU^\dagger x = x, \ x \in \mathcal{R}_U.$$

3 Approximative Frames

We begin this section with the following definition of approximative frame

Definition 3.1. Let $\mathcal{H}$ be a Hilbert space and $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}}$ be a sequence in $\mathcal{H}$, where $\{m_n\}$ be an increasing sequence of positive integers. Then, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}}$ is called an *approximative frame* for $\mathcal{H}$ if there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \lim_{n\to\infty} \sum_{i=1}^{m_n} |\langle x, x_{n,i}\rangle|^2 \leq B\|x\|^2, \ \ x \in \mathcal{H}. \tag{3.1}$$

The positive constants A and B are called *lower* and *upper* approximative frame bounds.

An approximative frame $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}}$ is said to be

- *tight* if we can choose $A = B$ as approximative frame bounds.
- *Parseval* if we can choose $A = B = 1$ as approximative frame bounds.

A sequence $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n \\ n\in\mathbb{N}}}$ is said to be an *approximative Bessel sequence* if righthand side of inequality (3.1) is satisfied.

Regarding the existence of approximative frames, we give the following examples:

Example 3.2. Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$. Define a sequence $\{x_{n,i}\}_{\substack{i=1,2,\cdots,2n \\ n\in\mathbb{N}}}$ in $\mathcal{H}$ by

$$x_{n,i} = \begin{cases} e_i, & \text{if } \ i = 1, 2, \cdots, n \\ e_r, & \text{if } \ i > n \ \text{ and } r \equiv i \bmod n. \end{cases}$$

Then, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,2n \\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$ with approximate frame bounds $A = 1$ and $B = 2$.

Next, we give an example of a Parseval approximative frame.

Example 3.3. Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$. Define a sequence $\{x_{n,i}\}_{\substack{i=1,2,\cdots,2n\\ n\in\mathbb{N}}}$ in $\mathcal{H}$ by

$$\begin{cases} x_{n,1} = e_1, \\ x_{n,2i+1} = \sqrt{1-\dfrac{1}{i^2}}\, e_i, \ \ i=1,2,\cdots,n-1, \ n\in\mathbb{N}, \\ x_{n,2i} = \dfrac{e_{i+1}}{i+1}, \ \ i=1,2,\cdots,n, \ n\in\mathbb{N}. \end{cases}$$

Then, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,2n\\ n\in\mathbb{N}}}$ is a Parseval approximative frame for $\mathcal{H}$.

Remark 3.4. One may observe that if a Hilbert space $\mathcal{H}$ has a frame, then it also has an approximative frame. Indeed, if $\{x_n\}_{n\in\mathbb{N}}$ is a frame for Hilbert space $\mathcal{H}$ with lower and upper frame bounds A and B respectively. Then for $x_{n,i} = x_i, \ i = 1, 2, \cdots, n; \ n \in \mathbb{N}$, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,n\\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$ with the same lower and upper approximative frame bounds A and B. Infact, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n}|\langle x, x_{n,i}\rangle|^2 = \sum_{i=1}^{\infty}|\langle x, x_i\rangle|^2, \ \ x\in\mathcal{H}.$$

If we have a Bessel sequence for a given Hilbert space $\mathcal{H}$ which does not satisfy lower frame inequality then in order handle such sequences couple of ways are

- to add more elements to the sequence.
- to sparse the elements of the sequence in such a way that it forces the sequence to satisfy the lower frame inequality and at the same time keeps the upper frame inequality intact in some sense.

As the notion of approximative frames has sparsity in its behavior, due to which it is possible to make a Bessel sequence which is not a frame for $\mathcal{H}$ an approximative frame for $\mathcal{H}$. In the following examples, we shall demonstrate this construction.

Example 3.5. Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$. Recall that $\{e_n + e_{n+1}\}_{n\in\mathbb{N}}$ is a Bessel sequence in $\mathcal{H}$ which is not a frame for $\mathcal{H}$. Define a sequence $\{x_{n,i}\}_{\substack{i=0,1,2,\cdots,2n\\ n\in\mathbb{N}}}$ in $\mathcal{H}$ by

$$\begin{cases} x_{n,0} = e_1, \\ x_{n,2i-1} = e_i, \ \ i=1,2,\cdots,n, \ n\in\mathbb{N}, \\ x_{n,2i} = e_i + e_{i+1}, \ \ i=1,2,\cdots,n, \ n\in\mathbb{N}. \end{cases}$$

Then, we have

$$\|x\|^2 \ \le \ \lim_{n\to\infty}\sum_{i=0}^{2n}|\langle x, x_{n,i}\rangle|^2 \ \le \ 5\|x\|^2, \ \ x\in\mathcal{H}.$$

Hence, $\{x_{n,i}\}_{\substack{i=0,1,2,\cdots,2n\\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$ with approximate frame bounds $A = 1$ and $B = 5$.

Now, we sparse the elements of a Bessel sequence (constructed by Casazza and Christensen [11]) which is not a frame for $\mathcal{H}$ and construct an approximative frame for $\mathcal{H}$.

Example 3.6. Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$. Define a sequence $\{x_n\}$ by $x_n = \dfrac{e_n}{\sqrt{n}}$, $n\in\mathbb{N}$. Then it is elementary to observe that $\{x_n\}$ is not a frame for $\mathcal{H}$. Define a sequence $\{x_{n,i}\}_{\substack{i=1,2,\cdots,\frac{n(n+1)}{2}\\ n\in\mathbb{N}}}$ in $\mathcal{H}$ by

$$\begin{cases} x_{n,1} = e_1, \\ x_{n,2} = x_{n,3} = \frac{e_2}{\sqrt{2}}, \\ x_{n,4} = x_{n,5} = x_{n,6} = \frac{e_3}{\sqrt{3}}, \\ \vdots \\ x_{n,\frac{n(n-1)}{2}+1} = x_{n,\frac{n(n-1)}{2}+2} = \cdots = x_{n,\frac{n(n+1)}{2}} = \frac{e_n}{\sqrt{n}}, \ \ n\in\mathbb{N}. \end{cases}$$

Then, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,\frac{n(n+1)}{2}\\ n\in\mathbb{N}}}$ is a tight approximative frame for $\mathcal{H}$ with approximative frame bound $A = B = 1$.

Let $\{m_n\}$ be an increasing sequence of positive integers. Define the space

$$\ell^2\left(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\right)=\Big\{\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}:\alpha_{n,i}\in\mathbb{K},\ \lim_{n\to\infty}\sum_{i=1}^{m_n}|\alpha_{n,i}|^2<\infty\Big\}.$$

Then, $\ell^2\left(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\right)$ is a Hilbert space with the norm induced by the inner-product given by

$$\left\langle\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}},\{\beta_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\right\rangle=\lim_{n\to\infty}\sum_{i=1}^{m_n}\alpha_{n,i}\overline{\beta_{n,i}},$$
$$\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}},\{\beta_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\in\ell^2\left(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\right).$$

Next, we give a characterization for an approximative Bessel sequence.

Theorem 3.7. *Let $\{m_n\}$ be an increasing sequence of positive integers and $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be a sequence in $\mathcal{H}$. Then, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ is an approximative Bessel sequence with bound B if and only if the operator $T_{\{m_n\}}:\ell^2\left(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\right)\to\mathcal{H}$ defined by*

$$T_{\{m_n\}}(\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})=\lim_{n\to\infty}\sum_{i=1}^{m_n}\alpha_{n,i}x_{n,i},\ \ \{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\in\ell^2\left(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\right)$$

is well-defined bounded linear operator from $\ell^2\left(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\right)$ into $\mathcal{H}$ with $\|T_{\{m_n\}}\|\le B$.

Proof. Let $\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\in\ell^2\left(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\right)$. Then, for $p,q\in\mathbb{N},\ p>q$, we have

$$\begin{aligned}\Big\|\sum_{i=1}^{m_p}\alpha_{p,i}x_{p,i}-\sum_{i=1}^{m_q}\alpha_{q,i}x_{q,i}\Big\| &= \sup_{\|g\|=1}\Big|\Big\langle\Big(\sum_{i=1}^{m_p}\alpha_{p,i}x_{p,i}-\sum_{i=1}^{m_q}\alpha_{q,i}x_{q,i}\Big),g\Big\rangle\Big|\\ &\le \sup_{\|g\|=1}\Big[\sum_{i=1}^{m_p}|\alpha_{p,i}\langle x_{p,i},g\rangle|\ +\sum_{i=1}^{m_q}|\alpha_{q,i}\langle x_{q,i},g\rangle|\Big]\\ &\le \sup_{\|g\|=1}\Big[\Big(\sum_{i=1}^{m_p}|\alpha_{p,i}|^2\Big)^{\frac12}\Big(\sum_{i=1}^{m_p}|\langle x_{p,i},g\rangle|^2\Big)^{\frac12}+\\ &\qquad\Big(\sum_{i=1}^{m_q}|\alpha_{q,i}|^2\Big)^{\frac12}\Big(\sum_{i=1}^{m_q}|\langle x_{q,i},g\rangle|^2\Big)^{\frac12}\Big]\\ &\le \sqrt{B}\Big[\Big(\sum_{i=1}^{m_p}|\alpha_{p,i}|^2\Big)^{\frac12}\ +\Big(\sum_{i=1}^{m_q}|\alpha_{q,i}|^2\Big)^{\frac12}\Big].\end{aligned}$$

Thus $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{m_n}\alpha_{n,i}x_{n,i}$ exists in $\mathcal{H}$. Therefore $T_{\{m_n\}}:\ell^2\left(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\right)\to\mathcal{H}$ is a well-defined operator. It is easy to verify that $T_{\{m_n\}}$ is linear and for each $\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\in\ell^2\left(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\right)$, we have

$$\begin{aligned}\Big\|T_{\{m_n\}}(\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})\Big\| &= \sup_{\|g\|=1}\Big|\Big\langle T_{\{m_n\}}(\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})\ ,\ g\Big\rangle\Big|\\ &= \sup_{\|g\|=1}\lim_{n\to\infty}\Big|\Big\langle\sum_{i=1}^{m_n}\alpha_{n,i}x_{n,i}\ ,\ g\Big\rangle\Big|\\ &\le \sqrt{B}\lim_{n\to\infty}\Big(\sum_{i=1}^{m_n}|\alpha_{n,i}|^2\Big)^{\frac12}\\ &\le \sqrt{B}\ \big\|\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\big\|.\end{aligned}$$

Hence $T_{\{m_n\}}$ is bounded operator with $\|T_{\{m_n\}}\|<\sqrt{B}$.

Conversely, the adjoint of linear operator $T_{\{m_n\}}$ is defined by

$$T^*_{\{m_n\}}(x)=\{\langle x,x_{n,i}\rangle\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}},\ \ x\in\mathcal{H}.$$

Note that,

$$\begin{aligned}\left\langle x,\ T_{\{m_n\}}(\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})\right\rangle_{\mathcal{H}} &= \left\langle x,\ \lim_{n\to\infty}\sum_{i=1}^{m_n}\alpha_{n,i}\ x_{n,i}\right\rangle_{\mathcal{H}}\\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n}\overline{\alpha_{n,i}}\ \langle x, x_{n,i}\rangle_{\mathcal{H}}\\ &= \left\langle \{\langle x, x_{n,i}\rangle\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}},\ \{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\right\rangle\end{aligned}$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x, x_{n,i}\rangle|^2 &= \|T^*_{\{m_n\}}(x)\|^2\\ &\le \|T^*_{\{m_n\}}\|^2\ \|x\|^2,\ \ x\in\mathcal{H}. \qquad \square\end{aligned}$$

Let $\{m_n\}$ be an increasing sequence of positive integers and $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be an approximative frame for $\mathcal{H}$. Then the operator $T_{\{m_n\}}:\ell^2(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}})\to\mathcal{H}$ given by

$$T_{\{m_n\}}(\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}) = \lim_{n\to\infty}\sum_{i=1}^{m_n}\alpha_{n,i}\ x_{n,i},\ \ \{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\in\ell^2(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}})$$

is bounded by Theorem 3.7. The operator $T_{\{m_n\}}$ is called ($\{m_n\}$-) *pre-approximative frame operator* or *($\{m_n\}$-) approximative synthesis operator*, the adjoint operator $T^*_{\{m_n\}}$ is called *($\{m_n\}$-)approximative analysis operator*. By composing $T_{\{m_n\}}$ and $T^*_{\{m_n\}}$, we obtain the ($\{m_n\}$-)*approximative frame operator* $S_{\{m_n\}}:\mathcal{H}\to\mathcal{H}$ given by

$$\begin{aligned}S_{\{m_n\}}(x) &= T_{\{m_n\}}T^*_{\{m_n\}}(x)\\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n}\langle x,\ x_{n,i}\rangle x_{n,i},\ \ x\in\mathcal{H}.\end{aligned}$$

In the next result we discuss some properties of ($\{m_n\}$)-approximative frame operator

Theorem 3.8. *Let $\{m_n\}$ be an increasing sequence of positive integers and $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be an approximative frame for $\mathcal{H}$ with approximative frame bounds A and B and approximative frame operator $S_{\{m_n\}}$. Then $S_{\{m_n\}}$ is a bounded, invertible, self-adjoint and positive operator. Further, $\{S^{-1}_{\{m_n\}}(x_{n,i})\}_{\substack{1,2,\cdots m_n\\ n\in\mathbb{N}}}$ is also an approximative frame for $\mathcal{H}$ with approximative frame bounds $B^{-1},\ A^{-1}$ and approximative frame operator $S^{-1}_{\{m_n\}}$.*

Proof. Clearly $S_{\{m_n\}}$ is bounded and

$$\|S_{\{m_n\}}\| = \|T_{\{m_n\}}\ T^*_{\{m_n\}}\| \le \|T_{\{m_n\}}\|^2 \le B.$$

Since $S^*_{\{m_n\}} = \left(T_{\{m_n\}}\ T^*_{\{m_n\}}\right)^* = T_{\{m_n\}}\ T^*_{\{m_n\}} = S_{\{m_n\}}$, the operator $S_{\{m_n\}}$ is self-adjoint. Also

$$\begin{aligned}\langle S_{\{m_n\}}(x),\ x\rangle &= \left\langle \lim_{n\to\infty}\sum_{i=1}^{m_n}\langle x,\ x_{n,i}\rangle x_{n,i},\ x\right\rangle\\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x,\ x_{n,i}\rangle|^2,\ \ x\in\mathcal{H}.\end{aligned}$$

So, by approximative frame inequality (3.1), we have

$$A\ \langle I(x),\ x\rangle \le \langle S_{\{m_n\}}(x),\ x\rangle \le B\ \langle I(x), x\rangle,\ \ x\in\mathcal{H},$$

where, I is the identity operator. This gives $A\ I \le S_{\{m_n\}} \le B\ I$. Thus, $S_{\{m_n\}}$ is a positive operator and

$$0 \le I - B^{-1}S_{\{m_n\}} \le \frac{B-A}{B}I$$

This gives $\|I - B^{-1}S_{\{m_n\}}\| < 1$. Hence $S_{\{m_n\}}$ is invertible. Further, note that

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x,\ S_{\{m_n\}}^{-1}(x_{n,i})\rangle|^2 &= \lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle S_{\{m_n\}}^{-1}(x_{n,i}),\ x_{n,i}\rangle|^2\\ &\le B\,\|S_{\{m_n\}}^{-1}(x)\|^2\\ &\le B\,\|S_{\{m_n\}}^{-1}\|^2\,\|x\|^2,\ \ x\in\mathcal{H}.\end{aligned}$$

Since $\{S_{\{m_n\}}^{-1}(x_{n,i})\}_{\substack{1,2,\cdots m_n\\ n\in\mathbb{N}}}$ is a Bessel sequence, the approximative frame operator for $\{S_{\{m_n\}}^{-1}(x_{n,i})\}_{\substack{1,2,\cdots m_n\\ n\in\mathbb{N}}}$ is well-defined. Also, we have $B^{-1}\,I \le S_{\{m_n\}}^{-1} \le A^{-1}\,I$. This gives

$$B^{-1}\,\|x\|^2 \le \lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x,\ S_{\{m_n\}}^{-1}(x_{n,i})\rangle|^2 \le A^{-1}\|x\|^2,\ \ x\in\mathcal{H}.$$

Thus, $\{S_{\{m_n\}}^{-1}(x_{n,i})\}_{\substack{1,2,\cdots m_n\\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$ with approximative frame bounds B^{-1} and A^{-1}. Finally,

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{m_n}\langle x,\ S_{\{m_n\}}^{-1}(x_{n,i})\rangle\ S_{\{m_n\}}^{-1}(x_{n,i}) &= S_{\{m_n\}}^{-1}\left(\lim_{n\to\infty}\sum_{i=1}^{m_n}\langle S_{\{m_n\}}^{-1}(x),\ x_{n,i}\rangle\ x_{n,i}\right)\\ &= S_{\{m_n\}}^{-1}\ S_{\{m_n\}}\ S_{\{m_n\}}^{-1}(x)\\ &= S_{\{m_n\}}^{-1}(x).\end{aligned}$$

Hence the approximative frame operator for approximative frame $\{S_{\{m_n\}}^{-1}(x_{n,i})\}_{\substack{1,2,\cdots m_n\\ n\in\mathbb{N}}}$ is $S_{\{m_n\}}^{-1}$ □.

We call the approximative frame $\{S_{\{m_n\}}^{-1}(x_{n,i})\}_{\substack{1,2,\cdots m_n\\ n\in\mathbb{N}}}$ as the *dual approximative frame* for $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$.

Corollary 3.9. *Let $\{m_n\}$ be an increasing sequence of positive integers and $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be an approximative frame for $\mathcal{H}$ with the frame operator $S_{\{m_n\}}$. Then*

$$x = \lim_{n\to\infty}\sum_{i=1}^{m_n}\langle x, S_{\{m_n\}}^{-1}(x_{n,i})\rangle\ x_{n,i},\quad x\in\mathcal{H}. \tag{3.2}$$

Proof. Follows in view of Theorem 3.8. □

Next we give a necessary and sufficient condition for an approximative frame

Theorem 3.10. *Let $\{m_n\}$ be an increasing sequence of positive integers and $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be a sequence in $\mathcal{H}$. Then, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$ if and only if $T_{\{m_n\}} : \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big) \to \mathcal{H}$ defined by*

$$T_{\{m_n\}}(\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}) = \lim_{n\to\infty}\sum_{i=1}^{m_n}\alpha_{n,i}x_{n,i},\ \ \{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} \in \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big)$$

is a well-defined bounded linear operator on $\ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big)$ onto $\mathcal{H}$.

Proof. By Theorem 3.8, $T_{\{m_n\}}$ is a well-defined, bounded linear operator and $S_{\{m_n\}}(= T_{\{m_n\}}\,T_{\{m_n\}}^*)$ is surjective. Thus, $T_{\{m_n\}}$ is surjective.

Conversely, suppose that $T_{\{m_n\}}$ is a well-defined mapping on $\ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big)$ onto $\mathcal{H}$. For each $n\in\mathbb{N}$, define a sequence of bounded linear operators $T_{m_n} : \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big) \to \mathcal{H}$ by

$$T_{m_n}(\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}) = \sum_{i=1}^{m_n}\alpha_{n,i}\,x_{n,i},\ \ \{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} \in \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big).$$

Clearly $T_{m_n} \to T_{\{m_n\}}$ point-wise as $n\to\infty$. So by Banach-Steinhaus Theorem, $T_{\{m_n\}}$ is bounded. Also, the adjoint operator of $T_{\{m_n\}}$ given by $T_{\{m_n\}}^* : \mathcal{H} \to \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big)$ is $T_{\{m_n\}}^*(x) =$

$\{\langle x, x_{n,i}\rangle\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$, $x \in \mathcal{H}$. Therefore

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x,\ x_{n,i}\rangle|^2 &= \|T^*_{\{m_n\}}(x)\|^2\\ &\leq \|T_{\{m_n\}}\|^2\ \|x\|^2,\ \ x\in\mathcal{H}.\end{aligned}$$

Let $T^\dagger_{\{m_n\}} : \mathcal{H} \to \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big)$ be the pseudo inverse of $T_{\{m_n\}}$. Then

$$\begin{aligned}x &= T_{\{m_n\}}\ T^\dagger_{\{m_n\}}(x)\\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n}(T^\dagger_{\{m_n\}})_{n,i}(x)\ x_{n,i},\ \ x\in\mathcal{H}.\end{aligned}$$

Thus

$$\begin{aligned}\|x\|^4 &= \left|\left\langle \lim_{n\to\infty}\sum_{i=1}^{m_n}(T^\dagger_{\{m_n\}})_{n,i}(x)\ x_{n,i},\ x\right\rangle\right|\\ &\leq \|T^\dagger_{\{m_n\}}\|\ \|x\|^2\ \lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x,\ x_{n,i}\rangle|^2,\ \ x\in\mathcal{H}.\end{aligned}$$

Hence, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$. $\square$

In the next result, we prove that if a sequence is an approximative frame for a dense subset of a Hilbert space, then it is also an approximative frame for the Hilbert space with the same bounds.

Theorem 3.11. *Let $\{m_n\}$ be an increasing sequence of positive integers, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be a sequence in $\mathcal{H}$ and there exist positive constants A and B such that*

$$A\|x\|^2 \ \leq\ \lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x, x_{n,i}\rangle|^2\ \leq\ B\|x\|^2,\ \textit{for all } x \textit{ in a dense subset } \mathcal{V} \textit{ of } \mathcal{H}.$$

Then, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$ with approximative frame bounds A and B.

Proof. Suppose that there exist $x \in \mathcal{H}$ such that

$$\lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x, x_{n,i}\rangle|^2 > B\|x\|^2.$$

Then, there exists some $n \in \mathbb{N}$ such that

$$\sum_{i=1}^{m_n}|\langle x, x_{n,i}\rangle|^2 > B\|x\|^2$$

Since $\mathcal{V}$ is dense in $\mathcal{H}$, there exist $y \in \mathcal{V}$ such that

$$\sum_{i=1}^{m_n}|\langle y, x_{n,i}\rangle|^2 > B\|y\|^2.$$

This a contradiction. Further, we know that

$$A\|x\|^2 \leq \|T^*_{\{m_n\}}(x)\|^2,\ \ x\in V. \tag{3.3}$$

Since $T^*_{\{m_n\}}$ is bounded and $\mathcal{V}$ is dense in $\mathcal{H}$, it follows that (3.3) holds for all $x \in \mathcal{H}$. $\square$

Finally, we give a characterization of approximative frame in terms of a bounded linear operator.

Theorem 3.12. *Let $\{m_n\}$ be an increasing sequence of positive integers, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be an approximative frame for $\mathcal{H}$ with approximative frame bounds A and B. Then $\{T(x_{n,i})\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$ if and only if there exists a positive constant γ such that adjoint operator T^* satisfies*

$$\|T^*x\|^2 \geq \gamma\|x\|^2, \ \ x \in \mathcal{H}.$$

Proof. For each $x \in \mathcal{H}$, we have

$$\begin{aligned}\gamma A\|x\|^2 &\leq A\|T^*x\| \\ &\leq \lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle T^*x,\ x_{n,i}\rangle| = \lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle x,\ T(x_{n,i})\rangle|.\end{aligned}$$

Also,

$$\lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle T^*x,\ x_{n,i}\rangle| \leq B\ \|T^*\|^2\ \|x\|^2, \ \ x \in \mathcal{H}.$$

Thus, $\{T(x_{n,i})\}$ is an approximative frame for $\mathcal{H}$.

Conversely, assume that $\{T(x_{n,i})\}$ is approximative frame for $\mathcal{H}$ with approximative lower frame bound $\mathfrak{A}$. Then, for each $x \in \mathcal{H}$

$$\begin{aligned}\mathfrak{A}\|x\|^2 &\leq \lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle T^*x,\ x_{n,i}\rangle|^2 \\ &\leq B\|T^*x\|^2. \qquad \Box\end{aligned}$$

Note: In case $\{m_n\} = \{n\}$ and $x_{n,i} = x_i,\ i = 1, 2, \cdots, n,\ n \in \mathbb{N}$, Theorem 3.12 reduces to a result due to Aldroubi [2].

The following result states that if a Hilbert space has an approximative frame, then it also has a Parseval approximative frame.

Theorem 3.13. *Let $\{m_n\}$ be an increasing sequence of positive integers, $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be an approximative frame for $\mathcal{H}$ with $\{m_n\}-$approximative frame operator $S_{\{m_n\}}$ and $S_{\{m_n\}}^{-1/2}$ be the square root of $S_{\{m_n\}}^{-1}$. Then $\{S_{\{m_n\}}^{-1/2}x_{n,i}\}_{\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ is a Parseval approximative frame.*

Proof. Straight forward. $\Box$

4 Perturbation of Approximative Frames

Christensen[13] gave a version of Paley-Wiener Theorem for frames. We give a similar version for approximative frames.

Theorem 4.1. *Let $\{m_n\}$ be an increasing sequence of positive integers and $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be an approximative frame for $\mathcal{H}$ with bounds A and B. Assume that $\{y_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be a sequence in $\mathcal{H}$ such that there exist constants $\lambda, \mu \geq 0$ such that $\left(\lambda + \frac{\mu}{\sqrt{A}}\right) < 1$ and*

$$\lim_{n\to\infty}\left\|\sum_{i=1}^{m_n}\alpha_{n,i}(x_{n,i}-y_{n,i})\right\| \leq \lambda \lim_{n\to\infty}\left\|\sum_{i=1}^{m_n}\alpha_{n,i}x_{n,i}\right\| + \mu\left\|\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\right\|,$$

$$\text{for all finite scalar sequences } \{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}. \tag{4.1}$$

Then, $\{y_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ is also an approximative frame for $\mathcal{H}$ with approximative frame bounds $A\left(1-\left(\lambda+\frac{\mu}{\sqrt{A}}\right)\right)^2$ and $B\left(1+\lambda+\frac{\mu}{\sqrt{B}}\right)^2$.

Proof. By Theorem 3.7, $\|T_{\{m_n\}}\| \leq B$. Also, for all finite scalar sequences $\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$, we have

$$\begin{aligned}\lim_{n\to\infty}\Big\|\sum_{i=1}^{m_n}\alpha_{n,i}y_{n,i}\Big\| &= \lim_{n\to\infty}\Big\|\sum_{i=1}^{m_n}\alpha_{n,i}(x_{n,i}-y_{n,i})\Big\| + \lim_{n\to\infty}\Big\|\sum_{i=1}^{m_n}\alpha_{n,i}x_{n,i}\Big\|\\ &\leq (1+\lambda)\lim_{n\to\infty}\Big\|\sum_{i=1}^{m_n}\alpha_{n,i}x_{n,i}\Big\| + \mu\|\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\|.\end{aligned}$$

Let $p, q \in \mathbb{N}$ with $p > q$. Then

$$\begin{aligned}\Big\|\sum_{i=1}^{m_p}\alpha_{n,i}y_{n,i} - \sum_{i=1}^{m_q}\alpha_{n,i}y_{n,i}\Big\| \leq (1+\lambda)\Big\|\sum_{i=1}^{m_p}\alpha_{n,i}x_{n,i}\Big\| + (1+\lambda)\Big\|\sum_{i=1}^{m_q}\alpha_{n,i}x_{n,i}\Big\|\\ + 2\mu\|\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\|.\end{aligned}$$

Since $\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} \in \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big)$ and $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{m_n}\alpha_{n,i}x_{n,i}$ exists, $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{m_n}\alpha_{n,i}y_{n,i}$ exists. Thus the ($\{m_n\}$-) pre-approximative frame operator $U_{\{m_n\}} : \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big) \to \mathcal{H}$ defined by $U_{\{m_n\}}(\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}) = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{m_n}\alpha_{n,i}y_{n,i}$ is well-defined. So, we have

$$\lim_{n\to\infty}\Big\|\sum_{i=1}^{m_n}\alpha_{n,i}y_{n,i}\Big\| \leq (1+\lambda)\lim_{n\to\infty}\Big\|\sum_{i=1}^{m_n}\alpha_{n,i}x_{n,i}\Big\| + \mu\|\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\|.$$

This gives

$$\begin{aligned}\Big\|U_{\{m_n\}}(\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})\Big\| &\leq (1+\lambda)\|T_{\{m_n\}}(\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}})\|\\ &\qquad + \mu\|\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\|\\ &\leq \Big((1+\lambda)\sqrt{B}+\mu\Big)\|\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\|.\end{aligned}$$

Therefore by Theorem 3.7, $\{y_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ is an approximative Bessel sequence for $\mathcal{H}$ with bound $B\Big(1+\lambda+\frac{\mu}{\sqrt{B}}\Big)^2$. Define $T^\dagger_{\{m_n\}} : \mathcal{H} \to \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big)$ by

$$\begin{aligned}T^\dagger_{\{m_n\}}(x) &= T^*_{\{m_n\}}(T_{\{m_n\}}T^*_{\{m_n\}})^{-1}(x)\\ &= \Big\{\Big\langle x,\ T_{\{m_n\}}T^*_{\{m_n\}}(x_{n,i})\Big\rangle\Big\}_{\substack{n=1,2,\cdots m_n\\ n\in\mathbb{N}}},\quad x\in\mathcal{H}\end{aligned}$$

Then

$$\begin{aligned}\|T^\dagger_{\{m_n\}}(x)\| &= \lim_{n\to\infty}\sum_{i=1}^{m_n}\Big|\Big\langle x,\ \big(T_{\{m_n\}}T^*_{\{m_n\}}\big)^{-1}(x_{n,i})\Big\rangle\Big|\\ &\leq \frac{1}{A}\|x\|^2,\quad x\in\mathcal{H}.\end{aligned}$$

Since $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{m_n}\alpha_{n,i}x_{n,i}$ and $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{m_n}\alpha_{n,i}y_{n,i}$ exist for all $\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} \in \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big)$, the inequality (4.1) holds for all $\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} \in \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big)$. Also, for each $\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} \in \ell^2\big(\substack{1,2,\cdots,m_n\\ n\in\mathbb{N}}\big)$, we have

$$\begin{aligned}\Big\|T_{\{m_n\}}\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}} - U_{\{m_n\}}\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\Big\| \leq\\ \lambda\,\|T_{\{m_n\}}\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\| + \mu\,\|\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}\|.\end{aligned}\tag{4.2}$$

Note that for each $x \in \mathcal{H}$, $T_{\{m_n\}}T^{\dagger}_{\{m_n\}}(x) = x$ and

$$\begin{aligned} U_{\{m_n\}}T^{\dagger}_{\{m_n\}}(x) &= \lim_{n\to\infty}\sum_{i=1}^{m_n}(T^{\dagger}_{\{m_n\}}x)_{n,i}y_{n,i} \\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n}\Big\langle x,\ (T_{\{m_n\}}T^{*}_{\{m_n\}})^{-1}(x_{n,i})\Big\rangle y_{n,i} \\ &= T^{\dagger}_{\{m_n\}}(x). \end{aligned}$$

Using (4.2), we have

$$\begin{aligned} \|x - U_{\{m_n\}}T^{\dagger}_{\{m_n\}}(x)\| &\le \lambda\|x\| \ + \ \mu\|T^{\dagger}_{\{m_n\}}(x)\| \\ &\le \left(\lambda + \frac{\mu}{\sqrt{A}}\right)\|x\|, \ \ x \in \mathcal{H}. \end{aligned}$$

This gives $\|U_{\{m_n\}}T^{\dagger}_{\{m_n\}}\| \ \le 1 + \lambda + \frac{\mu}{\sqrt{A}}$. Since $\left(\lambda + \frac{\mu}{\sqrt{A}}\right) < 1$, the operator $U_{\{m_n\}}T^{\dagger}_{\{m_n\}}$ is invertible and

$$\|(U_{\{m_n\}}T^{\dagger}_{\{m_n\}})\| \ \le \ \frac{1}{1 - \left(\lambda + \frac{\mu}{\sqrt{A}}\right)}.$$

Also, every $x \in \mathcal{H}$ can be written as

$$\begin{aligned} x &= U_{\{m_n\}}T^{\dagger}_{\{m_n\}}(U_{\{m_n\}}T^{\dagger}_{\{m_n\}})^{-1}(x) \\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n}\Big\langle (U_{\{m_n\}}T^{\dagger}_{\{m_n\}})^{-1}(x),\ (T_{\{m_n\}}T^{*}_{\{m_n\}})^{-1}(x_{n,i})\Big\rangle y_{n,i}. \end{aligned}$$

Therefore

$$\begin{aligned} \|x\|^4 &= \left|\lim_{n\to\infty}\sum_{i=1}^{m_n}\Big\langle (U_{\{m_n\}}T^{\dagger}_{\{m_n\}})^{-1}(x),\ (T_{\{m_n\}}T^{*}_{\{m_n\}})^{-1}(x_{n,i})\Big\rangle\langle y_{n,i},\ x\rangle\right|^2 \\ &\le \lim_{n\to\infty}\frac{1}{A}\ \|(U_{\{m_n\}}T^{\dagger}_{\{m_n\}})^{-1}(x)\|^2\ \sum_{i=1}^{m_n}|\langle y_{n,i},\ x\rangle|^2 \\ &\le \frac{1}{A}\ \frac{1}{\left[1 - \left(\lambda + \frac{\mu}{\sqrt{A}}\right)\right]^2}\ \|x\|^2\ \ \lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle y_{n,i},\ x\rangle|^2, \quad x \in \mathcal{H}. \end{aligned}$$

This gives

$$\lim_{n\to\infty}\sum_{i=1}^{m_n}|\langle y_{n,i},\ x\rangle|^2 \ \ge \ A\left[1 - \left(\lambda + \frac{\mu}{\sqrt{A}}\right)\right]^2\|x\|^2.$$

Hence, $\{y_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$. □

Finally, we give the following sufficient condition for a perturbed sequence to be an approximative frame for $\mathcal{H}$.

Theorem 4.2. *Let $\{m_n\}$ be an increasing sequence of positive integers and $\{x_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be an approximative frame for $\mathcal{H}$ with approximative frame bounds A and B. Let $x_0 \neq 0$ be any element in $\mathcal{H}$ and $\{\alpha_{n,i}\}_{\substack{i=1,2,\cdots,m_n\\ n\in\mathbb{N}}}$ be any sequence of scalars. Then, $\{x_{n,i} + x_0\alpha_{n,i}\}_{\substack{i=1,2,\cdots m_n\\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$ if*

$$\lim_{n\to\infty}\sum_{i=1}^{m_n}|\alpha_{n,i}|^2 < \frac{A}{\|x_0\|^2}.$$

Proof. Let $y_{n,i} = x_{n,i} + \alpha_{n,i} x_0,\ \ i = 1, 2, \cdots, m_n,\ n \in \mathbb{N}$. Then, for any $x \in \mathcal{H}$, we have

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{m_n} |\langle x,\ x_{n,i} - y_{n,i}\rangle|^2 &= \lim_{n\to\infty} \sum_{i=1}^{m_n} |\langle x,\ \alpha_{n,i} x_0\rangle| \\ &\leq \lim_{n\to\infty} \sum_{i=1}^{m_n} |\alpha_{n,i}|^2\ \|x\|^2\ \|x_0\|^2.\end{aligned}$$

Therefore, $\{x_{n,i} + x_0\alpha_{n,i}\}_{\substack{i=1,2,\cdots m_n \\ n\in\mathbb{N}}}$ is an approximative frame for $\mathcal{H}$ if $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{m_n} |\alpha_{n,i}|^2\ \|x_0\|^2 < A$. $\square$

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Author information

S.K. Sharma, Department of Mathematics, Kirori Mal College, University of Delhi, Delhi 110 007, INDIA.
E-mail: sumitkumarsharma@gmail.com

A. Zothansanga, Department of Mathematics, University of Delhi, Delhi 110 007, INDIA.
E-mail: azothansanga26@yahoo.com

S.K. Kaushik, Department of Mathematics, Kirori Mal College, University of Delhi, Delhi 110 007, INDIA.
E-mail: shikk2003@yahoo.co.in

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